Chapter 2: (Infinite) Sequences and Series

A(n infinite) *sequence* is a list of numbers $a_1, a_2, a_3, \ldots$, indexed by $\mathbb{N}$ (or $a_0, a_1, a_2, a_3, \ldots$, indexed by the nonnegative integers $\mathbb{N}_0$) — so there is a first one, a second one, a third one, etc., Repetitions are allowed: We may have $a_1 = a_2 \neq a_3 = a_4$, for example. A(n infinite) *series* is an indicated "summing" of a sequence,

$$a_1 + a_2 + a_3 + \ldots \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n \ ,$$

whether or not the sum makes sense. (We'll talk more about that later.)

Calculus was invented to deal with motion (speeds, etc.). Sequences are a tool to describe motion, at least in discrete steps: first, second, third, etc. Later we will take up functions, to describe continuous motion, but sequences are a step on the way. Note that both pure and applied math use sequences in this way, as better and better approximations to a solution.

Example.

$$\left(\frac{1}{n}\right)_{n=1}^{\infty} \ , \qquad (2(-1)^n)_{n=1}^{\infty} \ , \qquad (n)_{n=1}^{\infty} \ , \qquad ((-1)^n(n+2))_{n=1}^{\infty} \ , \qquad \left(\frac{2n+1}{3n-2}\right)_{n=1}^{\infty}$$

So really, a sequence is a function from $\mathbb{N}$ (or $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) into $\mathbb{R}$.

Example. (A sequence defined "by induction" or "defined recursively") $a_1 = 1 = 1/1$ and if $a_n = b_n/c_n$ where $b_n, c_n \in \mathbb{Z}$, then

$$a_{n+1} = \frac{b_n + 2c_n}{b_n + c_n} \ .$$

So

$$a_1 = \frac{1}{1} = 1 \ , \quad a_2 = \frac{3}{2} = 1.5 \ , \quad a_3 = \frac{7}{5} = 1.4 \quad a_4 = \frac{17}{12} = 1.41\overline{6}$$

Guess the limit?

To describe "a sequence gets close to a limit" in algebraic terms, (i.e., without reference to motion), we make a sequence of definitions — the last one is the official one, because we can write proofs with it.

Definition. A sequence $(x_n)_{n=1}^{\infty}$ (of real numbers) *converges* to the *limit* L in $\mathbb{R}$ (or just is *convergent* if the value of L is unknown or unimportant) iff

(1) as we go along the sequence, the x_n's get closer to L.

(2) for large enough n, the x_n's become as close as desired to L.

(3) for any tolerance around L, the x_n's are within that tolerance if n is large enough.

(4)
$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |x_n - L| < \varepsilon$$
$$(\text{or } -\varepsilon < x_n - L < \varepsilon \ , \qquad \text{or } L - \varepsilon < x_n < L + \varepsilon \ , \qquad \text{or } x_n \in (L-\varepsilon, L+\varepsilon) \ ).$$

Example. Prove that $\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}} = 2$.

Proof. Let $\varepsilon > 0$ be given. [Here is the reasoning that we will need to finish the proof: We need to find N so that, for $n > N$,
$$\left|\frac{2n}{\sqrt{n^2+1}} - 2\right| < \varepsilon \ .$$
Now
$$\begin{aligned}
\left|\frac{2n}{\sqrt{n^2+1}} - 2\right| &= \left|\frac{2n - 2\sqrt{n^2+1}}{\sqrt{n^2+1}}\right| \\
&= \left|\frac{4n^2 - (4n^2+4)}{\sqrt{n^2+1}(2n + 2\sqrt{n^2+1})}\right| \\
&= \frac{4}{\sqrt{n^2+1}(2n + 2\sqrt{n^2+1})}
\end{aligned}$$
Now for any n in $\mathbb{N}$, $2n + 2\sqrt{n^2+1} \geq 2 + 2 = 4$, so $4/(2n + 2\sqrt{n^2+1}) < 1$. So to get the inequality we need, we only have to arrange $1/\sqrt{n^2+1} < \varepsilon$, or $\sqrt{(1/\varepsilon^2) - 1} < n$. But what if $\varepsilon > 1$, so that the square root doesn't make sense? We can just rule that out, because if we choose a big tolerance, the terms of the sequence will always be within it. So here is the proof written forward:] Take N in $\mathbb{N}$ so that $N > \sqrt{\max(0, (1/\varepsilon^2) - 1)}$. Then for $n \geq N$, we have $n^2 + 1 > 1/\varepsilon^2$, so $1/\sqrt{n^2+1} < \varepsilon$, and hence
$$\left|\frac{2n}{\sqrt{n^2+1}} - 2\right| = \frac{4}{\sqrt{n^2+1}(2n + 2\sqrt{n^2+1})} = \frac{4}{2n + 2\sqrt{n^2+1}} \cdot \frac{1}{\sqrt{n^2+1}} < \varepsilon \ .$$
Therefore, $\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}} = 2$. $\square$

`[At this point students are ready to do the second problem set.]`

Of course we don't want to go through all that every time we need to find a limit, so we prove the Algebraic Limit Theorem, i.e., the "Limit Theorems" that everyone learns and forgets in Calc 1. The sum and difference parts of the proof are easy, the quotient part is harder, and the square root is an exercise. Let's prove the product part:

