Chapter 3: Topology of $\mathbb{R}$

Dictionary: Recall $V_\varepsilon(x)$ is the open interval $(x-\varepsilon, x+\varepsilon)$.

(a) A subset U of $\mathbb{R}$ is *open* iff, $\forall\, x \in U$, $\exists\, \varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq U$.

(b) A subset F of $\mathbb{R}$ is *closed* iff, for every convergent sequence (x_n) with $x_n \in F$ $\forall\, n \in \mathbb{N}$, the limit of (x_n) is also in F.

(c) The *interior* A° of a subset A of $\mathbb{R}$ is the largest open subset of A. (It is obtained from A by taking those elements that have a neighborhood inside A.)

(d) The *closure* $\overline{A}$ of a subset A of $\mathbb{R}$ is the smallest closed subset of $\mathbb{R}$ that contains A (as a subset). (It is obtained by adding to A all the points that are limits of sequences in A.)

(e) If $A \subseteq B \subseteq \overline{A}$, then A is *dense in* B.

(f) For a subset A of $\mathbb{R}$ and $x \in \mathbb{R}$, x is an *isolated point* if $\exists\, \varepsilon > 0$ s.t. $V_\varepsilon(x) \cap A = \{x\}$.

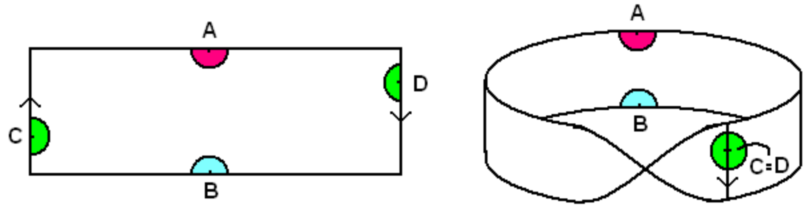
(g) For a subset A of $\mathbb{R}$ and $x \in \mathbb{R}$ (not necessarily in A), x is an *accumulation point* of A iff there is a sequence (a_n) with $a_n \in A \backslash \{x\}$ $\forall\, n \in \mathbb{N}$ s.t., $\lim a_n = x$. (The book calls such an x a *limit point* of A; other books allow isolated points to be limit points.)

Example.

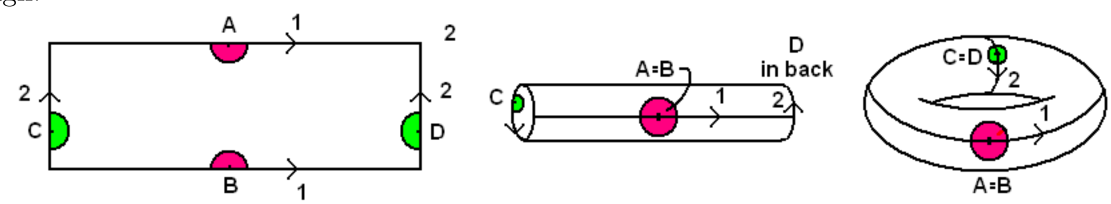
- $\mathbb{R}$ and $\emptyset$ are both open and closed. (In $\mathbb{R}$, they are the only such sets.)
- An open interval (a, b) is an open set. So are (a, ∞) and $(-\infty, a)$.
- A closed interval $[a, b]$ is a closed set. So are $[a, \infty)$ and $(-\infty, a]$. So is any finite set.
- A half-open interval $(a, b]$ is neither open nor closed.
- $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{Q}$ is dense in $\mathbb{R}$.
- $\mathbb{Q}^\circ = \emptyset$.
- $[0, 1]^\circ = (0, 1)$.
- $\overline{(0, 1]} = [0, 1]$.
- $\overline{\{1/n : n \in \mathbb{N}\}} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$.
- $\{1/n : n \in \mathbb{N}\}^\circ = \emptyset$.

Remark. These ideas of topology make sense in many "spaces" other than $\mathbb{R}$; for example, in the plane $\mathbb{R}^2$, or more generally in real n-space $\mathbb{R}^n$, where the neighborhoods of a point are discs (in $\mathbb{R}^2$) or n-dimensional "spheres" (in $\mathbb{R}^n$). It is possible to extend the ideas even to strange "spaces" like the Moebius strip, the torus, or the Klein bottle, just by saying what the neighborhoods of points are. For the Moebius strip, which is physically built by taking a strip of paper, giving it a half twist, and taping the ends, this amounts to saying that, in the following diagram, A and B will remain separate points, with half-disc neighborhoods around them, but that C and D will end up the same point, with a neighborhood around it that is a disc consisting of a half-disc around C

