October 27, 2009

Math 323 — Exam II

1. *(20 points)* True or false? Give a short explanation.
   (a) $\sum(-1)^{n+1}(1/n^p)$ converges for all $p > 0$.
   (b) The set of irrational numbers is a closed set.
   (c) If the sequence (b_n) converges to b, then the set $B = \{b, b_1, b_2, b_3, \ldots\}$ is a closed set.
   (d) The set $\{(-1)^n(1 - \frac{1}{n}) : n \in \mathbb{N}\}$ is an open set.

2. *(20 points)* Suppose (y_n) is defined by $y_{n+1} = (2y_n + 3)/(y_n + 2)$ (and some specified value of y_1).
   (a) If (y_n) converges to the limit ℓ, find all possible values of ℓ. (Which one is the real limit, or even if it exists, may depend on the choice of y_1.)
   (b) Prove that, if $y_1 > \sqrt{3}$, then the sequence (y_n) is bounded below by $\sqrt{3}$.
   (c) Prove that, if $y_1 > \sqrt{3}$, then the sequence (y_n) is decreasing.
   (d) Suppose $y_1 = 2$. Find and prove the limit of (y_n) if it exists. If it does not exist, say so. And in either case, explain how you know.

3. *(10 points)* (A stronger version of the Ratio Test, with a similar proof) For a sequence (a_n) of nonzero terms, assume that there is a number r between 0 and 1 for which the ratio $|a_{n+1}/a_n|$ is eventually $\leq r$. Prove that $\sum a_n$ converges absolutely.

4. *(15 points)* (This was a homework problem; I hope the results on (a) are better here.) Let A, B be subsets of $\mathbb{R}$.
   (a) Prove that, if y is a limit point of $A \cup B$, then y is either a limit point of A or a limit point of B.
   (b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (c) Does the result about closures in (b) extend to infinite unions of sets?

5. *(15 points)* Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. We proved that f is continuous iff, for every open set U of $\mathbb{R}$, $f^{-1}(U)$ is open in A. Prove that, if f is continuous, then for every closed set F in $\mathbb{R}$, $f^{-1}(F)$ is closed in A.

6. *(20 points)* We define the distance $d(A, B)$ between two (nonempty) subsets A, B of $\mathbb{R}$ to be the infimum of the set of (nonnegative) numbers $|a - b|$ for all $a \in A$ and $b \in B$. If $A \cap B \neq \emptyset$, then for any $c \in A \cap B$ we have $|c - c| = 0$, so $d(A, B) = 0$.
   (a) Give an example to show that, if A, B are open intervals with $A \cap B = \emptyset$, we may still have $d(A, B) = 0$.
   (b) (Challenge) Give an example to show that, if A, B are closed nonempty sets with $A \cap B = \emptyset$, then we may <u>still</u> have $d(A, B) = 0$.
   (c) Prove that a point a is in the closure $\overline{B}$ of a set B if and only if $d(\{a\}, B) = 0$.
   (d) Prove that, if A, B are nonempty subsets of $\mathbb{R}$ with $A \cap B = \emptyset$, B closed and A compact, then $d(A, B) > 0$. (Hint: For each $a \in A$, $d(\{a\}, B) = \varepsilon(a) > 0$. Consider the open cover $\mathcal{U} = \{V_{\varepsilon(a)/2}(a) : a \in A\}$ of A.)

Math 323 — Solutions to Exam II

1. (a) True, by the Alternating Series Test. (It converges absolutely only for $p > 1$.)
   (b) False: For example, $(\sqrt{2}/n)$ is a sequence of irrationals converging to the rational 0.
   (c) True: The only limit point of B is b, which is in the set.
   (d) False: It does not contain open intervals around any of its elements.

2. (a) Because ℓ is the limit, we have $\ell = (2\ell + 3)/(\ell + 2)$, and solving for ℓ gives $\ell = \pm\sqrt{3}$.
   (b) Proof by induction: The case $n = 1$ is given. Make the induction hypothesis that $y_n > \sqrt{3}$. Then we want to show that $y_{n+1} > \sqrt{3}$. The following statements are either all true or all false (because when we multiply both sides of an inequality by the same quantity, we know that quantity is positive):

$$\begin{aligned} y_{n+1} = \frac{2y_n + 3}{y_n + 2} &> \sqrt{3} \\ 2y_n + 3 &> \sqrt{3}y_n + 2\sqrt{3} \\ (2 - \sqrt{3})y_n &> 2\sqrt{3} - 3 \\ y_n &> \frac{2\sqrt{3} - 3}{2 - \sqrt{3}} = \sqrt{3} \end{aligned}$$

   The last is true by hypothesis, so the first is true as well.
   (c) To show $y_{n+1} = (2y_n + 3)/(y_n + 2) < y_n$ is to show that $2y_n + 3 < y_n^2 + 2y_n$, or $3 < y_n^2$; but we know that is true by (b).
   (d) Because (y_n) is decreasing and bounded below by $\sqrt{3}$, it converges by the Monotone Convergence Theorem; and its limit, which must be either $\sqrt{3}$ or $-\sqrt{3}$ by (a), is $\sqrt{3}$ because none of the terms are negative.