Theorem. (Product Limit Theorem) *If $\lim x_n = x$ and $\lim y_n = y$, then $\lim(x_n y_n) = xy$. (Here and elsewhere, of course, lim means the limit as $n \to \infty$, as long as we are talking about limits of sequences. When we get to limits of functions, we will have to write more.)*

Proof. Let $\varepsilon > 0$ be given. [We want $|x_n y_n - xy| < \varepsilon$; and it will be useful to add and subtract the same quantity inside the the absolute value:
$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n||y_n - y| + |x_n - x||y|$$
Now we know that we can make the differences $|y_n - y|$ and $|x_n - x|$ as small as we need, and $|y|$ doesn't change with n. But $|x_n|$ may change as we change n, so we need to be sure that it doesn't change too much. One way to arrange that is note that, for large enough values of n, $|x_n - x| < 1$, so that $|x_n| - |x| < 1$, or $|x_n| < |x| + 1$. Thus $|x_n|$ at least doesn't get any larger than the fixed value $|x| + 1$. Now all we need to arrange is to have the sum of the two terms come out less than ε

by making the differences that we can control small enough:] Pick N in $\mathbb{N}$ large enough so that all three of the following hold for all $n \geq N$:

$$|x_n - x| < 1 \ , \qquad |x_n - x| < \frac{\varepsilon}{2|y|} \ , \qquad |y_n - y| < \frac{\varepsilon}{2(|x|+1)} \ .$$

(What if $y = 0$? In that case, we can just drop the middle condition.) Then for all $n \geq N$,

$$|x_n y_n - xy| \leq |x_n||y_n - y| + |x_n - x||y| < (|x|+1)\frac{\varepsilon}{2(|x|+1)} + \frac{\varepsilon}{2|y|}|y| = \varepsilon \ ,$$

and the proof of the limit is complete. □

Similarly:

- $\lim(a_n + b_n) = (\lim a_n) + (\lim b_n)$
- $\lim(ca_n) = c(\lim a_n)$, where c is a constant
- If $\lim(b_n) \neq 0$, then $\lim(a_n/b_n) = (\lim a_n)/(\lim b_n)$
- If $a_n \leq b_n$ for all n (and the limits of $(a_n), (b_n)$ exist), then $\lim a_n \leq \lim b_n$.

Example. Let's redo the example that we did earlier directly from the definition of limit; this time, we'll use the Algebraic Limit Theorem. First, we divide both numerator and denominator by n; we can put n under the radical as n^2 because we know it is positive. Also, we know that the limit of a constant sequence is that constant, and the limit of $1/n$ is 0:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}} &= \lim \frac{2}{\sqrt{1+\frac{1}{n^2}}} \\
&= \frac{2}{\lim \sqrt{1+\frac{1}{n^2}}} = \frac{2}{\sqrt{1+\left(\lim \frac{1}{n}\right)^2}} \\
&= \frac{1}{\sqrt{1+0^2}} = 2
\end{aligned}$$

The next example is not important as a theorem; we include it as an example of how to do one of the homework problems:

Example. Proposition: If $\lim a_n = a$ and $|b_n - b| \leq 3(a_n - a)^2$ for all n in $\mathbb{N}$, then $\lim b_n = b$.

Proof. Let $\varepsilon > 0$ be given. Pick N in $\mathbb{N}$ s.t., for $n \geq N$, $|a_n - a| < \sqrt{\varepsilon/3}$. Then for $n \geq N$,

$$|b_n - b| \leq 3(a_n - a)^2 < 3(\sqrt{\varepsilon/3})^2 = \varepsilon \ .$$

Therefore, $\lim b_n = b$. □

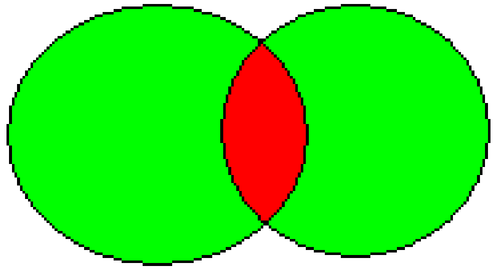
Here is why we can't just assume something (though of course we can make a guess) from the first few terms of a sequence:

(1) It is possible to cook up examples that look clear and then go crazy: Let

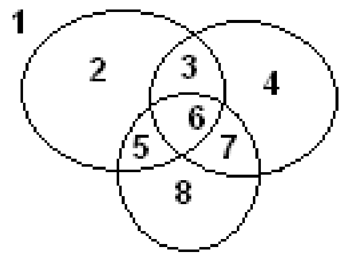
$$a_n = n + (n-1)(n-2)(n-3)(n-4)(n-5) \ .$$

$$a_1 = 1 \ , \quad a_2 = 2 \ , \quad a_3 = 3 \ , \quad a_4 = 4 \ , \quad a_5 = 5 \ , \quad a_6 = 126$$

(2) Artificial examples aside, here is a "real" one: Everyone has seen "Venn diagrams" used to show intersections and unions of sets:



So, into how many regions (inside and outside) is the plane divided by n circles? Let r_n denote that number. It is easy to count that $r_0 = 1$, $r_1 = 2$, $r_2 = 4$, and $r_3 = 8$.