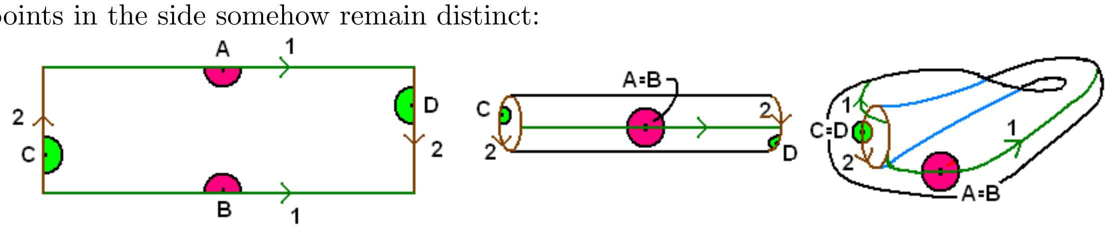
and a half-disc around D:



If we don't do the twist, but glue the top and bottom, first, in the same direction (yielding a tube), and then the left and right edges in the same direction, we end up with a torus, a surface where every point has a neighborhood that looks like a neighborhood in $\mathbb{R}^2$, if you stand close enough:



If you glue top and bottom to get a tube, and then glue the left and right ends in <u>opposite</u> directions, you get a Klein bottle, a surface that cannot be built inside $\mathbb{R}^3$, even though, again, every point has a neighborhood that looks like a neighborhood in $\mathbb{R}^2$. Here is a misleading attempt to draw it. Somewhere in that blue expanse of "bottleneck," the "neck" passes through the "side" of the bottle, so that the ends can be glued in the correct direction; but the points in the neck and the points in the side somehow remain distinct:



Note that, although we can't realize a Klein bottle in $\mathbb{R}^3$, it makes sense as a glued rectangle.

Proposition. *(a) Any union of open sets is open.*

(b) An intersection of <u>finitely many</u> open sets is open.

(c) A set A is open iff its complement $A^c = \mathbb{R}\backslash A$ is closed.

(d) Any intersection of closed set is closed.

(e) A union of <u>finitely many</u> closed sets is closed.

Proof. (a) and (b): These proofs are exercises.

(c) ($\Rightarrow$): Suppose A is open, and let (b_n) be a convergent sequence in A^c, with limit x. If $x \in A$, then there is an neighborhood $V_\varepsilon(x)$ of x contained in A. So $\forall\, n \in \mathbb{N}$, $|x_n - x| \not< \varepsilon$, $\rightarrow\!\leftarrow$. Hence, $x \in A^c$. Thus, A^c is closed.

($\Leftarrow$): Suppose A^c is closed, and let $x \in A$. Assume BWOC that there is no $\varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq A$. Then $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in A^c$ s.t. $|x_n - x| < 1/n$. The sequence (x_n) in A^c has limit x, which must also be in A^c because A^c is closed. So $x \notin A$, $\rightarrow\!\leftarrow$.

(d) and (e): These now follow from (a) and (b). $\square$

Remark. A subset of $\mathbb{R}$ is open iff it is a union of (possibly infinitely many) open intervals.

Proof. ($\Leftarrow$) The last proposition. ($\Rightarrow$) Let U be open. Then for all x in U, $\exists V_\varepsilon(x) \subseteq U$, so $U = \bigcup_{x\in U} V_\varepsilon(x)$. $\square$

Example. (Cantor set) (an interesting subset of $\mathbb{R}$)

$$C = \bigcap_{n=1}^{\infty} (\left[0, \frac{1}{3^n}\right] \cup \left[\frac{2}{3^n}, \frac{3}{3^n}\right] \cup \left[\frac{4}{3^n}, \frac{5}{3^n}\right] \cup \cdots \cup \left[\frac{3^n-1}{3^n}, 1\right]$$

or equivalently, C is the set of all x in $[0,1]$ that can be written as a base-3 "decimal" without using the digit 1. (For example, $1/3 = 0.022\ldots$ in base 3, as well as 0.1, so $1/3$ is in the Cantor set.) It is closed, because each of the sets being intersected (a finite union of closed intervals) is closed (or, because we are starting with a closed set and dropping out, i.e., taking the complement, of the open "middle third" intervals. Note that the "length" of C is

$$1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \frac{8}{81} - \cdots = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - \frac{1/3}{1-(2/3)} = 0 .$$

But C is uncountably infinite, because it can be set in 1-1 correspondence with the interval $[0,1]$, with its elements expressed in base 2, just by replacing the 2's in the base-3 representations of the elements of C by 1's and interpreting the result as a base-2 representation of an element of $[0,1]$.

