

Chapter 4: Functional Limits and Continuity

Definition. Let $S \subseteq \mathbb{R}$ and $f : S \rightarrow \mathbb{R}$.

- (a) For an accumulation point a of S , the number ℓ is the *limit of $f(x)$ as x approaches a* , or $\lim_{x \rightarrow a} f(x) = \ell$, iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t., if } 0 < |x - a| < \delta \text{ and } x \in S, \text{ then } |f(x) - \ell| < \varepsilon .$$

- (b) For $s \in S$, f is *continuous at s* iff,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t., if } |x - s| < \delta \text{ and } x \in S, \text{ then } |f(x) - f(s)| < \varepsilon .$$

- (c) If f is continuous at every point of S , then f is *continuous on S* .

Thus, relative to (b), if s is an isolated point of S , then f is automatically continuous at s — it just has to be defined there, but we assumed f has domain all of S , including at s . But if s is an accumulation point of S , then to say that f is continuous at s means:

- (1) $f(s)$ is defined (which, again, is automatic),
- (2) $\lim_{x \rightarrow s} f(x)$ exists, and
- (3) $\lim_{x \rightarrow s} f(x) = f(s)$.

The point of the next result is to relate limits of functions to limits of sequences.

Proposition. Let $S \subseteq \mathbb{R}$, $f : S \rightarrow \mathbb{R}$ and a be an accumulation point of S . Then $\lim_{x \rightarrow a} f(x) = \ell$ iff, for every sequence (s_n) in $S \setminus \{a\}$ s.t. $\lim s_n = a$, we have $\lim f(s_n) = \ell$.

Proof. (\Rightarrow) Take a sequence (s_n) in $S \setminus \{a\}$ with limit a , and let $\varepsilon > 0$ be given. Pick $\delta > 0$ as in the definition of function limit, and pick N in \mathbb{N} so that, $\forall n \geq N$, $|s_n - a| < \delta$. Then $\forall n \geq N$, $|f(s_n) - \ell| < \varepsilon$. So $\lim f(s_n) = \ell$.

(\Leftarrow) Assume $\lim_{x \rightarrow a} f(x) \neq \ell$. Negating the definition of function limit gives

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists s \in S \text{ s.t. } 0 < |s - a| < \delta \text{ and } |f(s) - \ell| \geq \varepsilon .$$

Picking the δ -values $1/n$ and picking a corresponding s_n for each of them, we get

$$\exists \varepsilon > 0 \text{ s.t. } \forall n \in \mathbb{N}, \exists s_n \in S \text{ s.t. } 0 < |s_n - a| < \frac{1}{n} \text{ and } |f(s_n) - \ell| \geq \varepsilon .$$

But then the sequence (s_n) in $S \setminus \{a\}$ satisfies $\lim s_n = a$ and $\lim f(s_n) \neq \ell$, contradicting the hypothesis about sequence limits. \square

Recall that, for any sets A, B , any function $f : A \rightarrow B$, and any subsets X of A and Y of B , we write $f(X) = \{f(x) : x \in X\}$ (a subset of B) and call it the *image of X* (under f), and we write $f^{-1}(Y) = \{a \in A : f(a) \in Y\}$ (a subset of A) and call it the *inverse image of Y* (under f). So f^{-1} makes sense as a mapping of subsets of B to subsets of A , even if f , as a function from the elements of A to the elements of B , is not even close to being one-to-one and onto B .

Proposition. Let $S \subseteq \mathbb{R}$ and $f : S \rightarrow \mathbb{R}$. Then f is continuous on S iff, for all open sets U in \mathbb{R} , $f^{-1}(U)$ is open in S (i.e., “the inverse image of every open set is open”).

Proof. (\Rightarrow) Let U be open in \mathbb{R} ; we want to show that $f^{-1}(U) = V \cap S$ for some open set V in \mathbb{R} , i.e., if $s \in f^{-1}(U)$, then $\exists \delta > 0$ s.t. $V_\delta(s) \cap S \subseteq f^{-1}(U)$. Now for $s \in f^{-1}(U)$, $f(s) \in U$, so $\exists \varepsilon > 0$ for which $V_\varepsilon(f(s)) \subseteq U$. Pick $\delta > 0$ so that, if $|x - s| < \delta$ and $x \in S$, then $|f(x) - f(s)| < \varepsilon$. Translating this into neighborhoods says that $V_\delta(s) \cap S \subseteq f^{-1}(V_\varepsilon(f(s))) \subseteq f^{-1}(U)$, as we wanted.