So the formula is $r_n = 2^n$, right? Nope. You can check that $r_4 = 14$. (Each new circle hits each of the old circles in at most two points, so the $(n+1)$-st new circle is divided into $2n$ arcs by these intersections, and each of these arcs cuts an old region into two, increasing the count of regions by $2n$. So a recursive definition of the sequence (r_n) is given by: $r_0 = 1$, $r_1 = 2$, and $r_{n+1} = r_n + 2n$. For $n = 3$, we get $r_4 = 8 + 2(3) = 14$, as claimed.)

(3) The polynomial $n^2 - n + 41$ gives prime numbers for $n = 0, 1, 2, \ldots, 40$, but not for $n = 41$.

`[At this point students are ready to do the third problem set.]`

Definition. A sequence $(x_n)_{n=1}^{\infty}$ is *increasing* if $x_1 \le x_2 \le x_3 \le \ldots$, i.e., if $n < m \implies x_n \le x_m$ [respectively *decreasing* if if $x_1 \ge x_2 \ge x_3 \ge \ldots$, i.e., if $n < m \implies x_n \ge x_m$]. It is *monotone* if it is either increasing or decreasing.

Theorem. (Monotone Convergence Theorem) *An increasing sequence that is bounded above converges (to the supremum of its terms). [Writing the "decreasing" version is left to you.]*

Proof. Let $(x_n)_{n=1}^{\infty}$ be an increasing sequence s.t. $\{x_n : n \in \mathbb{N}\}$ is bounded above, and let $x = \sup\{x_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ be given. Then $x - \varepsilon$ is not an upper bound for $\{x_n : n \in \mathbb{N}\}$, so $\exists\, N \in \mathbb{N}$ s.t. $x_N > x - \varepsilon$. Now for $n \ge N$ we have $x_N \le x_n \le x$, so $|x_n - x| \le |x_N - x| < \varepsilon$. Therefore $x = \lim x_n$. $\square$

Example. Recall $a_1 = 1/1$ and if $a_n = b_n/c_n$, then $a_{n+1} = (b_n + 2c_n)/(b_n + c_n) = (a_n + 2)/(a_n + 1)$. Now this sequence is not monotone; in fact, if a term is less than $\sqrt{2}$, then the next term is greater than $\sqrt{2}$, and vice versa.

(Verifying the last claim: Suppose $a_n < \sqrt{2}$; then the following statements are all true or all false, using the fact that $a_n + 1 > 0$:

$$\begin{aligned} a_{n+1} = \frac{a_n + 2}{a_n + 1} &> \sqrt{2} \\ a_n + 2 &> \sqrt{2}a_n + \sqrt{2} \\ 2 - \sqrt{2} &> (\sqrt{2} - 1)a_n \\ \sqrt{2} &> a_n \end{aligned}$$

But the last statement is true by hypothesis, so the first is also true. Similarly, if I had started by supposing that $a_n > 2$, then I would have found that $a_{n+1} < 2$.)

Do the alternating terms (i.e., every other term in the sequence — the odd ones and the even ones) form monotone sequences? Well,

$$a_{n+2} = \frac{(a_n + 2)/(a_n + 1) + 2}{(a_n + 2)/(a_n + 1) + 1} = \frac{3a_n + 4}{2a_n + 3}$$

Now suppose $a_n < \sqrt{2}$ (which is true for $n = 1$, say). We know that $a_{n+2} < \sqrt{2}$ again because we know the a_n's are alternately above and below $\sqrt{2}$; so the odd-numbered a_n's are bounded above, by $\sqrt{2}$. Do we necessarily have $a_{n+2} \geq a_n$? Again, the following statements are all true or all false (because $2a_n + 3 > 0$); I'm starting with the statement of which I'm not sure and seeing if I reach a statement that I know is true:

$$\begin{aligned} a_n &< \frac{3a_n + 4}{2a_n + 3} \\ 2a_n^2 + 3a_n &< 3a_n + 4 \\ a_n^2 &< 2 \\ a_n &< \sqrt{2} \end{aligned}$$

[Warning: Going from the second last statement to the last one here is an example showing that this kind of "algebraic logic" can in general be misleading. Here, I know that a_n is a positive number, so taking square roots on both sides is safe; but if I don't know that, the statements $a_n^2 > 2$ and $a_n > \sqrt{2}$ are not necessarily both true or both false.] All the statements are true, so the odd-numbered a_n's form an increasing sequence. Similarly, the even-numbered a_n's form a decreasing sequence bounded below by $\sqrt{2}$. By the Monotone Convergence Theorem, both "subsequences" converge. Do they have the same limit? (If so, it must be $\sqrt{2}$.) Well, because $a_n + 1 > 1$, if we drop it from a denominator (of a positive fraction), the result will get larger:

$$\begin{aligned} |a_{n+1} - \sqrt{2}| = \left| \frac{a_n + 2}{a_n + 1} - \sqrt{2} \right| &= \left| \frac{a_n + 2 - \sqrt{2}a_n - \sqrt{2}}{a_n + 1} \right| \\ &< |a_n + 2 - \sqrt{2}a_n - \sqrt{2}| = (\sqrt{2} - 1)|a_n - \sqrt{2}| \end{aligned}$$

Repeating this n times, we see that

$$|a_{n+1} - \sqrt{2}| < (\sqrt{2} - 1)^n |a_1 - \sqrt{2}| = (\sqrt{2} - 1)^{n+1}$$

Because $\sqrt{2} - 1 < 1$, its powers approach 0, so $\lim a_n = \sqrt{2}$.