Questions:

1. Can a nonempty countable set be open?
2. Can a set of length 0 (like the Cantor set) contain any open sets?

`[The answers are both no]`

I introduce the following terminology with some trepidation, because I know it can be misunderstood if it is not used carefully:

Definition. Let A be a subset of $\mathbb{R}$. Then a subset B of A is *open* [respectively *closed] in* A iff $B = A \cap S$ where S is open [respectively closed] in $\mathbb{R}$.

Example. Because of the equation $(1/2, 1] = [0,1] \cap (1/2, 3/2)$, we see that $(1/2, 1]$ is open in $[0,1]$ but closed in $(1/2, 3/2)$.

Definition.
- An *open cover* of a subset A of $\mathbb{R}$ is a family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of open sets in $\mathbb{R}$ for which $A \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda = \bigcup \mathcal{U}$.
- A *subcover* $\mathcal{V}$ of $\mathcal{U}$ is a subfamily of $\mathcal{U}$ for which $A \subseteq \bigcup \mathcal{V}$. It is a *finite subcover* if there are only finitely many open sets in $\mathcal{V}$.
- A subset C of $\mathbb{R}$ is *compact* iff every open cover of C has a finite subcover.

Theorem. (Heine-Borel) *A subset of* $\mathbb{R}$ *is compact iff it is closed and bounded.*

Example. A counterexample to ($\Leftarrow$) in a "metric space" different from $\mathbb{R}$: Let ℓ_1 be the set of all sequences (with terms from $\mathbb{R}$) whose sums are absolutely convergent, with the distance between two sequences (a_n) and (b_n) in ℓ_1 given by (the real number) $\sum |a_n - b_n|$. Consider the set

$$F = \{((1,0,0,0,\dots)\ ,\ (0,1,0,0,\dots)\ ,\ (0,0,1,0,\dots)\ ,\dots)\}\ .$$

Because the distance between any two elements of F is 2, there are no Cauchy sequences in F except for those that are eventually constant; so F is closed. And all elements of F are 1 unit away from $(0,0,0,\dots)$, so F is bounded. Now for each element x of F, notice that $V_{1/2}(x)$ contains only one element of F, namely x; but the union of all the $V_{1/2}(x)$'s, as x varies over F, contains F. So $\{V_{1/2}(x) : x \in F\}$ is an open cover of F with no finite subcover; so F is not compact.

Proof. (of Heine-Borel)

($\Rightarrow$) (works in every metric space) Suppose C is compact. Assume first, BWOC, that C is not closed, i.e., has a limit point x that is not in it. For $\varepsilon > 0$, let $U_\varepsilon = \{y \in \mathbb{R} : |y - x| > \varepsilon\}$, an open set in $\mathbb{R}$. Then $\bigcup_{\varepsilon>0} U_\varepsilon = \mathbb{R}\backslash\{x\}$, so $\{U_\varepsilon : \varepsilon > 0\}$ is an open cover of C. But for any finite subcover

$$\mathcal{V} = \{U_{\varepsilon(1)}, U_{\varepsilon(2)}, \dots, U_{\varepsilon(n)}\}\ ,$$

the union of $\mathcal{V}$ is U_ε where $\varepsilon = \min(\varepsilon(1), \varepsilon(2), \dots, \varepsilon(n))$. Now because x is a limit point of C, there is an element c of C for which $|c - x| < \varepsilon$, so $c \notin U_\varepsilon = \bigcup \mathcal{V}$, so $\mathcal{V}$ is not a subcover of C, $\rightarrow\!\leftarrow$. So C is closed.

Now assume BWOC that the compact set C is not bounded. Then $U_n = (-n, n)$ is an open cover of C with no finite subcover, $\rightarrow\!\leftarrow$. So C is bounded.

($\Leftarrow$) (needs the completeness of $\mathbb{R}$) Let C be a closed and bounded subset of $\mathbb{R}$, and assume BWOC that it has an open cover $\mathcal{U}$ with no finite subcover. Because C is bounded, it is contained in some closed interval $[a, b]$. Now at least one of

$$\left[a, \frac{1}{2}a + \frac{1}{2}b\right] \cap C\ ,\ \left[\frac{1}{2}a + \frac{1}{2}b, b\right] \cap C$$

has no finite subcover by elements of $\mathcal{U}$, because if both had finite subcovers, then their union would be a finite subcover of C. Suppose the first of these has no finite subcover. Then at least one of

$$\left[a, \frac{3}{4}a + \frac{1}{4}b\right] \cap C\ ,\ \left[\frac{3}{4}a + \frac{1}{4}b, \frac{1}{2}a + \frac{1}{2}b\right] \cap C$$

has no finite subcover by elements of $\mathcal{U}$, say the latter. Continuing in this way, we get a sequence of intervals

