Math 323 — Exam II

1. (20 points) True or false? Give a short explanation.

(a) \(\sum (-1)^{n+1}(1/n^p)\) converges for all \(p > 0\).
(b) The set of irrational numbers is a closed set.
(c) If the sequence \((b_n)\) converges to \(b\), then the set \(B = \{b, b_1, b_2, b_3, \ldots\}\) is a closed set.
(d) The set \(\{(-1)^n(1 - \frac{1}{n}) : n \in \mathbb{N}\}\) is an open set.

2. (20 points) Suppose \((y_n)\) is defined by \(y_{n+1} = (2y_n + 3)/(y_n + 2)\) (and some specified value of \(y_1\)).

(a) If \((y_n)\) converges to the limit \(\ell\), find all possible values of \(\ell\). (Which one is the real limit, or even if it exists, may depend on the choice of \(y_1\).)
(b) Prove that, if \(y_1 > \sqrt{3}\), then the sequence \((y_n)\) is bounded below by \(\sqrt{3}\).
(c) Prove that, if \(y_1 > \sqrt{3}\), then the sequence \((y_n)\) is decreasing.
(d) Suppose \(y_1 = 2\). Find and prove the limit of \((y_n)\) if it exists. If it does not exist, say so. And in either case, explain how you know.

3. (15 points) The “Cauchy Condensation Test” states: If \((a_n)\) is a decreasing sequence of positive numbers, then

\[
\sum a_n \text{ converges if and only if } \sum 2^n a_{2^n} \text{ converges.}
\]

Prove that if the second series converges, then so does the first one. [Hint: This proof includes elements both of the proof given in class of the fact that \(\sum 1/n\) diverges and the proof of the Integral Test. Look at the diagram:

![Diagram of Cauchy Condensation Test]

The lower graph is the graph of \((a_n)\), and the upper one is the graph of

\[a_1, a_2, a_2, a_4, a_4, a_4, a_4, a_8, a_8, a_8, a_8, a_8, a_8, a_8, \ldots,\]

of which the sum is \(\sum 2^n a_{2^n}\).]

TURN OVER! There are 6 questions.
4. (15 points) (This was a homework problem.) Let $A, B$ be subsets of $\mathbb{R}$.

(a) Prove that, if $y$ is a limit point of $A \cup B$, then $y$ is either a limit point of $A$ or a limit point of $B$.

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) Does the result about closures in (b) extend to infinite unions of sets?

5. (20 points) For each of the following series, tell whether it converges absolutely, converges conditionally, or diverges, and give a reason:

(a) \[ \sum_{n=1}^{\infty} (-1)^n \frac{n + 3}{\sqrt{2n^3 - 5}} \]

(b) \[ \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!} \]

6. (10 points) We know that a closed and bounded subset of $\mathbb{R}$ is compact. Give an example of a bounded subset of $\mathbb{Q}$ that is closed in $\mathbb{Q}$ (i.e., has the form $\mathbb{Q} \cap F$ where $F$ is closed in $\mathbb{R}$) that is not compact. Use either the open cover definition or the sequential definition of compactness to show that your set is not compact.
Math 323 — Solutions to Exam II

1. (a) True, by the Alternating Series Test. (It converges absolutely only for \( p > 1 \).)
(b) False: For example, \( (\sqrt{2}/n) \) is a sequence of irrationals converging to the rational 0.
(c) True: The only limit point of \( B \) is \( b \), which is in the set.
(d) False: It does not contain open intervals around any of its elements.

