October 25, 2011

Math 323 — Exam II

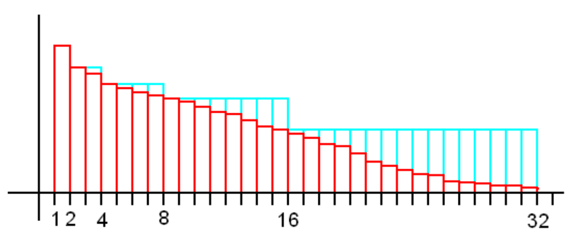
1. *(20 points)* True or false? Give a short explanation.
   (a) $\sum(-1)^{n+1}(1/n^p)$ converges for all $p > 0$.
   (b) The set of irrational numbers is a closed set.
   (c) If the sequence (b_n) converges to b, then the set $B = \{b, b_1, b_2, b_3, \ldots\}$ is a closed set.
   (d) The set $\{(-1)^n(1 - \frac{1}{n}) : n \in \mathbb{N}\}$ is an open set.

2. *(20 points)* Suppose (y_n) is defined by $y_{n+1} = (2y_n + 3)/(y_n + 2)$ (and some specified value of y_1).
   (a) If (y_n) converges to the limit ℓ, find all possible values of ℓ. (Which one is the real limit, or even if it exists, may depend on the choice of y_1.)
   (b) Prove that, if $y_1 > \sqrt{3}$, then the sequence (y_n) is bounded below by $\sqrt{3}$.
   (c) Prove that, if $y_1 > \sqrt{3}$, then the sequence (y_n) is decreasing.
   (d) Suppose $y_1 = 2$. Find and prove the limit of (y_n) if it exists. If it does not exist, say so. And in either case, explain how you know.

3. *(15 points)* The "Cauchy Condensation Test" states: If (a_n) is a decreasing sequence of positive numbers, then

$$\sum a_n \text{ converges} \qquad \text{iff} \qquad \sum 2^n a_{2^n} \text{ converges.}$$

Prove that if the second series converges, then so does the first one. [Hint: This proof includes elements both of the proof given in class of the fact that $\sum 1/n$ diverges and the proof of the Integral Test. Look at the diagram:



The lower graph is the graph of (a_n), and the upper one is the graph of

$$a_1, a_2, a_2, a_4, a_4, a_4, a_4, a_8, a_8, a_8, a_8, a_8, a_8, a_8, a_8 \ldots ,$$

of which the sum is $\sum 2^n a_{2^n}$.]

TURN OVER! There are 6 questions.

4. *(15 points)* (This was a homework problem.) Let A, B be subsets of $\mathbb{R}$.

   (a) Prove that, if y is a limit point of $A \cup B$, then y is either a limit point of A or a limit point of B.

   (b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (c) Does the result about closures in (b) extend to infinite unions of sets?

5. *(20 points)* For each of the following series, tell whether it converges absolutely, converges conditionally, or diverges, and give a reason:

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^n \frac{n+3}{\sqrt{2n^3 - 5}} \qquad\qquad \text{(b)} \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$$

6. *(10 points)* We know that a closed and bounded subset of $\mathbb{R}$ is compact. Give an example of a bounded subset of $\mathbb{Q}$ that is closed <u>in $\mathbb{Q}$</u> (i.e., has the form $\mathbb{Q} \cap F$ where F is closed in $\mathbb{R}$) that is not compact. Use either the open cover definition or the sequential definition of compactness to show that your set is not compact.

Math 323 — Solutions to Exam II

1. (a) True, by the Alternating Series Test. (It converges absolutely only for $p > 1$.)
   (b) False: For example, $(\sqrt{2}/n)$ is a sequence of irrationals converging to the rational 0.
   (c) True: The only limit point of B is b, which is in the set.
   (d) False: It does not contain open intervals around any of its elements.

