Def: Let f be a <u>bounded</u> function on a <u>compact</u> interval $[a, b]$.

(a) A *partition* P of $[a, b]$ is a finite subset of $[a, b]$ containing a, b. If Q is a second partition of $[a, b]$ and $P \subseteq Q$, then Q is called a *refinement* of P.

(b) Write $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \ .$$

Then the *upper* and *lower sums* of f with respect to P are:

$$\begin{aligned} U(f, P) &:= \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\ L(f, P) &:= \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \end{aligned}$$

(Notation used in text: The sup here is denoted M_i, the inf is m_i, and $x_i - x_{i-1} =: \Delta_i x$.)

(c) The *upper* and *lower integrals* of f on $[a, b]$ are:

$$\begin{aligned} U(f) &:= \inf\{U(f, P) : P \text{ is a partition of } [\mathrm{a}, \mathrm{b}]\} \\ L(f) &:= \sup\{L(f, P) : P \text{ is a partition of } [\mathrm{a}, \mathrm{b}]\} \end{aligned}$$

(d) If $U(f) = L(f)$, then f is *integrable* on $[a, b]$, and the common value of $U(f) = L(f)$ is denoted $\int_a^b f$ or $\int_a^b f(x)\, dx$.