Theorem: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Euler's mnemonic: Suppose the polynomial $p(x)$ has roots $a_1, a_2 \ldots, a_n$ and constant term equal to 1. Then we have

$$p(x) = (1 - \frac{x}{a_1})(1 - \frac{x}{a_2}) \cdots (1 - \frac{x}{a_n}) \ .$$

What is true of polynomials must also be true of power series! (Wrong: If it were right, e^x, having no roots, must be a constant.) So because the roots of $(\sin x)/x$ are $\pm k\pi$ for each positive integer k, we have

$$\begin{aligned} \frac{\sin x}{x} &= (1 - \frac{x}{\pi})(1 - \frac{x}{-\pi})(1 - \frac{x}{2\pi})(1 - \frac{x}{-2\pi})(1 - \frac{x}{3\pi})(1 - \frac{x}{-3\pi}) \cdots \\ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned}$$

Expanding the product on the right side shows that the coefficient of x^2 is $-\sum(1/(k^2\pi^2))$. On the left, the coefficient of x^2 is $-1/6$. Equating these two gives the desired result.

Proof: Consider first

$$\begin{aligned} \sin\frac{(2k+1)x}{2} - \sin\frac{(2k-1)x}{2} &= \sin kx \cos\frac{1}{2}x + \sin\frac{1}{2}x \cos kx \\ &\qquad - \sin kx \cos\frac{1}{2}x + \sin\frac{1}{2}x\cos kx \\ &= 2\sin\frac{1}{2}x\cos kx \ . \end{aligned}$$

Thus, using the fact that we have a telescoping sum, we get

$$\begin{aligned} \sin\frac{(2n+1)x}{2} - \sin\frac{x}{2} &= \sum_{k=1}^{n}\left(\sin\frac{(2k+1)x}{2} - \sin\frac{(2k-1)x}{2}\right) \\ &= 2\sin\frac{x}{2}\left(\sum_{k=1}^{n}\cos kx\right) \ . \end{aligned}$$

Define

$$f_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos kx \ ;$$

so that, by the last computation, we have, except at $x = 2k\pi$,

$$f_n(x) = \frac{\sin\dfrac{(2k+1)x}{2}}{2\sin\dfrac{x}{2}} \ .$$

Now define

$$E_n = \int_0^{\pi} x f_n(x)\,dx = \frac{\pi^2}{4} + \sum_{k=1}^{n}\frac{(-1)^k - 1}{k^2} \ .$$

Thus,

$$E_{2n-1} = \frac{\pi^2}{4} - \sum_{k=1}^{n}\frac{2}{(2k-1)^2} \ .$$

But in view of the other way of writing $f_n(x)$ (valid on the interval $[0, \pi]$ except for the right endpoint), we can also write

$$E_{2n-1} = \int_0^\pi \frac{x/2}{\sin(x/2)} \sin \frac{(4n-1)x}{2}\, dx$$

Using integration by parts with

$$u = h(x) = \begin{cases} \dfrac{x/2}{\sin(x/2)} & \text{if } 0 < x \leq \pi \\ 1 & \text{if } x = 0 \end{cases}$$

and $dv = \sin \dfrac{(4n-1)x}{2}\, dx$ (you should check that $h'(0) = 0 = \lim\limits_{x \to 0} h'(x)$, to be sure that h' is continuous), we get

$$\begin{aligned} E_{2n-1} &= \left[\frac{-2}{4n-1} h(x) \cos \frac{(4n-1)x}{2} \right]_0^\pi + \frac{2}{4n-1} \int_0^\pi h'(x) \cos \frac{(4n-1)x}{2}\, dx \\ &= \frac{2}{4n-1} \left[1 + \int_0^\pi h'(x) \cos \frac{(4n-1)x}{2}\, dx \right] . \end{aligned}$$

Now we have

$$\left| \int_0^\pi h'(x) \cos \frac{(4n-1)x}{2}\, dx \right| \leq \int_0^\pi |h'(x)| \left| \cos \frac{(4n-1)x}{2} \right| dx \leq \int_0^\pi |h'(x)|\, dx$$

and the last expression does not change with n. So we see that E_{2n-1} is the fraction $\frac{2}{4n-1}$ times a quantity that is no larger than $1 + \int_0^\pi |h'(x)| dx$; and hence, as $n \to \infty$, we have that $E_{2n-1} \to 0$. Therefore,

$$\frac{\pi^2}{4} = \sum_{k=1}^\infty \frac{2}{(2k-1)^2} .$$

But by breaking up the original sum into odd and even terms, we also have

$$\sum_{k=1}^\infty \frac{1}{k^2} = \sum_{k=1}^\infty \frac{1}{(2k)^2} + \sum_{k=1}^\infty \frac{1}{(2k-1)^2} = \frac{1}{4} \sum_{k=1}^\infty \frac{1}{k^2} + \frac{\pi^2}{8} ,$$

so we get

$$\frac{3}{4} \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{8}$$

and the result follows.