Chapter 1: The Real Numbers

Let's begin by listing my (Lantz's) notations that I will use on the board. If <u>any</u> of them are not clear, please feel free to ask for more explanation.

//	end of proof (QED)
$\not\!\!\!\!\times$	contradiction
BWOC	by way of contradiction
WLOG	without loss of generality
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3 \ldots\}$
$(\mathbb{Z}$	the set of integers $\{0, 1, -1, 2, -2, 3, -3, \ldots\})$
$\mathbb{Q}$	the set of rational numbers (quotients of two integers)
$\mathbb{R}$	the set of real numbers
$(\mathbb{C}$	the set of complex numbers)
$\forall$	for all (for every)
$\exists$	there exists (there is)
s.t.	such that
$\emptyset$	the empty set
$A \backslash B$	the set of elements of A that are not in B

The defining property of $\mathbb{N}$ is sometimes called the induction principle: "Let B be a set of numbers. If $1 \in B$ and if the implication $n \in B \implies n+1 \in B$ holds, then $\mathbb{N} \in B$." A consequence is that $\mathbb{N}$ is "well-ordered", i.e., every nonempty subset of $\mathbb{N}$ has a smallest element. (Notice that this is not true of the positive rationals or the positive reals, or the set of even integers, if we include negatives.)

Proof. (Of the well-ordering of $\mathbb{N}$) Let $A \subset \mathbb{N}$, and assume that A has no smallest element; we will show that $A = \emptyset$, by showing that the set B of all n's in $\mathbb{N}$ for which <u>none</u> of $1, 2, 3, \ldots, n$ are in A is equal to $\mathbb{N}$. (We have to be a little fussy about the description of B — it can't just be the complement of A in $\mathbb{N}$ because of the way the argument runs.): If $1 \in A$, then 1 is the smallest element of A, $\not\!\!\!\!\times$, so $1 \notin A$, and so $1 \in B$. Suppose $n \in B$, so that $1, 2, 3, \ldots, n$ are not in A. If $n+1 \in A$, then $n+1$ is the smallest element in A, $\not\!\!\!\!\times$; so $n+1 \notin A$, $\not\!\!\!\!\times$; so $n+1$ is also not in A. Thus, $n+1 \in B$. It follows that $\mathbb{N} \subseteq B \subseteq \mathbb{N}$, so $\mathbb{N} = B$. $\square$

The sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are "fields," i.e., within each of them it is possible to add, subtract, multiply and divide (except by 0). (Some subtractions are impossible in $\mathbb{N}$; some divisions are impossible in $\mathbb{Z}$; so these are not fields.) The "algebraic numbers" $\mathbb{A}$, consisting of all the complex numbers that are roots of polynomials with integer coefficients, is also a field.

The rationals and the reals are "ordered fields", i.e., they have a working sense of "greater than" and "less than" between any pair of their elements. (The complexes does <u>not</u> have that, because squares have to be greater than 0, but both 1 and -1 are squares in $\mathbb{C}$.) The "algebraic reals" $\mathbb{A} \cap \mathbb{R}$ is an ordered field, as is the "surd field" $\mathbb{S}$ that we will talk about in geometry. (It's the smallest field where all the positive elements have square roots.)

What is special about $\mathbb{R}$ is the fact that it is "complete." To make sense of completeness, we need the following definitions:

Definition. (1) Let A be a set of numbers. A number b is an *upper bound* of A if $a \leq b$ for every a in A. If b^* is an upper bound of A and $b^* \leq b$ for every upper bound b of A, then b^* is the *least upper bound* (lub) or *supremum* (sup) of A. [The use of the word "the" here assumes that a set has only one sup; this can and should be proved.] If b^* is an element of A, as well as being the lub of A, then b^* is called the *maximum* (max) of A.

(2) Let A be a set of numbers. A number b is a *lower bound* of A if $a \geq b$ for every a in A. If b^* is a lower bound of A and $b^* \geq b$ for every upper bound b of A, then b^* is the *greatest lower bound* (glb) or *infimum* (inf) of A. If b^* is an element of A, as well as being the glb of A, then b^* is called the *minimum* (min) of A.

Example. (1) If $P = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}$, then $\text{lub}(P) = \pi$.

(2) $A = \{a \in \mathbb{Q} : a^2 \leq 2\} = \{a \in \mathbb{Q} : -\sqrt{2} \leq a \leq \sqrt{2}\}$ has upper and lower bounds in $\mathbb{Q}$: -2 is a lower bound and 2 is an upper bound. But it has no lub or glb in $\mathbb{Q}$. In $\mathbb{R}$, $\text{lub}(A) = \sqrt{2}$ and $\text{glb}(A) = -\sqrt{2}$. But A has no maximum or minimum.

