November 29, 2011

Math 323 — Exam III

Make sure your reasoning is clear. Points are specified. The cubing function and/or the "unit hyperbola" $y^2 - x^2 = 1$ may be useful somewhere.

1. *(16 points)* Prove that a differentiable function $f : I \to \mathbb{R}$ on an interval I is decreasing if and only if $f'(x) \leq 0$ for all x in I.

2. *(20 points)* Let f be a continuous function from an interval I into $\mathbb{R}$; denote its range $f(I)$ by J. Assume that f is strictly increasing, i.e., if $x, y \in I$ and $x < y$, then $f(x) < f(y)$. Then clearly f is one-to-one, so it has an inverse $f^{-1} : J \to I$.

   (a) Prove that f^{-1} is also strictly increasing.

   (b) Assume that f is differentiable at a in I and f^{-1} is differentiable at $f(a) = c$ in J. Find a formula for $(f^{-1})'(c)$ in terms of f, f', a and/or c. (The Chain Rule may be useful.)

   (c) Even though f may be differentiable at a point a in I, it is possible that f^{-1} is not differentiable at $f(a) = c$ in J. How could this happen? (What does the derivative of f mean geometrically?)

3. *(20 points)* Recall that a function f is called *Lipschitz* if there is a constant M for which the slope of the segment joining any two points on the graph of f has absolute value at most M.

   (a) Prove that a Lipschitz function is continuous.

   (b) Prove or give a counterexample: A continuous function is Lipschitz if its domain is a closed, bounded interval.

4. *(20 points)* Suppose f, g share a common domain in $\mathbb{R}$, that $f(x) \geq g(x)$ for x in that domain, that a is is a limit point of that domain, and that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Using the ε–δ definition of functional limit (not the Order Limit Theorem for sequences), prove that $L \geq M$. (Hint: Assume not and take $\varepsilon = (M - L)/2$; then use

$$|(M - g(x)) - (L - f(x)| \leq |M - g(x)| + |L - f(x)| \text{ .)}$$

5. *(24 points)* For each of the following statements, tell whether it is (necessarily) true or (possibly) false. If true, give a quick proof. If false, give a counterexample.

   (a) If f is differentiable on an interval I, then f is continuous on I.

   (b) If f is differentiable on an interval I, then f' is continuous on I.

   (c) If f is continuous on a subset A of $\mathbb{R}$, then we can extend f to a continuous function $\overline{f}$ from the closure $\overline{A}$ of A to $\mathbb{R}$.

   (d) (Challenge) If f is continuous on the interval I in $\mathbb{R}$ and $|f(x_2) - f(x_1)| < |x_2 - x_1|$ for all x_1, x_2 in I, then f is contractive on I (i.e., $\exists\, s \in (0, 1)$ s.t. $|f(x_2) - f(x_1)| < s|x_2 - x_1|$ for all x_1, x_2 in I).

Math 323 — Solutions to Exam III

1. ($\Rightarrow$) Because f is decreasing, whenever $x, y \in I$ and $x < y$, we have $f(x) \geq f(y)$, so the difference quotient $(f(y) - f(x))/(y - x)$ is nonpositive (the denominator is positive and the numerator is negative or 0); so its limit as $y \to x$ is also nonpositive, and that limit is $f'(x)$.

   ($\Leftarrow$) Let $x, y \in I$ and $x < y$. Then by the MVT $(f(y) - f(x))/(y - x) = f'(z)$ for some z between x and y. Now $f'(z) \leq 0$ and $y - x > 0$, so $f(y) - f(x) \leq 0$, i.e., $f(y) \leq f(x)$. Therefore, f is decreasing.

2. (a) Take c, d in J with $c < d$. Assume BWOC $f^{-1}(c) \geq f^{-1}(d)$. Then because f is strictly increasing, we have $c = f(f^{-1}(c)) \geq f(f^{-1}(d) = d$, a contradiction. So $f^{-1}(c) < f^{-1}(d)$.

   (b) We have $f(f^{-1}(y)) = y$ for all y in J, so by the Chain Rule,

   $$1 = \frac{d}{dy}\left(f(f^{-1}(y)\right)|_{y=c} = f'(f^{-1}(c))(f^{-1})'(c) = f'(a)(f^{-1})'(c) \ ,$$

   and hence $(f^{-1})'(c) = 1/f'(a)$.

   (c) If $f'(a) = 0$, then the formula in (b) fails; and indeed, if f has a horizontal tangent at some point a, then f' must have a vertical tangent at $f(a)$ and hence not be differentiable there. (Example: The function $f(x) = x^3$ is strictly increasing, but its horizontal tangent at $x = 0$ means that its inverse, the cube root function, has a vertical tangent at 0 and is not differentiable there.)

3. (a) Pick a in the domain A of the Lipschitz function f, which has corresponding constant M; and let $\varepsilon > 0$ be given. Set $\delta = \varepsilon/M$. Then because $|f(x) - f(a)| \leq M|x - a|$ for all x in A, if $|x - a| < \delta$, we have $|f(x) - f(a)| < M \cdot (\varepsilon/M) = \varepsilon$.

   (b) A counterexample is the square root function, which is continuous but has a vertical tangent at 1, so it is continuous but not Lipschitz on the interval $[0, 1]$. (The cube root function also works on the interval $[-1, 1]$, so that the vertical tangent comes at an interior point.)

4. Assume BWOC that $M > L$, and take $\varepsilon = (M - L)/2$. Then $\exists\, \delta > 0$ such that $|x - a| < \delta$ (and x in the common domain of f and g) implies $|f(x) - L| < \varepsilon$ and $|g(x) - M| < \varepsilon$. But then for such x, $f(x) - g(x) \geq 0$, so

   $$\begin{aligned} M - L &\leq (M - L) + (f(x) - g(x)) \leq |(M - g(x)) - (L - f(x))| \\ &\leq |M - g(x)| + |L - f(x)| < 2\varepsilon = M - L \ , \end{aligned}$$

   the desired contradiction.

5. (a) True: A differentiable function must first be continuous.

   (b) False: f' can't have any jump discontinuities, but it can be discontinuous, like the derivative of $x^2 \sin(1/x)$ (assigned the value 0 at $x = 0$).

   (c) False: If $A = \mathbb{R}\backslash\{0\}$, then $f(x) = 1/x$ is continuous on A, but it cannot be extended to a continuous function on $\overline{A} = \mathbb{R}$. (If f is <u>uniformly</u> continuous on A, then it <u>can</u> extended to $\overline{A}$.)

   (d) False: The function $y = f(x) = \sqrt{x^2 + 1}$ has derivative $f'(x) = x/\sqrt{x^2 + 1}$, which has values < 1 but as $x \to \infty$, $f'(x) \to 1$. So, even though, by the MVT, we always have $|f(x_2) - f(x_1)| < |x_2 - x_1|$, there is no $s < 1$ for which $|f(x_2) - f(x_1)| < s|x_2 - x_1|$ for all x_1, x_2. (Remember that with a contractive function g and any point a, the sequence $a, g(a), g^2(a), g^3(a), \ldots$ was Cauchy and approached a fixed point of g. This f has no fixed points, because its graph does not cross the line $y = x$; and for any a the corresponding sequence is not Cauchy — it diverges to ∞, though very slowly.)