Let's try a different approach to the end of the argument above, one that the text uses repeatedly in the exercises. Suppose we have reached the point above where we know that the odd-numbered

a_n's form an increasing sequence that is bounded above. Then we know that it converges; but we don't (officially) know the limit. To <u>find</u> the limit, let's call it a. Then as $n \to \infty$, both a_n and a_{n+2} must get closer and closer to a, and they are related by $a_{n+2} = (3a_n+4)/(2a_n+3)$; so a must be related to itself by $a = (3a+4)/(2a+3)$. A little algebra gives $a = \pm\sqrt{2}$, and because all the a_n's are positive, the limit must be the positive square root. Similarly, as soon as we know that the even-numbered a_n's are decreasing and bounded below, we can find that the limit must be $\sqrt{2}$ by the same reasoning.

Lemma. *If* $|r| < 1$*, then* $\lim r^n = 0$.

Proof. We may assume WLOG that $r > 0$; then $r > r^2 > r^3 > \cdots \geq 0$, so $\lim r^n = L$ exists. Assume BWOC that $L > 0$, and pick N in $\mathbb{N}$ s.t. $|r^N - L| < (L/r) - L$ (a positive number, so a possible choice for ε), i.e., $r^N < L/r$. Then $r^{N+1} < L$, so L is not a lower bound for $\{r_n : n \in \mathbb{N}\}$, so L is not $\lim r_n$, $-\!\!\times\!\!-$. Therefore $L = 0$. $\square$

Challenge. In one of the text exercises, the author challenges us to find the limit of the sequence defined recursively by $x_1 = 2$, $x_{n+1} = (x_n + 2/x_n)/2$ converges to $\sqrt{2}$. His reasoning is that, as $n \to \infty$ both x_{n+1} and x_n go to the same limit x, so x must satisfy $x = (x+2/x)/2$, which simplifies to $x = \sqrt{2}$. Now, given $c > 0$, find a sequence that converges to $\sqrt{c}$.
[Tom Dinitz proposed $x_{n+1} = ((c-1)/c)(x_n + c/((c-1)x_n))$, because again, $x = ((c-1)/c)(x + c/((c-1)x))$ simplifies to $x = \sqrt{c}$. This does seem to work for many c -- for $c = 0.5$ the next term is 0 and the following terms are meaningless; and for $c = 0.6$ the sequence seems to diverge. But $x_{n+1} = (1/2)(x_n + c/x_n)$ -- essentially reasoning that if x_n is too small to have $x_n^2 = c$, then c/x_n will be too large, and the average of the two will be closer to right -- seems to work.]

As mentioned earlier, for a sequence $(a_n)_{n=1}^{\infty}$, the indicated sum symbol $\sum_{n=1}^{\infty} a_n$ is a *series* — but that doesn't imply that the symbol means anything. Or if it does mean something, then it means the sequence of "partial sums":

Definition. We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* to L in $\mathbb{R}$ if the *sequence of partial sums*

$$s_1 = a_1 \ , \quad s_2 = a_1 + a_2 \ , \quad s_3 = a_1 + a_2 + a_3 \ , \quad \ldots$$

converges to L (as a sequence). (The m-th partial sum can be written $s_m = \sum_{n=1}^{m} a_n$.)

Example. $\sum_{n=1}^{\infty} (-1)^n$: The partial sums are $-1, 0, -1, 0, -1, 0, \ldots$, so the series doesn't converge.

Example. (Geometric series) $\sum_{n=0}^{\infty} ar^n$. The n-th (or maybe the $(n+1)$-st, depending on how you count) partial sum is $a(1-r^{n+1})/(1-r)$ (unless $r = 1$), so the series converges to $a/(1-r)$ if $|r| < 1$ and diverges if $|r| \geq 1$.

(Proof of the partial sum formula: Suppose $r \neq 1$. Then

$$s_n = a + ar + ar^2 + \cdots + ar^n \ ,$$

and so

$$rs_n = ar + ar^2 + \cdots + ar^n + ar^{n+1} \ .$$

Subtracting gives $s_n - rs_n = a - ar^{n+1}$, and dividing both sides by $1-r$ gives the result.)

But for almost any other series, even if we know it converges, we can't tell what it converges to.