(3) The half-open interval $B = [2, 3)$ has $\inf(B) = \min(B) = 2$ and $\sup(B) = 3$, but B has no maximum.

(4) The infinite interval $C = \{x \in \mathbb{R} : x > 5\}$ has no upper bound and hence no sup, We might write $\sup(C) = \infty$ just as notation, and write $C = (5, \infty)$. Similarly, $\inf((-\infty, 3)) = -\infty$. But ∞ and $-\infty$ are just symbols, not real numbers and not in the field — we won't do arithmetic with them.

With this notation, $\text{lub}(\emptyset) = -\infty$ and $\text{glb}(\emptyset) = \infty$.

Axiom. (Completeness) *For any subset A of $\mathbb{R}$, if A has a (finite) upper bound, then A has a sup in $\mathbb{R}$.*

Corollary. *For any subset B of $\mathbb{R}$, if B has a finite lower bound, then B has an inf in $\mathbb{R}$.*

Clearly the corollary is just the mirror image of the axiom, but its proof would require some simple algebra of inequalities. So let's collect some facts about that:

Algebra for real analysis:

1. Inequalities

   - If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
   - If $a < b$ and $c > 0$, then $ac < bc$.
   - If $a < b$ and $c < 0$, then $ac > bc$.
   - If $0 < a < b$, then $0 < (1/b) < (1/a)$.

2. Absolute values

   - $|ab| = |a||b|$
   - $|a + b| \leq |a| + |b|$ (the "Triangle Inequality")
   - $|a - b| = |b - a|$
   - The (unsigned) distance between a, b on the real line is $|a - b|$.
   - (from the Triangle Inequality) $|a - b| + |b - c| \geq |a - c|$.

Proposition. $|a-b| \geq ||a|-|b||$.

Proof. $|a-b|+|b| \geq |a-b+b| = |a|$, the inequality by the Triangle Inequality; so $|a-b| \geq |a|-|b|$. And $|a-b|+|a| = |b-a|+|a| \geq |b-a+a| = |b|$, so again $|a-b| \geq |b|-|a| = -(|a|-|b|)$. So $|a-b| \geq ||a|-|b||$. □

Or, just to give an example using WLOG:

Proof. Because neither side changes value if we reverse a and b, WLOG we may assume that $|a| \geq |b|$, so that $||a|-|b|| = |a|-|b|$. Then $|a-b|+|b| \geq |a-b+b| = |a|$, the inequality by the Triangle Inequality; so $|a-b| \geq |a|-|b| = ||a|-|b||$. □

Note: If $a < b < c$, then $|b| \leq \max\{|a|, |c|\}$.

You are now challenged to write a proof for the corollary to the completeness axiom. We'll go on to collect some consequences of completeness.

Theorem. (Archimedean Property) *For any positive $\varepsilon, M \in \mathbb{R}$ [no matter how small ε is and how large M is], $\exists\ n \in \mathbb{N}$ s.t. $n\varepsilon > M$.*

Proof. It is enough to find an n in $\mathbb{N}$ for which $n > M/\varepsilon$, so we need to show that it is <u>not</u> true that $n \leq M/\varepsilon\ \forall n \in \mathbb{N}$, i.e., that M/ε is not an upper bound for $\mathbb{N}$. It is enough to show that there is no upper bound for $\mathbb{N}$ in $\mathbb{R}$.

So assume BWOC that $\mathbb{N}$ has an upper bound in $\mathbb{R}$. The it has a sup, say L, in $\mathbb{R}$. Now because L is a sup, and $L - \frac{1}{2} < L$, $L - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$, so there is an n in $\mathbb{N}$ for which $L - \frac{1}{2} < n$. But then $n+1 \in \mathbb{N}$ and

$$n + 1 > (L - \frac{1}{2}) + 1 = L + \frac{1}{2} > L\ ,$$

$\not\leftrightarrow$. □

Corollary. *$\forall r \in \mathbb{R}$, $\exists z \in \mathbb{Z}$ s.t. $z < r \leq z+1$.*

Corollary. *$\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $0 < (1/n) < \varepsilon$.*

Proof. Pick n so that $n\varepsilon > 1$. □

Theorem. ($\mathbb{Q}$ is "dense" in $\mathbb{R}$) *$\forall a, b \in \mathbb{R}$ with $a < b$, $\exists q \in \mathbb{Q}$ s.t. $a < q < b$.*

Proof. We have $b - a > 0$, so $\exists n \in \mathbb{N}$ s.t. $0 < (1/n) < b - a$. Then $nb - na > 1$. Now $\exists z \in \mathbb{Z}$ s.t. $z < nb \leq z + 1$, and $nb - z \leq 1 < nb - na$, so $z > na$. Thus, $na < z < nb$, so $q = z/n$ satisfies $a < q < b$. □

Example. Let A, B be subsets of $\mathbb{R}$ such that $\forall a \in A\ \exists b \in B$ s.t. $a < b$. Prove that $\sup(A) \leq \sup(B)$, and show by example that it may <u>not</u> be true that $\sup(A) < \sup(B)$.