2. (a) Because \( \ell \) is the limit, we have \( \ell = (2\ell + 3)/(\ell + 2) \), and solving for \( \ell \) gives \( \ell = \pm \sqrt{3} \).
(b) Proof by induction: The case \( n = 1 \) is given. Make the induction hypothesis that \( y_n > \sqrt{3} \). Then we want to show that \( y_{n+1} > \sqrt{3} \). The following statements are either all true or all false (because when we multiply both sides of an inequality by the same quantity, we know that quantity is positive):

\[
y_{n+1} = \frac{2y_n + 3}{y_n + 2} > \sqrt{3} \]
\[
2y_n + 3 > \sqrt{3}y_n + 2\sqrt{3} \]
\[
(2 - \sqrt{3})y_n > 2\sqrt{3} - 3 \]
\[
y_n > \frac{2\sqrt{3} - 3}{2 - \sqrt{3}} = \sqrt{3} \]

The last is true by hypothesis, so the first is true as well.
(c) To show \( y_{n+1} = (2y_n + 3)/(y_n + 2) < y_n \) is to show that \( 2y_n + 3 < y_n^2 + 2y_n \), or \( 3 < y_n^2 \); but we know that is true by (b).
(d) Because \( (y_n) \) is decreasing and bounded below by \( \sqrt{3} \), it converges by the Monotone Convergence Theorem; and its limit, which must be either \( \sqrt{3} \) or \( -\sqrt{3} \) by (a), is \( \sqrt{3} \) because none of the terms are negative.

3. Following the hint: If we let \( (b_n) \) denote the second sequence in the hint, then each \( b_n \) is an \( a_m \) where \( m \leq n \) (to be specific, \( m \) is the largest power of 2 that is \( \leq n \)). Since \( a_n \) is a decreasing sequence, \( a_n \leq a_m = b_n \). Therefore, if the second series converges, then the first series converges by the Comparison Test.

4. (a) Because \( y \) is a limit point of \( A \cup B \), there is a sequence \( (y_n) \) in \( A \cup B \) with limit \( y \). Either \( y_n \in A \) for infinitely many \( n \in \mathbb{N} \), or \( y_n \in B \) for infinitely many \( n \in \mathbb{N} \), or both; so in at least one of \( A \) or \( B \) there is a sequence with limit \( y \), i.e., \( y \) is a limit point of either \( A \) or \( B \).
(b) Clearly a limit point of either \( A \) or \( B \) is a limit point of \( A \cup B \), so together with (a) we see that the set \( L \) of limit points of \( A \cup B \) is equal to the set \( L_A \) of limit points of \( A \) union the set \( L_B \) of limit points of \( B \). Thus

\[
\overline{A \cup B} = A \cup B \cup L = A \cup B \cup L_A \cup L_B = (A \cup L_A) \cup (B \cup L_B) = \overline{A} \cup \overline{B}.
\]

(c) No: Single-point sets are closed, but \( \bigcup_{n \in \mathbb{N}} \{1/n\} = \{1/n : n \in \mathbb{N}\} \) is not closed.

5. (a) The Alternating Series Test says that the series converges conditionally. But because the absolute value of the \( n \)-th term of the sequence of has degree \(-1/2\), we should expect
that it should not converge absolutely. Specifically, because

\[
\frac{n + 3}{\sqrt{2n^3 - 5}} = \frac{1 + \frac{3}{n}}{\sqrt{2n - \frac{5}{n^2}}} > \frac{1}{2} \frac{1}{\sqrt{n}}
\]

for sufficiently large \( n \), and \( \frac{1}{2} \sum 1/\sqrt{n} \) diverges, the series of absolute values of the given series also diverges.

(b) 
\[
\left| \frac{(-1)^n e^{2^{n+1}}}{(n + 1)!} \right| = \frac{2}{n + 1} \to 0 < 1,
\]
so the series converges absolutely by the Ratio Test.

6. Let \( S = \mathbb{Q} \cap [0, 1] \), a bounded subset of \( \mathbb{Q} \) that is closed in \( \mathbb{Q} \). If we take a sequence \( (s_n) \) in \( S \) that has limit \( 1/\sqrt{2} \) in \( \mathbb{R} \), then no subsequence of \( (s_n) \) converges to an element of \( S \) (because they all converge to \( 1/\sqrt{2} \), not in \( S \)); so \( S \) is not compact by the sequential definition. Similarly, \( U = \{(-\infty, 1/\sqrt{2} - 1/n) \cup (1/\sqrt{2} + 1/n, \infty) : n \in \mathbb{N}\} \) is an open cover of \( S \) with no finite subcover; so \( S \) is not compact by the open cover definition.