Proposition. *If a series $\sum a_n$ converges, then the sequence of terms a_n converges to 0; but not conversely.*

Proof. Suppose $\sum a_n$ converges to L, and let $\varepsilon > 0$ be given. Then $\exists\, N \in \mathbb{N}$ s.t. $\forall\, m \geq N$,

$$\left| \sum_{n=1}^{m} a_n - L \right| < \frac{\varepsilon}{2} \; .$$

In particular, $\forall\, m \geq N+1$,

$$|a_m - 0| = \left| \sum_{n=1}^{m} a_n - \sum_{n=1}^{m-1} a_n \right| \leq \left| \sum_{n=1}^{m} a_n - L \right| + \left| \sum_{n=1}^{m-1} a_n - L \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \; .$$

So $\lim a_n = 0$.

For the "but not conversely" part, we need a sequence a_n that converges to 0 but $\sum a_n$ diverges. The simplest example is the *harmonic series* $\sum(1/n)$: We know that $\lim(1/n) = 0$, so we need to show that the partial sums of $\sum(1/n)$ don't converge. Because the $(1/n)$'s are all positive, the partial sums are increasing, so we just need to show they are not bounded above. So assume BWOC that they are bounded above, say by B. Pick $N \in \mathbb{N}$ s.t. $N > 2B$, and consider the sum of the first 2^N terms of $\sum(1/n)$. In the sum below, we replace the numbers $1/n$ with n between successive powers of 2 with the reciprocal of the next larger power of 2; so $1/3$ gets replaced by $1/4$; each of $1/5$, $1/6$ and $1/7$ get replaced by $1/8$; each of $1/9$ through $1/15$ get replaced by $1/16$, and so on. The result, of course, is smaller than before because each of the terms is replaced by something smaller (or equal).

$$\begin{aligned}
\sum_{n=1}^{2^N} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{2^N} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^N} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= 1 + \frac{1}{2} N > 1 + B > B
\end{aligned}$$

Because we have 2 copies of $1/4$, 4 copies of $1/8$, 8 copies of $1/16$, and so on, it adds up to N copies of $1/2$. So the partial sums of the harmonic series are not bounded above by B, $\rightarrow\!\leftarrow$. $\square$

`[At this point students are ready to do the fourth problem set.]`

We will return to the subject of series later, but let's get back to sequences:

Definition. A *subsequence* of a sequence $(a_n)_{n=1}^{\infty}$ is a sequence of the form $(a_{n_k})_{k=1}^{\infty}$ where $n_1 < n_2 < n_3 < \ldots$

So even divergent sequences can have subsequences that converge:

- $0, 1, 0, 2, 0, 3, 0, \ldots$ has the subsequence $0, 0, 0, \ldots$
- $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$ has subsequences $1, 1, 1, \ldots$ and $2, 2, 2, \ldots$ and $3, 3, 3, \ldots$

- $1, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{5}, \frac{5}{6}, \frac{1}{7}, \ldots$ has subsequences $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ and $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots$

Theorem. (Bolzano-Weierstrass) *A bounded sequence has a convergent subsequence.*

Proof. `(Picture first.)` Suppose all terms of (a_n) are in the interval $[C, D]$. Set $b_1 = a_1$. Now infinitely many of the a_n's are in the left subinterval $[C, \frac{1}{2}C + \frac{1}{2}D]$ <u>or</u> infinitely many are in the right subinterval $[\frac{1}{2}C + \frac{1}{2}D, D]$ (or maybe both). Say the former, and let b_2 be the first a_n after b_1 that lies in the left subinterval.

Now infinitely many of the a_n's are in the left subinterval $[C, \frac{3}{4}C + \frac{1}{4}D]$ of $[C, \frac{1}{2}C + \frac{1}{2}D]$ <u>or</u> infinitely many of the a_n's are in the right subinterval $[\frac{3}{4}C + \frac{1}{4}D, \frac{1}{2}C + \frac{1}{2}D]$ (or both). Suppose the latter, and let b_3 be the first a_n after b_2 that lies in $[\frac{3}{4}C + \frac{1}{4}D, \frac{1}{2}C + \frac{1}{2}D]$. Continue; then (b_n) is a subsequence of (a_n), and we want to show that it converges.

Consider the intervals $I_1 = [C, D]$, $I_2 = [C, \frac{1}{2}C + \frac{1}{2}D]$, $I_3 = [\frac{3}{4}C + \frac{1}{4}D, \frac{1}{2}C + \frac{1}{2}D]$, etc., that were used in choosing the b_n's. We have $I_1 \supset I_2 \supset I_3 \supset \ldots$, so their intersection is nonempty. But the length of I_n is $(D - C)/2^{n-1}$, approaching 0, so their intersection is a single point $\{b\}$.

We claim that $b = \lim b_n$: Let $\varepsilon > 0$ be given, and pick N in $\mathbb{N}$ so that $(D-C)/2^{n-1} < \varepsilon$. Then for all $n \geq N$, both b_n and b are in I_n, so $|b_n - b| \leq (D - C)/2^{n-1} < \varepsilon$. The result follows. $\square$

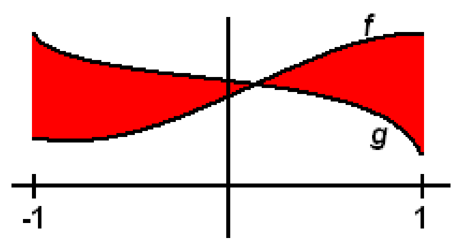
Challenge. (Chosen to be similar to a homework problem) Find a sequence with subsequences that converge to each natural number.
`One answer is` $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$.