2. (a) Because ℓ is the limit, we have $\ell = (2\ell + 3)/(\ell + 2)$, and solving for ℓ gives $\ell = \pm\sqrt{3}$.
   (b) Proof by induction: The case $n = 1$ is given. Make the induction hypothesis that $y_n > \sqrt{3}$. Then we want to show that $y_{n+1} > \sqrt{3}$. The following statements are either all true or all false (because when we multiply both sides of an inequality by the same quantity, we know that quantity is positive):

$$\begin{aligned} y_{n+1} = \frac{2y_n + 3}{y_n + 2} &> \sqrt{3} \\ 2y_n + 3 &> \sqrt{3}y_n + 2\sqrt{3} \\ (2 - \sqrt{3})y_n &> 2\sqrt{3} - 3 \\ y_n &> \frac{2\sqrt{3} - 3}{2 - \sqrt{3}} = \sqrt{3} \end{aligned}$$

   The last is true by hypothesis, so the first is true as well.
   (c) To show $y_{n+1} = (2y_n + 3)/(y_n + 2) < y_n$ is to show that $2y_n + 3 < y_n^2 + 2y_n$, or $3 < y_n^2$; but we know that is true by (b).
   (d) Because (y_n) is decreasing and bounded below by $\sqrt{3}$, it converges by the Monotone Convergence Theorem; and its limit, which must be either $\sqrt{3}$ or $-\sqrt{3}$ by (a), is $\sqrt{3}$ because none of the terms are negative.

3. Following the hint: If we let (b_n) denote the second sequence in the hint, then each b_n is an a_m where $m \le n$ (to be specific, m is the largest power of 2 that is $\le n$). Since a_n is a decreasing sequence, $a_n \le a_m = b_n$. Therefore, if the second series converges, then the first series converges by the Comparison Test.

4. (a) Because y is a limit point of $A \cup B$, there is a sequence (y_n) in $A \cup B$ with limit y. Either $y_n \in A$ for infinitely many $n \in \mathbb{N}$, or $y_n \in B$ for infinitely many $n \in \mathbb{N}$, or both; so in at least one of A or B there is a sequence with limit y, i.e., y is a limit point of either A or B.
   (b) Clearly a limit point of either A or B is a limit point of $A \cup B$, so together with (a) we see that the set L of limit points of $A \cup B$ is equal to the set L_A of limit points of A union the set L_B of limit points of B. Thus

$$\overline{A \cup B} = A \cup B \cup L = A \cup B \cup L_A \cup L_B = (A \cup L_A) \cup (B \cup L_B) = \overline{A} \cup \overline{B}\ .$$

   (c) No: Single-point sets are closed, but $\bigcup_{n\in\mathbb{N}}\{1/n\} = \{1/n : n \in \mathbb{N}\}$ is not closed.

5. (a) The Alternating Series Test says that the series converges conditionally. But because the absolute value of the n-th term of the sequence of has degree $-1/2$, we should expect

that it should not converge absolutely. Specifically, because

$$\frac{n+3}{\sqrt{2n^3-5}} = \frac{1+\frac{3}{n}}{\sqrt{2}\sqrt{n-\frac{5/2}{n^2}}} > \frac{1}{2}\frac{1}{\sqrt{n}}$$

for sufficiently large n, and $\frac{1}{2}\sum 1/\sqrt{n}$ diverges, the series of absolute values of the given series also diverges.

(b)

$$\left|\frac{(-1)^{n+1}2^{n+1}/(n+1)!}{(-1)^n 2^n/n!}\right| = \frac{2}{n+1} \to 0 < 1 \ ,$$

so the series converges absolutely by the Ratio Test.

6. Let $S = \mathbb{Q} \cap [0,1]$, a bounded subset of $\mathbb{Q}$ that is closed in $\mathbb{Q}$. If we take a sequence (s_n) in S that has limit $1/\sqrt{2}$ in $\mathbb{R}$, then no subsequence of (s_n) converges to an element of S (because they all converge to $1/\sqrt{2}$, not in S); so S is not compact by the sequential definition. Similarly, $\mathcal{U} = \{(-\infty, 1/\sqrt{2} - 1/n) \cup (1/\sqrt{2} + 1/n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover; so S is not compact by the open cover definition.