Proof. If $\sup(B) = \infty$, then this is clear, so suppose B has a finite sup, b^*. Then for all a in A, we can find a b in B for which $a < b \leq b^*$; so b^* is an upper bound for A, and hence $\sup(A) \leq b^*$.

But if $A = \{.9, .99, .999, .9999, \ldots\}$ and $B = \{1\}$, then $\sup(A) = 1 = \sup(B)$. (In fact, we could have let A be any set that does not contain its sup and set $B = A$ to get an example.) □

Example. Let A, B be subsets of $\mathbb{R}$ that are bounded below, and set $A+B = \{a+b : a \in A,\ b \in B\}$. Prove that $\text{glb}(A) + \text{glb}(B) = \text{glb}(A+B)$.

Proof. Set $glb(A) = a^*$ and $glb(B) = b^*$. Each element of $A + B$ has the form $a + b$ where $a \in A$ and $b \in B$, and we know $a* \leq a$ and $b^* \leq b$, so $a^* + b^* \leq a + b^* \leq a + b$. Thus, $a^* + b^*$ is a lower bound for $A + B$.

Now take any lower bound c for $A + B$. Then for any (temporarily fixed) $b \in B$, we know that for all a in A, $c \leq a + b$, so $c - b \leq a$; thus, $c - b$ is a lower bound for A, so $c - b \leq a^*$, or equivalently $c - a^* \leq b$. Thus (letting b vary through B again), we see that $c - a^* \leq b$ for every b in B, i.e., $c - a^*$ is a lower bound for B; so $c - a^* \leq b^*$. So we have $c \leq a^* + b^*$.

We might be a little trickier in the second paragraph: Take any lower bound c for $A + B$, and assume BWOC that $c > a^* + b^*$. Then $a * + \frac{1}{2}(c - (a^* + b^*))$ is greater than a^*, so it is not a lower bound for A; so there is an element a of A for which $a < a * + \frac{1}{2}(c - (a^* + b^*))$. Similarly, we can find an element b of B for which $b < b^* + \frac{1}{2}(c - (a^* + b^*))$. But then

$$a + b < a^* + \frac{1}{2}(c + (a^2 + b^*)) + b^* + \frac{1}{2}(c + (a^2 + b^*)) = c \ ,$$

so c is not a lower bound for $A + B$, $-\!\!\times\!\!-$.

Either way the proof ends as follows: Because $c \leq a^* + b^*$ for every lower bound c of $A + B$, we have $a^* + b^* = \text{glb}(A + B)$. □

Corollary. (Nested Interval Property) *Let $I_1 \supseteq I_2 \supseteq I_3 \subseteq \ldots$ be closed finite (nonempty) intervals in $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. Write $I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$. Then for $m \geq n$, $I_m \subseteq I_n$, so $a_1 \leq a_2 \leq \cdots \leq a_m \leq b_m \leq b_n$; so every b_n is an upper bound for $\{a_m : m \in \mathbb{N}\}$, and similarly every a_n is a lower bound for $\{b_m : m \in \mathbb{N}\}$. Set $a^* = \text{lub}(\{a_m : m \in \mathbb{N}\})$ and $b^* = \text{glb}(\{b_m : m \in \mathbb{N}\})$. We want to show that $a^* \leq b^*$, so that $I^* = [a^*, b^*]$ will be nonempty (even if it is a single point) and $I^* \subseteq I_n$ for all n in $\mathbb{N}$, so it is in the intersection.

So assume BWOC that $a^* > b^*$. Then b^* is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, so there is an n in $\mathbb{N}$ for which $a_n > b^*$. Thus, a_n is not a lower bound for $\{b_m : m \in \mathbb{N}\}$, $-\!\!\times\!\!-$, because we saw above that every a_n is a lower bound for this set. So the proof is complete. □

Example. We need the hypotheses of closedness and boundedness on the intervals in the Nested Interval Property:

$$\bigcap_{n=1}^{\infty} (0, \frac{1}{n}] = \emptyset \qquad , \qquad \bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$$

`[At this point students are ready to do the first problem set.]`