December 16, 2011

Math 323 — Final Exam

Make sure your reasoning is clear. Points are specified.

1. *(20 points)* Prove the Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then there is a point c in $[a, b]$ for which $f(c)$ is the "average value" $(\int_a^b f)/(b - a)$ of f on $[a, b]$. (Hint for one possible proof: Denote by m and M the extreme values of f on $[a, b]$, and use them to find bounds on $\int_a^b f$.)

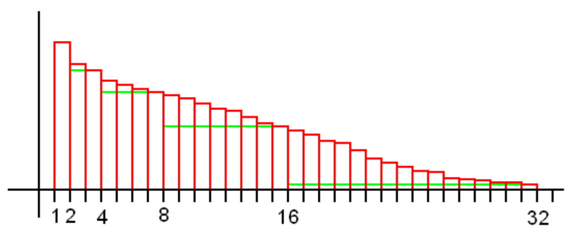
2. *(18 points)* Give examples to show the following:

   (a) The intersection of infinitely many nested, nonempty, bounded, open sets $U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots$ may be empty.

   (b) A continuous function on a bounded open set may be unbounded.

   (c) A sequence in an unbounded closed set may not have a convergent subsequence.

3. *(24 points)* Prove the other half of the Cauchy Condensation Test: If (a_n) is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=0}^{\infty} 2^k a_{2^k}$ also converges. (Hint: It is enough to show that the convergence of $\sum_{n=2}^{\infty} a_n$ implies the convergence of $\sum_{k=1}^{\infty} 2^{k-1} a_{2^k}$. Consider the diagram below.)



4. *(24 points)* True (in every case) or (possibly) false? If true, give a short proof. (For example, refer to a theorem.) If false, give a counterexample.

   (a) On a closed interval $[a, b]$, every continuous function is the derivative of some (differentiable) function.

   (b) On a closed interval $[a, b]$, every derivative of a (differentiable) function is a continuous function.

   (c) If f is integrable on $[a, b]$ and $F(x) = \int_a^x f$, then $F' = f$.

   (d) If f is integrable and g differs from f at only a finite number of points, then g is also integrable.

TURN OVER! One more question.

5. *(14 points)* Recall the Generalized Mean Value Theorem: If g, h are continuous on $[a, b]$ and differentiable on (a, b), then there exists c in (a, b) for which

$$(g(b) - g(a))h'(c) = (h(b) - h(a))g'(c) \ .$$

Let $f(x)$ be twice differentiable on $[0, b]$. Prove the second case of Lagrange's Remainder Theorem: There is a c in $(0, b)$ for which

$$f(b) = f(0) + f'(0)b + \frac{1}{2}f''(c)b^2 \ .$$

(The first case is the Mean Value Theorem.) Hint: Let $g(x) = f(x) - (f(0) + f'(0)x)$ and $h(x) = x^2$, and check $g(0) = g'(0) = 0$.

Math 323 — Solutions to Final Exam

1. There are two possible proofs; one is shorter, and the other avoids any mention of derivatives:

   (i) Because f is continuous on $[a, b]$, by the FTC we know that $\int_a^x f = F$ is a function whose derivative is f. By the Mean Value Theorem (for Derivatives), there is a c between a and b for which $F'(c)(b - a) = F(b) - F(a)$; and because $F'(c) = f(c)$, $F(b) = \int_a^b f$ and $F(a) = 0$, the result follows.

   (ii) Because f is continuous on $[a, b]$, by the Extreme Value Theorem it takes on its minimum and maximum values, say at $d, e \in [a, b]$ respectively. Then because $f(d) \le f(x) \le f(e)$ for all $x \in [a, b]$, we have $\int_a^b f(d) \le \int_a^b f \le \int_a^b f(e)$, i.e., $f(d)(b-a) \le \int_a^b f \le f(e)(b-a)$. Dividing through by $(b-a)$, we get $f(d) \le (\int_a^b f)/(b-a) \le f(e)$, and by the Intermediate Value Theorem we know there is an x-value c between d, e, and hence in $[a, b]$, for which $f(c) = (\int_a^b f)/(b - a)$.

2. Here are some possible examples:

   (a) $U_n = (0, 1/n)$.

   (b) $f(x) = 1/x$ on $(0, 1)$.

   (c) $x_n = n$ in $[0, \infty)$.

3. Let (b_k) denote the sequence

$$a_2, a_4, a_4, a_8, a_8, a_8, a_8, a_{16}, a_{16}, a_{16}, a_{16}, a_{16}, a_{16}, a_{16}, a_{16}, a_{32}, \ldots$$

   i.e., b_k is a_{2^n} where 2^n is the smallest power of 2 greater than k. Thus (b_k) consists of 2^{n-1} copies of a_{2^n}, so $\sum_{k=1}^{\infty} b_k = \sum_{n=0}^{\infty} 2^{n-1} a_{2^n}$. Now because the a_n's are nonnegative, so are the b_k; so to show that the b-series converges, it suffices by the Comparison Test to show $b_k \le a_k$ for each k. Now, $b_k = a_{2^n}$ where $2^n > k$, and because the a_n's are decreasing, it follows that $b_k = a_{2^n} \le a_k$. So the b-series converges, and hence $a_1 + 2\sum_{n=0}^{\infty} 2^{n-1} a_{2^n} = \sum_{n=0}^{\infty} 2^n a_{2^n}$ also converges.

4. (a) True, by the Fundamental Theorem of Calculus.

   (b) False: We know that $x^2 \sin(1/x)$ (defined to be 0 at $x = 0$) has a derivative that is discontinuous at 0.

   (c) False: Let $f(x) = 0$ if $x \neq 0$ and 1 if $x = 0$. Then $F(x) = \int_0^x f = 0$, the constant function, so F' is also the constant function 0, not f.

   (d) True: If P is a partition for which $U(f, P) - L(f, P) < \varepsilon/2$, then pick a partition Q that isolates the points where g differs from f in intervals of which the lengths add up to $\varepsilon/(2M)$ where M is a bound on $|g|$. Then $U(g, P \cup Q) - L(g, P \cup Q) < \varepsilon$.

5. As suggested in the hint, $g(0) = f(0) - f(0) - f'(0)0 = 0$ and $g'(0) = f'(0) - f'(0) = 0$. And of course $h(0) = 0$ and $h'(0) = 0$. Thus by the GMVT, there is an a in $(0, b)$ for which $(g(b) - g(0))(h'(a)) = (h(b) - h(0))g'(a)$, i.e., $g(b)h'(a) = h(b)g'(a)$; and there is a c in $(0, a)$ (and hence in $(0, b)$) for which $(g'(a) - g'(0))h'(c) = (h(a) - h(0))g'(c)$, i.e., $g'(a)h(c) = h'(a)g(c)$. Thus, because $h'(x)$ and $h''(x)$ are not 0 inside $(0, b)$, we have

$$\frac{f(b) - [f(0) + f'(0)b]}{b^2} = \frac{g(b)}{h(b)} = \frac{g'(a)}{h'(a)} = \frac{g''(c)}{h''(c)} = \frac{f''(c)}{2}\ .$$

   The result follows.