November 7, 2001

Math 323 — Exam IIB

1. *(25 points)* Using only the δ-ε definition of continuity, prove the following:
   (a) $f(x) = x^2 - 2x$ is continuous at $x = -1$. (Hint: Try $\delta = \min\{1, \varepsilon/5)\}$.)
   (b) If $f, g : S \to \mathbf{R}$ are continuous at the point c in S, then $f + 2g$ is also continuous at c.

2. *(15 points)* Prove that, if $p > 1$, then $\sum_{n=1}^{\infty}(1/n^p)$ converges. You may assume the "Cauchy Condensation Test": If (a_n) is a decreasing sequence of positive numbers, then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges iff} \quad \sum_{n=1}^{\infty} 2^n a_{2^n} \quad \text{converges.}$$

3. *(20 points)* Give an example of each of the following, or argue that such an example cannot exist.
   (a) A uncountably infinite set with empty interior.
   (b) A collection of compact sets whose union is neither open nor closed.
   (c) A finite collection of open sets whose intersection is nonempty and compact.
   (d) A continuous function $f : \mathbf{R} \to \mathbf{R}$ and a compact set A for which $f^{-1}(A)$ is not compact. (You may do this one with a sketch of a graph, but make clear what A and $f(A)$ are.)

4. *(20 points)* Prove that a subset A of $\mathbf{R}$ is open if and only if, for every convergent sequence (x_n) in R such that $\lim(x_n) \in A$, there are at most finitely many $n \in \mathbf{N}$ for which $x_n \notin A$. (Hint: If A is not open, then we can find an element $a \in A$ such that, for all $n \in \mathbf{N}$, $V_{1/n}(a) \not\subseteq A$, so $\exists\, x_n \notin A$ such that $|x_n - a| < 1/n$.)

5. *(20 points)* True or false? If true, prove it. If false, give a counterexample.
   (a) If each $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n/(1 + a_n)$ also converges.
   (b) If $(a_n) \to 0$ and and $|c_m - c_n| \leq a_n\ \forall\, m \geq n$, then (c_n) converges.
   (c) Suppose that $S \subseteq \mathbf{R}$, $c \in S$ and $f, g, h : S \to \mathbf{R}$ with $f(x) \leq g(x) \leq h(x)$ for all $x \in S$, and that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} h(x)$ both exist. Then $\lim_{x\to c} g(x)$ also exists and $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x) \leq \lim_{x\to c} h(x)$,
   (d) The uncountably infinite open cover $\{V_{0.1}(x) : x \in [0, 1]\}$ of the closed interval $[0, 1]$ has no finite subcover.

Math 323 — Solutions to Exam IIB

1. (a) Let $\varepsilon > 0$ be given, and pick $\delta = \min\{1, \varepsilon/5)\}$, as suggested in the hint. Then for $x \in \mathbf{R}$ with $|x-(-1)| < \delta \le 1$, we have $-2 < x < 0$, so $-5 < x-3 < -3$ and hence $|x+2| < 5$; so $|(x^2-2x)-((-1)^2-2(-1))| = |x^2-2x-3| = |x-(-1)||x+3| < (\varepsilon/5)5 = \varepsilon$. Therefore, f is continuous at -1.

   (b) Let $\varepsilon > 0$ be given, and let $\delta > 0$ be such that $x \in S$ and $|x-c| < \delta$ implies $|f(x)-f(c)| < \varepsilon/2$ and $|g(x)-g(c)| < \varepsilon/4$. Then $x \in S$ and $|x-c| < \delta$ implies

   $$|(f(x)+2g(x))-(f(c)+2g(c))| \le |f(x)-f(c)|+2|g(x)-g(c)| < \frac{\varepsilon}{2}+2\left(\frac{\varepsilon}{4}\right) = \varepsilon\ .$$

   Therefore, $f+g$ is continuous at c.

2. Clearly $1/n^p > 1/(n+1)^p$, so the terms of the given series are decreasing and positive. By the Cauchy Condensation Test, it suffices to show that $\sum 2^n(1/(2^n)^p)$ converges; and the latter series can be written $\sum 1/(2^{p-1})^n$. This is a geometric series, and because $p > 1$, the common ratio $1/2^{p-1}$ is less than one. So the series converges, and hence by the Cauchy Condensation Test $\sum_{n=1}^{\infty}(1/n^p)$ also converges.

3. (a) $\mathbf{I}$, the set of irrational numbers.

   (b) $K_n = [0, (n-1)/n]$: $\bigcup_{n=1}^{\infty} K_n = [0,1)$.

   (c) Impossible: The intersection of finitely many open sets is open, so if it is nonempty, it can't be closed and hence not compact.

   (d) One example is $f(x) = 2$ and $A = \{2\}$: $f^{-1}(A) = \mathbf{R}$.

4. Suppose A is open, and take a convergent sequence (x_n) with limit $a \in A$. Then there is an $\varepsilon > 0$ for which $V_\varepsilon(a) \subseteq A$, and there is an $N \in \mathbf{N}$ for which $n \ge N$ implies $|x_n - a| < \varepsilon$. Thus, for all n except for the finitely many $n < N$, we have $x_n \in V_\varepsilon(a) \subseteq A$.

   Conversely, suppose that every sequence converging to a limit in A has all but a finite number of its terms in A. Assume BWOC that A is not open; then there is an $a \in A$ for which there is no $V_\varepsilon(a) \subseteq A$. In particular, because for every $\varepsilon > 0$ we have $1/n < \varepsilon$ for some $n \in \mathbf{N}$, there is no n for which $V_{1/n}(a) \subseteq A$; so there is an element $x_n \in V_{1/n}(a)\backslash A$. But then the sequence (x_n) converges to a but has no terms at all in A. This contradiction shows that A is open.

5. (a) True: Because $a_n \ge 0$, we have $1 + a_n \ge 1$, so $a_n \ge a_n/(1+a_n)$. Thus the given series converges by the Comparison Test.

   (b) True: Let $\varepsilon > 0$ be given. Then $\exists\, N \in \mathbf{N}$ such that $n \ge N$ implies $|a_n - 0| < \varepsilon$. Thus, for $m > n \ge N$, $|c_m - c_n| \le |a_n| < \varepsilon$, so (c_n) is Cauchy and hence convergent.

   (c) False: The inequality holds if $\lim_{x\to c} g(x)$ exists, but this limit doesn't have to exist: Take $S = \mathbf{R}$, $c = 1$, $f(x) = 0$, $h(x) = 1$ (constant functions), and $g = \chi_{[1,\infty)}$, i.e., $g(x) = 0$ if $x < 1$ and 1 if $x \ge 1$.

   (d) False: It must be false, because $[0,1]$ is compact. In fact,

   $$\{V_{0.1}(x) : x = 0.09, 0.28, 0.47, 0.66, 0.85, 1\}$$

   is a subcover with 6 elements (which is the best we can do, because the $V_{0.1}(x)$'s have width 0.2 and do not contain their endpoints, while $[0,1]$ has length 1 and does contain its endpoints).