September 25, 2001

Math 323 — Exam I

1. *(25 points)* Using the definition of limit (i.e., not the Algebraic Limit Theorem), prove the following:
   (a) $\lim((3n+4)/(2n-1)) = 3/2$.
   (b) If $\lim x_n = x$ and $\lim y_n = y$, then $\lim(x_n - y_n) = x - y$.

2. *(20 points)* Use the Archimedean property to prove the following: If $A \subseteq \mathbb{R}$ is nonempty and bounded below, then there is a sequence (a_n) with $a_n \in A$ for all $n \in \mathbb{N}$ such that $\lim a_n = \inf A$. (Hint: Why, for each $n \in \mathbb{N}$, is there an element a_n for which $a_n - \inf A < 1/n$?)

3. *(10 points)* Recall that, in creating a definition for "convergence to ∞", the phrase "$|x_n - \ell| < \varepsilon$" in the usual definition of convergence to ℓ (where ε is thought of as small) is replaced with "$x_n > \varepsilon$" (where ε is thought of as large). Use this definition to prove that the sequence $(n - \frac{1}{n})_{n=1}^{\infty}$ converges to ∞.

4. *(20 points)* Let (x_n) be a sequence of real numbers, and define the sequence (y_n) (which may have terms of $-\infty$) by: $y_n = \inf\{x_m : m \geq n\}$.
   (a) Prove that $y_n \leq y_{n+1}$ for all $n \in \mathbb{N}$.
   (b) Prove that, if (x_n) converges to ℓ, then so does (y_n).

   You may use the fact that if $A \subseteq B$, then $\inf A \geq \inf B$. [Read this after the exam: Because (y_n) is increasing, it has a limit, which may be ∞ or $-\infty$. This limit is called the "limit inferior", or "lim inf" of (x_n). There is a parallel concept of "limit superior"; and (x_n) converges to ℓ iff
   $$\liminf x_n = \limsup x_n = \ell\ .]$$

5. *(25 points)* True or false? If true, prove it. If false, give a counterexample.
   (a) In any interval $I = \{x \in \mathbb{R} : a \leq x \leq b\}$ (where $a < b$, of course), there is no second-largest number, i.e., no number just below b.
   (b) If (y_n) is divergent and (x_n) is convergent with $\lim x_n \neq 0$, then $(x_n y_n)$ is divergent.
   (c) Let A, B be subsets of $\mathbb{R}$ that are nonempty and bounded below. If $\forall\, a \in A$, $\exists\, b \in B$ such that $a < b$, then we have $\inf A < \inf B$.
   (d) If A is a set of irrational real numbers that is bounded above, then $\sup A$ is irrational.

Math 323 — Solutions to Exam I

1. (a) Let $\varepsilon > 0$ be given. Pick $N \in \mathbb{N}$ so that $N > (1/4)((11/\varepsilon) + 2)$. Then for all $n \geq N$ we have :
$$\left|\frac{3n+4}{2n-1} - \frac{3}{2}\right| = \left|\frac{6n+8-(6n-3)}{2(2n-1)}\right| = \left|\frac{11}{4n-2}\right| < \varepsilon \ .$$
Therefore, $\lim((3n+4)/(2n-1)) = 3/2$.

   (b) Let $\varepsilon > 0$ be given. Pick $N \in \mathbb{N}$ so that for all $n > N$, $|x_n - x| < \varepsilon/2$ and $|y_n - y| < \varepsilon/2$. Then for all $n > N$ we have
$$|(x_n - y_n) - (x - y)| = |(x_n - x) - (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$
Therefore, $\lim(x_n - y_n) = x - y$.

2. Write $a^* = \inf A$. Now for each $n \in \mathbb{N}$, $a^* + (1/n)$ is greater than the greatest lower bound of A, so it is not a lower bound of A; hence $\exists\, a_n \in A$ such that $a_n < a^* + 1/n$, and hence $0 \leq a_n - a^* < 1/n$. We show (a_n) converges to a^*: Let $\varepsilon > 0$ be given, and pick $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then for all $n \geq N$, because $a_n - a^* \geq 0$, $|a_n - a^*| = a_n - a^*$, and by our choice of the a_n's, the last expression is less than $1/n$, which in turn is less than ε. Therefore $\lim(a_n) = a^*$.

3. Let $\varepsilon > 0$ be given, and pick $N \in \mathbb{N}$ such that $N > \varepsilon + 1$. Then for $n \geq N$ we have
$$n - \frac{1}{n} = \frac{n^2-1}{n} = (n-1)\frac{n+1}{n} > n - 1 > \varepsilon \ .$$
Therefore, $\lim(n - \frac{1}{n}) = \infty$.

4. (a) Because $\{x_m : m \geq n\} \supseteq \{x_m : m \geq n+1\}$, we have $y_n = \inf\{x_m : m \geq n\} \leq \inf\{x_m : m \geq n+1\} = y_{n+1}$.

   (b) Let $\varepsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that, for $n \geq N$, $|x_n - \ell| < \varepsilon/2$. Then for $n \geq N$, because all the x_m with $m \geq n$ are in the interval $(\ell - \varepsilon/2, \ell + \varepsilon/2)$, we have
$$\ell - \varepsilon/2 = \inf(\ell - \varepsilon/2, \ell + \varepsilon/2) \leq \inf\{x_m : m \geq n\} = y_n \leq x_n < \ell + \varepsilon/2 \ ,$$
so $|y_n - \ell| \leq \varepsilon/2 < \varepsilon$. Therefore, $\lim y_n = \ell$.

5. (a) True. Take any c in $[a, b]$ with $c < b$. Then there is a number q — indeed, a rational number, if we want one — for which $c < q < b$, so c is not the second-largest in the interval.

   (b) True. Assume BWOC that $(x_n y_n)$ converges. Then by the Algebraic Limit Theorem, $(1/x_n)$ also converges, so $(y_n) = ((1/x_n)(x_n y_n))$ also converges, a contradiction.

   (c) False. A counterexample is $A = \{1/(2n+1) : n \in \mathbb{N}\}$ and $B = \{1/(2n) : n \in \mathbb{N}\}$.

   (d) False. A counterexample is $A = \{-\sqrt{2}/n : n \in \mathbb{N}\}$.