Math 323 — Exam III

Make sure your reasoning is clear. Points are specified. The cubing function and/or the “unit hyperbola” $y^2 - x^2 = 1$ may be useful somewhere.

1. (16 points) Prove that a differentiable function $f : I \rightarrow \mathbb{R}$ on an interval $I$ is decreasing if and only if $f'(x) \leq 0$ for all $x$ in $I$.

2. (20 points) Let $f$ be a continuous function from an interval $I$ into $\mathbb{R}$; denote its range $f(I)$ by $J$. Assume that $f$ is strictly increasing, i.e., if $x, y \in I$ and $x < y$, then $f(x) < f(y)$. Then clearly $f$ is one-to-one, so it has an inverse $f^{-1} : J \rightarrow I$.
   
   (a) Prove that $f^{-1}$ is also strictly increasing.
   
   (b) Assume that $f$ is differentiable at $a$ in $I$ and $f^{-1}$ is differentiable at $f(a) = c$ in $J$. Find a formula for $(f^{-1})'(c)$ in terms of $f$, $f'$, $a$ and/or $c$. (The Chain Rule may be useful.)

   (c) Even though $f$ may be differentiable at a point $a$ in $I$, it is possible that $f^{-1}$ is not differentiable at $f(a) = c$ in $J$. How could this happen? (What does the derivative of $f$ mean geometrically?)

3. (20 points) Recall that a function $f$ is called Lipschitz if there is a constant $M$ for which the slope of the segment joining any two points on the graph of $f$ has absolute value at most $M$.

   (a) Prove that a Lipschitz function is continuous.

   (b) Prove or give a counterexample: A continuous function is Lipschitz if its domain is a closed, bounded interval.

4. (20 points) Suppose $f, g$ share a common domain in $\mathbb{R}$, that $f(x) \geq g(x)$ for $x$ in that domain, that $a$ is a limit point of that domain, and that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Using the $\varepsilon-\delta$ definition of functional limit (not the Order Limit Theorem for sequences), prove that $L \geq M$. (Hint: Assume not and take $\varepsilon = (M - L)/2$; then use
   
   $$|(M - g(x)) - (L - f(x))| \leq |M - g(x)| + |L - f(x)|.$$  

5. (24 points) For each of the following statements, tell whether it is (necessarily) true or (possibly) false. If true, give a quick proof. If false, give a counterexample.

   (a) If $f$ is differentiable on an interval $I$, then $f$ is continuous on $I$.

   (b) If $f$ is differentiable on an interval $I$, then $f'$ is continuous on $I$.

   (c) If $f$ is continuous on a subset $A$ of $\mathbb{R}$, then we can extend $f$ to a continuous function $\overline{f}$ from the closure $\overline{A}$ of $A$ to $\mathbb{R}$.

   (d) (Challenge) If $f$ is continuous on the interval $I$ in $\mathbb{R}$ and $|f(x_2) - f(x_1)| < |x_2 - x_1|$ for all $x_1, x_2$ in $I$, then $f$ is contractive on $I$ (i.e., $\exists s \in (0, 1)$ s.t. $|f(x_2) - f(x_1)| < s|x_2 - x_1|$ for all $x_1, x_2$ in $I$).
Math 323 — Solutions to Exam III

1. (⇒) Because \( f \) is decreasing, whenever \( x, y \in I \) and \( x < y \), we have \( f(x) \geq f(y) \), so the difference quotient \( (f(y) - f(x))/(y - x) \) is nonpositive (the denominator is positive and the numerator is negative or 0); so its limit as \( y \to x \) is also nonpositive, and that limit is \( f'(x) \).

(⇐) Let \( x, y \in I \) and \( x < y \). Then by the MVT \( (f(y) - f(x))/(y - x) = f'(z) \) for some \( z \) between \( x \) and \( y \). Now \( f'(z) \leq 0 \) and \( y - x > 0 \), so \( f(y) - f(x) \leq 0 \), i.e., \( f(y) \leq f(x) \). Therefore, \( f \) is decreasing.

