November 13, 2001

Math 323 — Exam III

1. *(20 points)*
   (a) Prove that, if $f : [a, b] \to \mathbf{R}$ is continuous and $f(x) > 0$ for all $x \in [a, b]$, then $1/f$ is bounded above on $[a, b]$.
   (b) Give an example to show that (a) is false if the closed interval $[a, b]$ is replaced by the open interval (a, b).

2. *(30 points)* In this problem, you may assume without proof that sin and cos are continuous functions on all of $\mathbf{R}$ and that $\lim_{t\to 0}((\sin t)/t) = 1$.
   (a) Use the identity $\sin^2 A + \cos^2 A = 1$ and the fact that $\cos 0 \neq -1$ to prove that $\lim_{t\to 0}(((\cos t) - 1)/t) = 0$.
   (b) Explain why the limit in (a) is also a consequence of the Interior Extremum Theorem — or as we called it in class, Fermat's Other Theorem. (Your explanation will need information about sin and/or cos beyond what is explicitly listed above. Hint: Interpret the limit as a derivative.)
   (c) Use the identity $\cos A - \cos B = -2\sin(\frac{1}{2}(A - B))\sin(\frac{1}{2}(A + B))$ to find the derivative of cos at the x-value c, from the limit definition of derivative. (In this process, of course, you are proving that cos is differentiable at c).
   (d) If we restrict the domain of the cos function to the interval $[0, \pi]$, then cos becomes one-to-one, with range $[-1, 1]$. Thus, it has an inverse, $\arccos : [-1, 1] \to [0, \pi]$, which we assume is also differentiable. Use the Chain Rule to find the derivative of arccos (with respect to y, if we think of the domain of cos as part of the x-axis and the domain of arccos as part of the y-axis). To simplify your answer, you may use without proving it the fact that that sin is ≥ 0 on $[0, \pi]$. (Hint: What is $\cos(\arccos(y))$?)

3. *(25 points)* Suppose that f is a function from an interval I into $\mathbf{R}$ that satisfies the Intermediate Value Property (on every subinterval of I).
   (a) Prove that, if there exist $x_1, x_2 \in I$ such that $f(x_1) \neq f(x_2)$, then $f(I)$ is uncountable.
   (b) Prove that, if f is monotone, then it is continuous. (You may just prove that it is continuous at each <u>interior</u> point c of I — the proof at an endpoint would be similar. Hint: Suppose f is decreasing. For a given $\varepsilon > 0$, find $a, b \in I$ such that $a < c < b$ and $f(a) < f(c) + \varepsilon$ and $f(b) > f(c) - \varepsilon$; then what δ works?)

4. *(25 points)* Suppose $f : [a, b] \to \mathbf{R}$ is such that f' is continuous on $[a, b]$ and f'' exists on (a, b). Then there is a $c \in (a, b)$ for which

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2 .$$

   (Hint: For the functions $g(x) = f(x) - (f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2)$ and $h(x) = (x - a)^2$, check that $g(a) = g'(a) = h(a) = h'(a) = 0$, and apply the Generalized Mean Value Theorem twice to get $g(b)/h(b) = g'(c_1)/h'(c_1) = g''(c)/h''(c)$ where $a < c < c_1 < b$. Read after the exam: We can extend this to show that, if $f(x)$ is k times differentiable, then it is approximated by its Taylor polynomial of degree k.)

Math 323 — Solutions to Exam III

1. (a) By the Extreme Value Theorem, there is a $c \in [a,b]$ for which $f(c) \le f(x)$ for all $x \in [a,b]$, and by hypothesis $f(c) > 0$. Thus, $1/f(x) \le 1/f(c)$ for all $x \in [a,b]$.

   (b) Let $a = 0$, $b = 1$, and $f(x) = x$. Then $1/x$ is not bounded above on $(0,1)$.

2. (a) Because $\lim_{t\to 0}((\sin t)/t)$ exists and the denominator has limit 0, the numerator must also have limit 0 (because otherwise the quotient becomes unbounded); and because cos is continuous, $\lim_{t\to 0}(1+\cos t) = 1 + \cos 0$, which we are told we may assume is not 0. Therefore:

$$\begin{aligned}
\lim_{t\to 0}\frac{(\cos t)-1}{t} &= -\lim_{t\to 0}\frac{1-\cos t}{t} = -\lim_{t\to 0}\left(\frac{1-\cos^2 t}{t}\cdot\frac{1}{1+\cos t}\right)\\
&= -\lim_{t\to 0}\left(\frac{\sin^2 t}{t}\cdot\frac{1}{1+\cos t}\right) = -\lim_{t\to 0}\left(\frac{\sin t}{t}\cdot\frac{\sin t}{1+\cos t}\right)\\
&= -1\cdot\frac{0}{1+\cos 0} = 0\ .
\end{aligned}$$

   (b) Because $\cos 0 = 1$, this limit is $\cos'(0)$, which we are assuming exists. Because cos does not take any values greater than 1, cos has a maximum at 0; so by the Interior Extremum Theorem, its derivative there is 0.