A "Cauchy sequence" is a sequence of objects in any "metric space" (where distances make sense) that is, in a sense, trying to converge; but if the space is not complete, there may not be a point for it to converge to. Here are some examples:

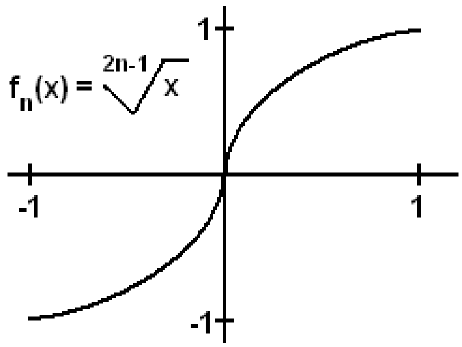
Example. (1) The now-familiar sequence $a_1 = 1$ and $a_{n+1} = (a_n + 2)/(a_n + 1)$ is a sequence in $\mathbb{Q}$, but its limit, $\sqrt{2}$, isn't there.

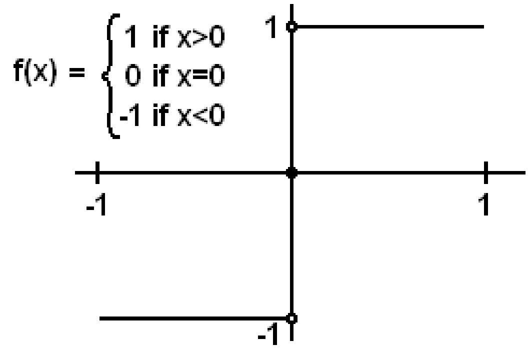
(2) Suppose we measure the distance between continuous functions f, g on the interval $[-1, 1]$ by

$$d(f, g) = \int_{-1}^{1} |f(x) - g(x)|\, dx$$



i.e., the area between the graphs of the functions. Then the continuous functions $f_n(x) = \sqrt[2n-1]{x}$ approach the step function $f(x)$ which is 1 if $x > 0$, 0 if $x = 0$ and -1 if $x < 0$; but that is not a continuous function.





But we will shortly show that, in $\mathbb{R}$, a sequence is Cauchy iff it is convergent. So for our purposes the Cauchy criterion is just a convenient way to handle sequences when we don't know the limit. In particular, because we usually don't know the sum of a series, it is particularly useful with series.

Theorem. *A sequence (a_n) in $\mathbb{R}$ is convergent iff it is Cauchy, i.e., iff, $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, m, n \geq N$, $|a_m - a_n| < \varepsilon$.*

Proof. ($\Leftarrow$) Write $\lim a_n =$. Let $\varepsilon > 0$ be given, and pick $N \in \mathbb{N}$ s.t. $\forall\, n \geq N$, $|a_n - a| < \varepsilon/2$. Then for all $m, n \geq N$,

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

($\Rightarrow$) We begin with a:

Lemma. *A Cauchy sequence is bounded.*

Proof. For (a_n) Cauchy, pick $N \in \mathbb{N}$ s.t. $\forall\, m, n \geq N$, $|a_m - a_n| < 1$. Then $\forall\, m \geq N$, $|a_m| - |a_N| \leq |a_m - a_N| < 1$, so $\forall\, n \in \mathbb{N}$,

$$|a_n| \leq \max(|a_1|, |a_2|, \ldots, |a_N|, |a_N| + 1) \ .$$

□

Now suppose (a_n) is Cauchy. By the lemma and Bolzano-Weierstrass, it has a convergent subsequence (b_n), with limit b, say. We claim that (a_n) converges to b also: Let $\varepsilon > 0$ be given, and pick $N \in \mathbb{N}$ such that

$$\forall\, m, n \geq N,\ |a_m - a_n| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \forall\, m \geq N,\ |b_m - b| < \frac{\varepsilon}{2} \ .$$

Then $\forall\, n \geq N$, because we can find a b_m for which $b_m = a_k$ with $k \geq N$, we have

$$|a_n - b| \leq |a_n - a_k| + |b_m - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

□

Corollary. (Cauchy criterion for series) *$\sum_{n=1}^{\infty} a_n$ converges iff, $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t., $\forall\, m > n \geq N$, $\sum_{k=n}^{m} a_k < \varepsilon$.*

Here are the basic theorems about the convergence of series:

Proposition. (Comparison test) *Suppose $0 \leq a_n \leq b_n$ $\forall\, n \in \mathbb{N}$. If $\sum b_n$ is converges, then $\sum a_n$ converges [or equivalently the contrapositive: If $\sum a_n$ diverges, then $\sum b_n$ diverges].*

The proof is an exercise, but it could be based on the Cauchy criterion, because $\sum_{k=n}^{m} a_k \leq \sum_{k=n}^{m} b_k$.

