December 14, 2001

Math 323 — Final Exam

1. *(20 points)* Let $f : [a, b] \to \mathbb{R}$. Prove that if f is integrable on $[a, b]$, then for each $\varepsilon > 0$, there is a partition P of $[a, b]$ for which $U(f, P) - L(f, P) < \varepsilon$. You need not prove all the facts that you use about upper and lower sums and integrals, but state them clearly.

2. *(16 points)* Let $f : [a, b] \to \mathbb{R}$ be increasing. Prove that f is integrable on $[a, b]$. (Hint: Use a partition with subintervals of equal length.)

3. *(20 points)*

   (a) Prove that, if $F, G : [a, b] \to \mathbb{R}$ are differentiable, $F(a) = G(a)$ and $F'(x) = G'(x)$ for all $x \in [a, b]$, then $F(x) = G(x)$ for all $x \in [a, b]$. (Hint: Assume not, and apply the MVT to $F - G = H$.)

   (b) Use (a) to prove the validity of the Integration by Parts formula: If f, g are continuously differentiable functions on an interval including its left endpoint a, then

$$\int_a^x fg' = f(x)g(x) - f(a)g(a) - \int_a^x f'g \ .$$

4. *(20 points)* If g is continuous on $[a, b]$, show that there exists a point $c \in (a, b)$ where

$$g(c)(b - a) = \int_a^b g \ .$$

   (Remarks: Because $(\int_a^b g)/(b - a)$ is defined to be the *average value* of g on $[a, b]$, this result is sometimes called the "Mean Value Theorem for Integrals", in which case the result that we called simply the MVT is the "Mean Value Theorem for Derivatives". It is not hard to use the MVT for Integrals to give a slightly shorter proof for the second version of the FTC.)

5. *(24 points)* Decide which of the following conjectures are true and supply a short proof. For those that are not true, give a counterexample.

   (a) Every integrable function is the derivative of some other function.

   (b) If $|f|$ is integrable on $[a, b]$, then so is f.

   (c)

$$\text{If} \qquad f(x) = \begin{cases} 1 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad x \geq 1 \end{cases} , \qquad \text{then} \qquad F(x) = \int_0^x f = \begin{cases} x & \text{if} \quad x < 1 \\ 0 & \text{if} \quad x \geq 1 \end{cases}$$

   (d) Let $f(x) = 1$ if $x < 1$ and 0 if $x \geq 1$. Then there is a partition P of $[0, 2]$ with only 4 points (3 subintervals) for which $U(f, P) - L(f, P) < 0.03$.

Math 323 — Solutions to Final Exam

1. Because f is integrable, the upper and lower integrals $U(f), L(f)$ are equal. Let $\varepsilon > 0$ be given. Now because $L(f) = \sup L(f,P)$ and $U(f) = \inf U(f,P)$, where the sup and inf are taken over all partitions P of $[a,b]$, we can find partitions P_1, P_2 of $[a,b]$ for which $L(f,P_1) > L(f) - \frac{\varepsilon}{2}$ and $U(f,P_2) < U(f) + \frac{\varepsilon}{2}$. Let $P = P_1 \cup P_2$; then P is a refinement of both P_1, P_2, so we have

$$L(f) - \frac{\varepsilon}{2} < L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2) < U(f) + \frac{\varepsilon}{2} ,$$

so

$$U(f,P) - L(f,P) < (U(f) + \frac{\varepsilon}{2}) - (L(f) - \frac{\varepsilon}{2}) = \varepsilon .$$

2. Let $P = \{x_i = a + (i(b-a)/n) : i = 0,1,2,\ldots,n\}$, the partition of $[a,b]$ into subintervals of equal length $x_i - x_{i-1} = (b-a)/n$. Then because f is increasing, it attains its inf and sup at the left and right ends respectively of each subinterval $[x_{i-1}, x_i]$, so that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) \qquad \text{and similarly} \\ L(f,P) &= \frac{b-a}{n}\sum_{i=1}^{n} f(x_{i-1}) . \end{aligned}$$

Thus, their difference has a "telescoping sum":

$$\begin{aligned} U(f,P) - L(f,P) &= \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) \\ &= \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)) . \end{aligned}$$

Because $b, a, f(b), f(a)$ are constants, we see that we can make this difference as small as desired by dividing $[a,b]$ into as large a number of (equal-length) subintervals as needed; so f is integrable.

3. (a) It suffices to prove that $F - G = H$ is the constant function 0, and $H(a) = F(a) - G(a) = 0$. Assume BWOC there is an x-value, say c, in $(a,b]$ for which $H(c) \neq 0$. Then there is an x-value d between a and c, and hence in (a,b), for which $H'(d)(c-a) = H(c) - H(a) = H(c) \neq 0$, so $H'(d) \neq 0$. But this contradicts the hypothesis that $H' = F' - G'$ is constantly 0, and the proof is complete.

   (b) Let $F(x) = \int_a^x fg'$ and $G(x) = f(x)g(x) - f(a)g(a) - \int_a^x f'g$. Then $F(a) = \int_a^a fg' = 0$ and $G(a) = f(a)g(a) - f(a)g(a) - \int_a^a f'g = 0$, so $F(a) = G(a)$. Now because f, g, f', g' are continuous, the FTC shows that

$$\begin{aligned} F'(x) &= f(x)g'(x) \qquad \text{and} \\ G'(x) &= (fg)'(x) - f'(x)g(x) = f'(x)g(x) + f(x)g'(x) - f'(x)g(x) = f(x)g'(x) , \end{aligned}$$

   the latter by the Product Rule, so $F' = G'$. Therefore, $F(x) = G(x)$ for all $x \in [a,b]$.

4. Because g is continuous on the compact interval $[a, b]$, by the EVT there exist $d, e \in [a, b]$ for which $g(d) \leq g(x) \leq g(e)$ for all $x \in [a, b]$. Thus,

$$\int_a^b g(d)\, dx \leq \int_a^b g(x)\, dx \leq \int_a^b g(e)\, dx \ ,$$

that is,

$$g(d)(b-a) \leq \int_a^b g \leq g(e)(b-a) \ .$$

We have $g(d) \leq (\int_a^b g)/(b-a) \leq g(e)$, and g is continuous, so by the IVT there is a c between d and e, and hence between a and b, for which $g(c) = (\int_a^b g)/(b-a)$.

5. (a) False: Let $f(x) = 0$ if $x < 0$ and 1 if $x \geq 0$. Then f is integrable, but because it has a jump discontinuity, it does not have the IVP and so is not the derivative of any function. (An obvious candidate for an antiderivative is $F(x) = \int_a^x f$ for any fixed a, but such an F is not differentiable at $x = 0$.)

   (b) False: For $x \in [0, 1]$, let $f(x) = 1$ if x is rational and -1 if x is irrational. Then f, like the Dirichlet function, is not integrable. (The Dirichlet function could be expressed as $\frac{1}{2}(f + 1)$, so if f were integrable, then the Dirichlet function would also be integrable.) But $|f|$ is the constant function 1, which is integrable.

   (c) False: Because F is continuous and $\lim_{x\to 1^-} F(x) = 1$, $F(x) = 1$ for $x \geq 1$.

   (d) True: $P = \{0, 0.99, 1.01, 2\}$ works:

$$\begin{aligned} U(f, P) - L(f, P) &= (1(0.99) + 1(0.02) + 0(0.99)) - (1(0.99) + 0(0.02) + 0(0.99)) \\ &= 0.02 < 0.03 \ . \end{aligned}$$