2. (a) Take \( c, d \) in \( J \) with \( c < d \). Assume BWOC \( f^{-1}(c) \geq f^{-1}(d) \). Then because \( f \) is strictly increasing, we have \( c = f(f^{-1}(c)) \geq f(f^{-1}(d)) = d \), a contradiction. So \( f^{-1}(c) < f^{-1}(d) \).

(b) We have \( f(f^{-1}(y)) = y \) for all \( y \) in \( J \), so by the Chain Rule,

\[
1 = \frac{d}{dy} (f(f^{-1}(y)))|_{y=c} = f'(f^{-1}(c))(f^{-1})'(c) = f'(a)(f^{-1})'(c),
\]

and hence \( (f^{-1})'(c) = 1/f'(a) \).

(c) If \( f'(a) = 0 \), then the formula in (b) fails; and indeed, if \( f \) has a horizontal tangent at some point \( a \), then \( f' \) must have a vertical tangent at \( f(a) \) and hence not be differentiable there. (Example: The function \( f(x) = x^3 \) is strictly increasing, but its horizontal tangent at \( x = 0 \) means that its inverse, the cube root function, has a vertical tangent at 0 and is not differentiable there.)

3. (a) Pick \( a \) in the domain \( A \) of the Lipschitz function \( f \), which has corresponding constant \( M \); and let \( \varepsilon > 0 \) be given. Set \( \delta = \varepsilon/M \). Then because \( |f(x) - f(a)| \leq M|x - a| \) for all \( x \) in \( A \), if \( |x - a| < \delta \), we have \( |f(x) - f(a)| < M \cdot (\varepsilon/M) = \varepsilon \).

(b) A counterexample is the square root function, which is continuous but has a vertical tangent at 1, so it is continuous but not Lipschitz on the interval \([-1, 1]\), so that the vertical tangent comes at an interior point.

4. Assume BWOC that \( M > L \), and take \( \varepsilon = (M - L)/2 \). Then \( \exists \delta > 0 \) such that \( |x - a| < \delta \) (and \( x \) in the common domain of \( f \) and \( g \)) implies \( |f(x) - L| < \varepsilon \) and \( |g(x) - M| < \varepsilon \). But then for such \( x \), \( f(x) - g(x) \geq 0 \), so

\[
M - L \leq (M - L) + (f(x) - g(x)) \leq |(M - g(x)) - (L - f(x))| \\
\leq |M - g(x)| + |L - f(x)| < 2\varepsilon = M - L,
\]

the desired contradiction.
5. (a) True: A differentiable function must first be continuous.

(b) False: \( f' \) can’t have any jump discontinuities, but it can be discontinuous, like the derivative of \( x^2 \sin(1/x) \) (assigned the value 0 at \( x = 0 \)).

(c) False: If \( A = \mathbb{R} \setminus \{0\} \), then \( f(x) = 1/x \) is continuous on \( A \), but it cannot be extended to a continuous function on \( \overline{A} = \mathbb{R} \). (If \( f \) is uniformly continuous on \( A \), then it \underline{can} extended to \( \overline{A} \).

(d) False: The function \( y = f(x) = \sqrt{x^2 + 1} \) has derivative \( f'(x) = x/\sqrt{x^2 + 1} \), which has values < 1 but as \( x \to \infty \), \( f'(x) \to 1 \). So, even though, by the MVT, we always have \( |f(x_2) - f(x_1)| < |x_2 - x_1| \), there is no \( s < 1 \) for which \( |f(x_2) - f(x_1)| < s|x_2 - x_1| \) for all \( x_1, x_2 \). (Remember that with a contractive function \( g \) and any point \( a \), the sequence \( a, g(a), g^2(a), g^3(a), \ldots \) was Cauchy and approached a fixed point of \( g \). This \( f \) has no fixed points, because its graph does not cross the line \( y = x \); and for any \( a \) the corresponding sequence is not Cauchy — it diverges to \( \infty \), though very slowly.)