Proposition. (Absolute convergence test) *If $\sum |a_n|$ converges, then $\sum a_n$ converges.*

Proof. Use the Cauchy criterion: $|\sum_{k=m}^{n} a_k| \leq \sum_{k=m}^{n} |a_k|$, so the result follows. □

Remark. Theorem 2.7.10 says that even if the terms in an absolutely convergent series are rearranged, the sum will be the same. But it is a horrible fact that, if a series $\sum a_n$ converges but does not converge absolutely, then for every B in $\mathbb{R} \cup \{\infty, \infty\}$, there is a rearrangement (b_n) of (a_n) for which $\sum b_n$ converges to B. Here is a sketch of the argument:

If we look at the sum of the positive a_n's (in order) and the sum of the negative ones, at least one of these sums must diverge (to ∞ or to $-\infty$, respectively), or else $\sum |a_n|$ would converge. But if one converges and the other diverges, then $\sum a_n$ diverges, which isn't true. So both diverge. Now fix a B — in $\mathbb{R}$, for now. Add up just enough positive a_n's to get a partial sum larger than B. Then add in just enough negative a_n's to make the sum less B. Then add in just enough more positive a_n's to get above B again. And so on. Now because $\sum a_n$ converges, its terms approach 0 (in absolute value); and the partial sums obtained in this way never differ from B by more than the last term of the other sign, so these partial sums will approach the limit B.

If $B = \infty$, first add enough positive a_n's to get above 1, then just enough negative a_n's to get below 1, then just enough positive a_n's to get above 2, then just enough b_n's to get below 2, and so on. The case when $B = -\infty$ is left to you.

It follows that the commutative law fails rather spectularly for infinite series.

Definition. If $\sum |a_n|$ converges, we say that $\sum a_n$ *converges absolutely.*

Example. The alternating harmonic series $\sum (-1)^{n+1} 1/n$ converges (by the next result), but not absolutely.

Proposition. (Alternating series test) *If the terms in $\sum a_n$ alternate signs and their absolute values are decreasing with limit 0, then $\sum a_n$ converges.*

Proof. Assume WLOG that $a_1 > 0$. Then because the signs of the a_n's alternate and their absolute values decrease, $a_2 + a_3$ is a negative number. Hence, letting s_n denote the n-th partial sum, we have

$$\begin{aligned} s_1 = a_1 > s_1 + a_2 + a_3 &= s_3 \qquad \text{and similarly} \\ s_3 > s_3 + a_4 + a_5 &= s_5 \\ s_5 > s_5 + a_6 + a_7 &= s_7 \end{aligned}$$

and so on. So the odd-numbered partial sums form a decreasing sequence. Similarly, the even-numbered partial sums form an increasing sequence. Moreover, every odd partial sum s_{2n-1} is greater than any odd partial sum s_{2m-1} with $m > n$, which is greater than s_{2m}, which is greater than any s_{2k} with $k < m$. So all the even-numbered partial sums are upper bounds for the set of odd-numbered partial sums. And vice versa. It follows that the limits of monotone sequences of even-numbered partial sums and odd-numbered partial sums both exist and lie between the sequences. It remains to show they are equal. But because the difference between them is bounded above by the difference between any even-numbered and any odd-numbered partial sum, and the difference between two <u>consecutive</u> partial sums, i.e., a term in the sequence, is approaching 0, it follows that the difference between the limits is 0, i.e., they are equal. □

Proposition. (Ratio test) *If* $\lim |a_{n+1}/a_n| < 1$*, then* $\sum a_n$ *converges absolutely. If the limit is greater than 1, the series diverges. If the limit is 1, the series could do either.*

The proof is an exercise, so we will not provide it here. The general idea is to set up a comparison test with a geometric series with a common ratio $r < 1$. Because it is usually easy to apply, the ratio test is usually the first choice to decide whether a series (of constants) converges. But the following test can sometimes give an answer when the ratio test fails.

Proposition. (Root test) *If* $\lim \sqrt[n]{|a_n|} < 1$*, then* $\sum a_n$ *converges absolutely. If the limit is greater than 1, the series diverges. If the limit is 1, the series could do either.*

Proof. Assume the limit is less than 1; pick r strictly between the limit and 1, and choose N in $\mathbb{N}$ such that, for all $n > N$, $\sqrt[n]{|a_n|} < r$. Then $|a_n| < r^n$, and $\sum r^n$ converges, so by the comparison test $\sum_{n=N}^{\infty} |a_n|$ also converges. Adding the finite sum $\sum_{n=1}^{N-1} |a_n|$ shows that $\sum_{n=1}^{\infty} a_n$ converges absolutely. □

Example. Let us apply the ratio and root tests to a few series to see if we can verify convergence:

(a) $\sum 5/(2^n - 1)$:

Ratio test:

$$\frac{5/(2^{n+1}-1)}{5/(2^n-1)} = \frac{2^n - 1}{2^{n+1}-1} = \frac{1-(1/2^n)}{2-(1/2^n)} \to \frac{1-0}{2-0} = \frac{1}{2} < 1$$

so the series converges absolutely.

Root test:

$$\sqrt[n]{\frac{5}{2^n-1}} = \frac{\sqrt[n]{5}}{2\sqrt[n]{1-(1/2^n)}}$$

The n-th root of 5 approaches 1, and because $1-(1/2)^n$ approaches 1, its n-th root approaches it even more quickly; so the limit is $1/2$ and the series converges absolutely.