$$\begin{aligned}
I_1 &= [a, b] \\
I_2 &= \left[a, \frac{1}{2}a + \frac{1}{2}b\right] \\
I_3 &= \left[\frac{3}{4}a + \frac{1}{4}b, \frac{1}{2}a + \frac{1}{2}b\right] \\
&\vdots
\end{aligned}$$

for which $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, the length of I_n is $(b-a)/2^{n-1}$, and $I_n \cap C$ has no finite subcover in $\mathcal{U}$. Now we use the completeness of $\mathbb{R}$: By the Nested Interval Property (and the fact that the lengths of the I_n's are approaching 0), we know $\bigcap_{n=1}^\infty I_n$ is a single-point set, $\{d\}$. And because each $I_n \cap C$ has no finite subcover, it is not empty, so we can pick an element c_n in it. Because both c_n and d are in I_n, we have $|c_n - d| \leq (b-a)/2^{n-1}$, so (c_n) converges to d. Because C_n is closed, $d \in C$, so there is an element U of $\mathcal{U}$ for which $d \in U$; and because U is open, there is an $\varepsilon > 0$ for which $V_\varepsilon(d) \subseteq U$. Pick N in $\mathbb{N}$ for which $(b-a)/2^{N-1} < \varepsilon$. Then because all of I_N is less than ε away from d, we see that

$$I_N \cap C \subseteq V_\varepsilon(d) \subseteq U \ .$$

So $\{U\}$ is a finite subcover of $\mathcal{U}$ for the set $I_N \cap C$, contradicting our choice of I_N. Therefore, C is compact. $\square$

The property in the following proposition is what the text takes as the definition of compact set, but it is not easy to show that they are equivalent in a general metric space.

Proposition. *A subset C of $\mathbb{R}$ is compact iff every sequence in C has a subsequence that has a limit in C.*

Proof. ($\Rightarrow$) Because C is bounded, every sequence in C has a convergent subsequence by Bolzano-Weierstrass; and the limit of that subsequence is in C because C is closed.

($\Leftarrow$) If C is not bounded, then we can find a sequence in C with limit ∞ or $-\infty$, and such a sequence has no convergent subsequence; so C <u>is</u> bounded. If C is not closed, then we can find a sequence in C approaching a limit point not in C, and such a sequence has no subsequence with a limit in C; so C is also closed. $\square$

As the text points out, compact sets should be thought of as generalizations of closed bounded intervals. It offers the following fact as a sample of the kind of statement that generalizes from closed bounded intervals (in this case, the Nested Interval Property) to compact sets.

Proposition. *A decreasing sequence $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$ of nonempty compact sets has nonempty intersection.*

Proof. If the intersection were empty, the complements $\mathbb{R} \backslash C_n$ would give an open cover of C_1, and there would be no finite subcover, because the union of any finite subcover would be the complement of one of the C_n's and hence would not include the elements of C_n, which are in C_1. $\square$

`[At this point students are ready to do the sixth problem set.]`

Optional: Topology in general: Topology (or "rubber-sheet geometry") is a major field of study in math. The most active areas are probably in "manifolds", where every point has a neighborhood that looks like a neighborhood in some $\mathbb{R}^n$. (The torus and Klein bottle mentioned earlier are examples of "2-manifolds without boundary." The Moebius strip is a 2-manifold with boundary, because of the edge.) But the general ideas of topology turn out to be useful in more general situations. Here is one way of describing a "topological space": For any set X of "points", take a family $\mathcal{U}$ of subsets of X that satisfy:

(1) X and $\emptyset$ are in $\mathcal{U}$;

(2) any union of sets in $\mathcal{U}$ is again in $\mathcal{U}$; and

(3) any finite intersection of sets in $\mathcal{U}$ is again in $\mathcal{U}$.

Then $(X, \mathcal{U})$ is a "topological space," the elements of $\mathcal{U}$ are the "open sets," and two points are "close" if they are in the same open set. The "closed sets" are just the complements of the open sets.

Example. $X = \mathbb{R}^2$ and the closed sets are the "algebraic varieties" defined by sets P of polynomials in two variables:

$$V(P) = \{(a, b) \in \mathbb{R}^2 : \forall\, p(x, y) \in P,\ p(a, b) = 0\}\ .$$

Then $\mathbb{R}^2$, with the complements of the algebraic varieties as open sets, forms a topological space. But it is not a metric space, because you can't "separate points": Given two distinct points $(a, b), (c, d)$, for any open sets U, V where $(a, b) \in U$ and $(c, d) \in V$, we have $U \cap V \neq \emptyset$. (If it were a metric space, taking ε to be half the distance between the two points would place the points in disjoint ε-neighborhoods.)