   (c) Using the definition of derivative, the given identity, the Algebraic Limit Theorem and the continuity of sin, we have:

$$\begin{aligned}
\cos'(c) &= \lim_{x\to d}\frac{\cos x - \cos c}{x-c} = \lim_{x\to c}\frac{-2\sin(\frac{1}{2}(x-c))\sin(\frac{1}{2}(x+c))}{x-c}\\
&= -\lim_{x\to c}\left[\frac{\sin(\frac{1}{2}(x-c))}{\frac{1}{2}(x-c)}\cdot\sin(\frac{1}{2}(x+c))\right] = -1\cdot\sin(\frac{1}{2}(c+c)) = -\sin c\ .
\end{aligned}$$

   (d) Because cos is the inverse of arccos, we have $\cos(\arccos(y)) = y$ for all $y \in [-1,1]$. Differentiating both sides with respect to y (using the Chain Rule), we get

$$-\sin(\arccos(y))\cdot(\arccos'(y)) = 1\ ,$$

   and hence $\arccos'(y) = -1/\sin(\arccos(y))$. And because sin is nonnegative on the range of arccos, the "Pythagorean" identity in (a) shows that

$$\sin(\arccos(y)) = \sqrt{1-\cos^2(\arccos(y))} = \sqrt{1-y^2}\ ,$$

   so $\arccos'(y) = -1/\sqrt{1-y^2}$.

3. (a) Because f has the Intermediate Value Property, $f(I)$ includes at least the whole interval with endpoints $f(x_1)$ and $f(x_2)$; and any nontrivial interval is uncountable.

   (b) WLOG, assume f is decreasing (for a change). Take c an interior point of I, and let $\varepsilon > 0$ be given. Pick $a, b \in I$ such that $a < c < b$; then $f(a) \ge f(c) \ge f(b)$. If $f(a) - f(c) < \varepsilon$, fine. Otherwise, pick $r > f(c)$ such that $r - f(c) < \varepsilon$; then $f(c) < r < f(a)$, so by the Intermediate Value Property there is an $a_1 \in (a,c)$ for which $f(a_1) = r$, and we can replace a with a_1. Similarly we can arrange to have $f(c) - f(b) < \varepsilon$. Let $\delta = \min\{c-a, b-c\}$. Then if $x \in I$ and $|x-c| < \delta$, we have $a < x < b$, so $f(a) \ge f(x) \ge f(b)$, so $f(a) - f(c) \ge f(x) - f(c) \ge f(b) - f(c)$, so $|f(x) - f(c)| \le \max\{f(a) - f(c), f(b) - f(c)\} < \varepsilon$.

4. By the definitions of g, h, they both have continuous first derivatives on $[a, b]$ and (existing) second derivatives on (a, b). Clearly $h(a) = (a-a)^2 = 0$, and because $h'(x) = 2(x-a)$ we also have $h'(a) = 0$. And, $g(a) = f(a) - (f(a) + f'(a)(a-a)^2 + \frac{1}{2}f''(a)(a-a)^2) = 0$, and because $g'(x) = f'(x) - (f'(a) + f''(a)(x-a))$ we also have $g'(a) = f'(a) - (f'(a) + f''(a)(a-a)) = 0$. Thus, applying the GenMVT to g, h, we see that there is a c_1 between a and b for which

$$\frac{g(b)}{h(b)} = \frac{g(b) - g(a)}{h(b) - h(a)} = \frac{g'(c_1)}{h'(c_1)} \ ;$$

and then applying the GenMVT to g', h' we see that there is a c between a and c_1 for which

$$\frac{g'(c_1)}{h'(c_1)} = \frac{g'(c_1) - g'(a)}{h'(c_1) - h'(a)} = \frac{g''(c)}{h''(c)} \ .$$

Putting the last two equations together and substituting the definitions of g, h, we have

$$\frac{f(b) - (f(a) + f'(a)(b-a) + \frac{1}{2}f''(a)(b-a)^2)}{(b-a)^2} = \frac{f''(c) - f''(a)}{2} \ ,$$

and the result follows by a little algebra.