(b) We know that $\sum 1/n$ diverges, but neither the ratio nor the root test give us that information:

$$\frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1 \qquad\qquad \sqrt[n]{\frac{1}{n}} = n^{-1/n} \to 1$$

The latter limit is found by taking the logarithm and appealing to L'Hôpital's rule from calculus: Because $\log(n^{-1/n}) = -(\log n/n)$, and letting $n \to \infty$ gives the indeterminate form $-\infty/\infty$, we can take the derivative in numerator and denominator; the quotient of the derivatives is $-1/n$, with limit 0, so that is the limit of $\log(n^{-1/n})$; so the limit we want is 1.

(c) We claim that the $\sum 1/n^2$ converges, but the ratio and root tests again fail:

$$\frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1 \qquad\qquad \sqrt[n]{\frac{1}{n^2}} = n^{-2/n} \to 1$$

Euler was the discoverer of the value of $\sum 1/n^2$; it is $\pi^2/6$. His method, however was flawed; a correct proof is linked to the course home page. But Euler's idea is interesting, so let us review it. He reasoned that, if a polynomial $p(x)$ has roots $a_1, a_2, \ldots, a_n$ (repeated by multiplicity), then it can be written as

$$p(x) = c(x-a_1)(x-a_2)\ldots(x-a_n) \ ,$$

where c is the leading coefficient of $p(x)$. If the constant term $\pm ca_1a_2 \dots a_n$ of $p(x)$ is 1, then we can divide through by it and write

$$p(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\dots\left(1 - \frac{x}{a_n}\right) .$$

Now: $**$What is true for polynomials is true for power series.$**$ (This is Euler's error. For example, the exponential function, which is given by a power series, has no roots, so by his reasoning it should be a constant function; but it isn't.) Consider the function

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots$$

It has zeros $\pm\pi, \pm 2\pi, \pm 3\pi, \dots$ and its constant term is 1, so by the reasoning above it can be written as an infinite product

$$\begin{aligned} \frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\dots \\ &= \left(1 - \frac{x}{\pi^2}\right)\left(1 - \frac{x}{4\pi^2}\right)\left(1 - \frac{x}{9\pi^2}\right)\dots \end{aligned}$$

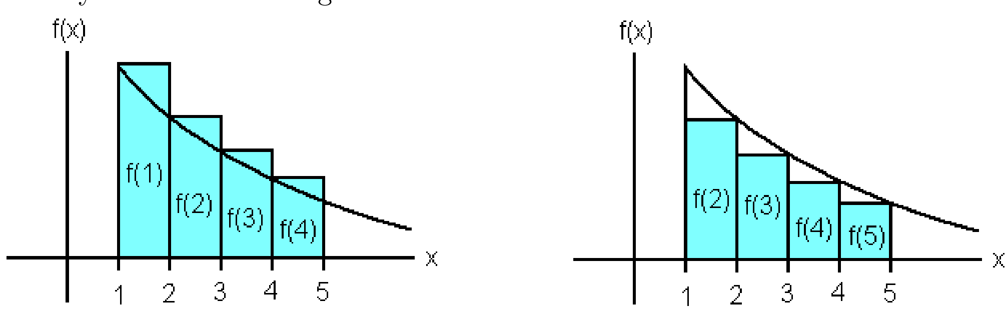
If we multiply this out, the coefficient of x^2 is $-\sum 1/n^2\pi^2$; while the coefficient of x^2 in the power series above is $-1/3!$. Setting these equal to each other gives $\sum 1/n^2 = \pi^2/6$. And that, miraculously, turns out to be correct.

`[At this point students are ready to do the fifth problem set.]`

Our text chooses not to include the following test, because we haven't talked about integrals yet. But it is usually placed at this point in calculus books, so let's include it here.

Proposition. (Integral test) *Suppose $f : [1, \infty) \to \mathbb{R}$ is a decreasing positive function. Then the improper integral* $\inf_1^\infty f$ *converges iff the series* $\sum_{n=1}^\infty f(n)$ *converges.*

Proof. It is easy to see from a diagram



that for every positive integer N

$$\sum_{n=1}^{N-1} f(n) \geq \int_1^N f(x)\,dx \geq \sum_{n=2}^{N} f(n) .$$

Passing to the limit as $N \to \infty$, we see that if either the integral or the series is bounded, then so is the other. (But be careful: If they exist, their limits are not equal. The improper integral is somewhere between the sum of the series and that sum minus its first term.) $\square$

Corollary. $\sum_{n=1}^{\infty} 1/n^p$ *converges iff* $p > 1$.

Proof. If $p \leq 0$, then the terms of the sequence do not approach 0, so the series does not converge. For $p = 1$, we have seen that the harmonic series diverges. For $p > 0$ and $p \neq 1$, $f(x) = x^p$ is a decreasing positive function on $[1, \infty)$, so the Integral Test applies:

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \left[\frac{x^{1-p}}{1-p}\right]_1^{\infty} = \frac{1}{1-p} \lim_{M \to \infty} \left(M^{1-p} - 1\right),$$

and the limit exists (as a finite number) iff $1 - p < 0$, i.e., $p > 1$ $\square$

`[At this point students are ready to do the sixth problem set.]`