September 27, 2011

Math 323 — Exam I

Make sure your reasoning is clear. Points are specified.

1. *(20 points)* Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Then it can be shown (but don't do it!) that there is a sequence (a_n) of terms in A for which $\lim a_n = \sup(A)$.

   (a) Give an example of a set A for which there is no sequence in A with limit $\sup A$ that is eventually constant.

   (b) Give an example of a set A for which every sequence in A with limit $\sup A$ <u>is</u> eventually constant.

   (c) Prove that if c is a <u>negative</u> real number, then the set $cA = \{ca \ : \ a \in A\}$ is bounded below and $\inf(cA) = c \cdot \sup(A)$.

2. *(25 points)* Let us say that a sequence $(c_n)_{n=1}^{\infty}$ of real numbers *"cervonges* to c" (where $c \in \mathbb{R}$) if and only if there is an $N \in \mathbb{N}$ such that, for all $n > N$ and all $\varepsilon > 0$, $|c_n - c| < \varepsilon$.

   (a) If a sequence (c_n) cervonges to c, does (c_n) converge to c? Explain, and if not, give an example.

   (b) If a sequence (c_n) converges to c, does (c_n) cervonge to c? Explain, and if not, give an example.

3. *(10 points)* A problem in our text instructs us to define the phrase "converges to ∞" — most texts would say "diverges to ∞". Define what it should mean to say "(a_n) diverges to $-\infty$". (Warning: "$|a_n + \infty| < \varepsilon$" is gibberish. What <u>should</u> "close to $-\infty$" mean?)

4. *(20 points)* Concerning the Algebraic Limit Theorem :

   (a) Prove the product part of the ALT: If $\lim a_n = a$ and $\lim b_n = b$, then $\lim(a_n b_n) = ab$. To eliminate a case in the proof, you may assume that $a \neq 0$.

   (b) Using the ALT, prove that the limit of $(3n + 1)/(2n - 5)$ is as expected.

5. *(10 points)* Prove that, if $x_n \leq y_n \leq z_n$ for all n in $\mathbb{N}$ and $\lim x_n = \ell = \lim z_n$, then $\lim y_n$ also exists and is ℓ. You may use without proof the fact that, if $a \leq b \leq c$, then $|b| \leq \max(|a|, |c|)$.

6. *(15 points)* True or false. If true, give a quick proof; if false, give a counterexample.

   (a) $\sup(AB) = (\sup A)(\sup B)$. (Here, $AB = \{ab : a \in A, \, b \in B\}$.)

   (b) $\lim(a_n/b_n) = (\lim a_n)/(\lim b_n)$.

   (c) If $a_n \leq b_n$ for all n in $\mathbb{N}$, and $\lim a_n = a$ and $\lim b_n = b$, then $a \leq b$.

Math 323 — Solutions to Exam I

1. (a) One such A is the open interval $(0,1)$, but any set that is bounded above and does not contain its sup would work.

   (b) One such A is $\{1\}$, but any set in which the sup is in A but is separated from the rest of A by an open gap would work. (The technical phrase is that that sup is an "isolated point" of the set.)

   (c) For each x in cA, $x = ca$ for some a in A, and $\sup(A) \geq a$, so $c \cdot \sup(A) \leq ca = x$; thus x is bounded below by $c \cdot \sup(A)$. Now let b be a lower bound of cA; then for all a in A, because $ca \in cA$, $b \leq ca$, so $b/c \geq a$. Thus b/c is an upper bound for A, so $b/c \geq \sup(A)$, and hence $b \leq c \cdot \sup(A)$. It follows that $c \cdot \sup(A)$ is the greatest lower bound of cA.

2. (a) Yes. The definition of "cervonges" implies that $c_n = c$ for all $n \geq N$; so of course for any $\varepsilon > 0$ there is an N for which $|c_n - c| = 0 < \varepsilon$ for all $n \geq N$.

   (b) No: The sequence $(1/n)$ converges to 0, but it does not cervonge to 0.

3. One version of the definition might be: a_n converges to $-\infty$ iff, for each $B > 0$, there is an $N \in \mathbb{N}$ for which, for all $n \geq N$, $a_n \leq -B$.

4. (a) Let $\varepsilon > 0$ be given, and choose N in $\mathbb{N}$ sufficiently large that for all $n \geq N$,

$$|a_n - a| < \frac{\varepsilon}{2(|b|+1)} \qquad \text{and} \qquad |b_n - b| < \frac{\varepsilon}{2|a|} \qquad \text{and} \qquad |b_n - b| < 1 \ .$$

   Then for such n we have $||b_n| - |b|| < 1$, so that $|b_n| < |b| + 1$, and so

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b_n + a(b_n - b)| \leq |a_n - a||b_n| + |a||b_n - b| \\ &< \frac{\varepsilon}{2(|b|+1)}(|b|+1) + |a|\frac{\varepsilon}{2|a|} = \varepsilon \ . \end{aligned}$$

   Therefore, $\lim(a_n b_n) = ab$.

   (b) Because $(3n+1)/(2n-5)$ is equal to $(3 + \frac{1}{n})/(2 - \frac{5}{n})$, the limit of the latter is equal to the limit of the former. But using the fact that $\lim(c/n) = 0$ for any constant c, we get using the ALT that $\lim(3 + \frac{1}{n}) = (\lim 3) + (\lim \frac{1}{n}) = 3 + 0 = 3$ and similarly $\lim(2 - \frac{5}{n}) = (\lim 2) - (\lim \frac{5}{n}) = 2 - 0 = 2$, so again by the ALT (and the fact that $2 \neq 0$), we have $\lim((3n+1)/(2n-5)) = 3/2$.

5. Let $\varepsilon > 0$ be given, and pick N in $\mathbb{N}$ so that, for all $n \geq N$, $|x_n - \ell| < \varepsilon$ and $|z_n - \ell| < \varepsilon$. Then because $x_n - \ell \leq y_n - \ell \leq z_n - \ell$, we have, for all $n \geq N$, $|y_n - \ell| \leq \max(|x_n - \ell|, |z_n - \ell|) < \varepsilon$. Therefore $\lim y_n = \ell$.

6. (a) False: A counterexample is $A = \{-1\}$ and $B = \{0, 1\}$, because $\sup(AB) = \sup\{-1, 0\} = 0$, but $(\sup A)(\sup B) = (-1)(1) = -1$.

   (b) False: The hypothesis that $\lim b_n \neq 0$ is necessary. For instance $a_n = 1/n$ and $b_n = 1/n$ would make the left side equal to 1 and the right undefined.

   (c) True, by the Order Limit Theorem.