3. We can mimic the proof of the Ratio Test: There is an $N \in \mathbb{N}$ for which $|a_{n+1}/a_n| \le r$ for all $n \ge N$. For all $n \ge N$ we have $|a_{n+1}| \le |a_n|r$, so for all $k > 0$,

$$|a_{N+k}| \le |a_{N+k-1}|r \le |a_{N+k-2}|r^2 \le \cdots \le |a_N|r^k \ .$$

Now $\sum_k |a_N|r^k$ is a geometric series with common ratio r less than 1 (in absolute value), so it converges. And then the displayed inequality shows that $\sum_k |a_{N+k}|$ converges by the Comparison Test. But this is the same as the series $\sum_n |a_n|$ except for a finite number ($N-1$, to be exact) of terms at the beginning; so $\sum_n |a_n|$ also converges, i.e., $\sum_n a_n$ converges absolutely.

4. (a) Because y is a limit point of $A \cup B$, there is a sequence (y_n) in $A \cup B$ with limit y. Either $y_n \in A$ for infinitely many $n \in \mathbb{N}$, or $y_n \in B$ for infinitely many $n \in \mathbb{N}$, or both; so in at least one of A or B there is a sequence with limit y, i.e., y is a limit point of either A or B.
   (b) Clearly a limit point of either A or B is a limit point of $A \cup B$, so together with (a) we see that the set L of limit points of $A \cup B$ is equal to the set L_A of limit points of A union the set L_B of limit points of B. Thus

$$\overline{A \cup B} = A \cup B \cup L = A \cup B \cup L_A \cup L_B = (A \cup L_A) \cup (B \cup L_B) = \overline{A} \cup \overline{B} \ .$$

   (c) No: Single-point sets are closed, but $\bigcup_{n\in\mathbb{N}}\{1/n\} = \{1/n : n \in \mathbb{N}\}$ is not closed.

5. One way is to use the result repeated in the question: Take a closed set F in $\mathbb{R}$; then F^c is open in $\mathbb{R}$, so $f^{-1}(F^c)$ is open in A, so $A\backslash f^{-1}(F^c)$ is closed in A; but

$$A\backslash f^{-1}(F^c) = \{x \in A : x \notin f^{-1}(F^c)\} = \{x \in A : f(x) \notin F^c\} = \{x \in A : f(x) \in F\} = f^{-1}(F)\,.$$

   Another way is to show that $f^{-1}(F)$ contains all its limit points in A, so suppose $x \in A$ and (x_n) is a sequence in $f^{-1}(F)$ with limit x. Then $(f(x_n))$ is a sequence in F and, because f is continuous, $f(x) = \lim f(x_n)$, so $f(x)$ is a limit point of F. Because F is closed, $f(x) \in F$, so $x \in f^{-1}(F)$.

6. (a) $A = (-1, 0)$, $B = (0, 1)$ works: $-1/n \in A$, $1/n \in B$, and $\inf\{|(-1/n)-1/n| : n \in \mathbb{N}\} = 0$.

   (b) $A = \mathbb{N}$, $B = \{n + \frac{1}{n+1} : n \in \mathbb{N}\}$ works: $\inf\{|n - (n + \frac{1}{n+1})| : n \in \mathbb{N}\} = 0$.

   (c) Suppose $a \in \overline{B}$. Then there is a sequence of elements (b_n) of B (which may eventually be a constant a) for which $|a - b_n|$ is less than any positive ε for sufficiently large n, so $\inf\{|a - b_n| : n \in \mathbb{N}\} = 0$, so $d(\{a\}, B) = \inf\{|a - b| : b \in B\} = 0$. Conversely, if $d(\{a\}, B) = 0$, then either $a \in B$ (so that $a \in \overline{B}$ and we are already finished) or, for each $n \in \mathbb{N}$, there is an element $b_n \in B$ for which $|a - b_n| < 1/n$. In the latter case, the sequence (b_n) in B converges to a, so a is a limit point of B, so $a \in \overline{B}$.

   (d) As suggested in the hint: For all $a \in A$, a is not the closed set $B = \overline{B}$, so by (c) we have that $d(\{a\}, B) = \varepsilon(a) > 0$. Then $\mathcal{U} = \{V_{\varepsilon(a)/2}(a) : a \in A\}$ is an open cover of A — every a in A is at least in $V_{\varepsilon(a)/2}(a)$ — so because A is compact, $\mathcal{U}$ has a finite subcover, say the sets $V_{\varepsilon(a_k)/2}(a_k)$ for $k = 1, 2, \ldots, n$. Then for every $a \in A$, a is in at least one of the sets $V_{\varepsilon(a_k)/2}(a_k)$, so $|a_k - a| < \varepsilon(a_k)/2$. Because a is at most $\varepsilon(a_k)/2$ away from a_k, it is at least $\varepsilon(a_k)/2$ away from every element b of B: $|b - a| = |b - a_k + a_k - a| \geq |b - a_k| - |a_k - a| > \varepsilon(a_k) - \varepsilon(a_k)/2 = \varepsilon(a_k)/2$. Therefore, for $\varepsilon = \min\{\varepsilon(a_k) : k = 1, 2, \ldots, n\}$, we see that $d(A, B) \geq \varepsilon/2 > 0$.