(\Leftarrow) Assume the inverse image of every open set in \mathbb{R} is open in S . Take s in S , and let $\varepsilon > 0$ be given. Then $V_\varepsilon(f(s))$ is open in \mathbb{R} , so $f^{-1}(V_\varepsilon(f(s))) = V \cap S$, where V is open in \mathbb{R} . Let $\delta > 0$ be such that $V_\delta(s) \subseteq V$. Then if $|x - s| < \delta$ and $x \in S$, we have $x \in V_\delta(s) \cap S \subseteq V \cap S \subseteq f^{-1}(V_\varepsilon(f(s))$, i.e., $|f(x) - f(s)| < \varepsilon$. So f is continuous at s ; and hence f is continuous on all of S . \square

Example. Proposition. $\lim_{x \rightarrow 2}(2x^2 - 3) = 5$

Proof. Note that the domain of $2x^2 - 3$ is all of \mathbb{R} , so 2 is an accumulation point of the domain — so this limit has a hope of making sense. Now let $\varepsilon > 0$ be given. [We need $\varepsilon > |(2x^2 - 3) - 5| = 2|x - 2||x + 2|$. We can make $|x - 2|$ small, so we only need to put a bound on $|x + 2|$ near to 2.] Let $\delta = \min(\varepsilon/10, 1)$. Then for $x \in \mathbb{R}$ s.t. $|x - 2| < \delta$, we have $1 < x < 3$ and so $|x + 2| < 3 + 2 = 5$, and hence

$$|(2x^2 - 3) - 5| = 2|x - 2||x + 2| < 2 \left(\frac{\varepsilon}{10} \right) (5) = \varepsilon ,$$

as required. \square

Of course we don't want to go through that every time, so we establish the Algebraic Limit Theorem for Functions as in the text. We won't prove it all — see the text for most of it — but we will prove the subtraction part two ways: Let $S \subseteq \mathbb{R}$, $f, g : S \rightarrow \mathbb{R}$ be continuous and a be an accumulation point of S . If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) .$$

Let's use the notation $\lim_{x \rightarrow a} f(x) = \ell$ and $\lim_{x \rightarrow a} g(x) = m$.

Proof. (I, from the definition of functional limit) Let $\varepsilon > 0$ be given. Pick $\delta > 0$ for which, if $0 < |x - a| < \delta$ and $x \in S$, then $|f(x) - \ell| < \varepsilon/2$ and $|g(x) - m| < \varepsilon/2$. If $0 < |x - a| < \delta$ and $x \in S$,

$$|(f(x) - g(x)) - (\ell - m)| = |(f(x) - \ell) - (g(x) - m)| \leq |f(x) - \ell| + |g(x) - m| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

The result follows. \square

Proof. (II, from the ALT for sequences, using the sequential version of function limits) Let (x_n) be any sequence of elements of S with limit a . Then because f and g are continuous, the sequences $(f(x_n))$ and $(g(x_n))$ have limits ℓ and m respectively. By the ALT for sequences, $\lim(f(x_n) - g(x_n)) = \ell - m$. The result follows. \square

Example. Two odd functions. Here are two examples that show that continuity of functions can happen at odd spots.

- The Dirichlet function, also called the salt-and-pepper function or the characteristic function of the rationals: Recall that, for any subset B of any set A , we can define the “characteristic function of B (in A)” to be the function $\chi_B : A \rightarrow \mathbb{R}$ given by $\chi_B(a) = 1$ if $a \in B$ and 0 if $a \notin B$. If we apply this to the rationals \mathbb{Q} as a subset of the reals \mathbb{R} , we get

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

We claim that $\chi_{\mathbb{Q}}$ is discontinuous at every real number. Suppose first that a is an irrational number. Then we know that $f(a) = 0$, but every $V_{\delta}(a)$, with $\delta > 0$, contains rational numbers b , for which $f(b) = 1$; so with $\varepsilon = 1/2$, there is no $\delta > 0$ for which $0 < |x - a| < \delta$ makes $|f(x) - f(a)| < \varepsilon$. We can make a similar argument if a is rational, because the irrationals are also dense in \mathbb{R} .

- The Thomae function or popcorn function:

$$t(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ where } m, n \in \mathbb{N}, n > 0, \gcd(m, n) = 1 \end{cases}$$

At every rational number m/n , the value of the Thomae function is $1/n$; but there are irrationals, with Thomae value 0, in every neighborhood of m/n , so if we set $\varepsilon = 1/(2n)$, then there is no δ that satisfies the definition of continuity at m/n , i.e., t is discontinuous at every rational number. But it is continuous at every irrational x : Given $\varepsilon > 0$, we can find a $\delta > 0$ for which $V_{\delta}(x)$ contains none of the rational numbers of the form m/n with $1/n \geq \varepsilon$ — such rationals are spaced discretely along the line, so we can exclude them all by taking δ small enough (though writing a formula for such a δ would be a mess). Then $|x - y| < \delta$ forces $|f(x) - f(y)| = f(y) < \varepsilon$.

Example. Exercise 4.3.11 from the text: Find $f : \mathbb{R} \rightarrow \mathbb{R}$ discontinuous at each point of S and continuous on S^c if

(a) $S = \mathbb{Z}$. [One answer is $\chi_{\mathbb{Z}}$.]

(c) (The shift in order is deliberate.) $S = [0, 1]$ [One answer is $\chi_{[0,1] \cap \mathbb{Q}}$.]

(b) $S = (0, 1)$. [The trick is to make the last example continuous at the endpoints, by dragging the value of the ends down to 0. One way to do that is to multiply the answer to (c) by a continuous function that is 0 at 0 and 1, say the parabola $p(x) = x(1 - x)$: $\chi_{[0,1] \cap \mathbb{Q}} \cdot p$.]

(d) $S = \{1/n : n \in \mathbb{N}\}$ [One answer is the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin S \\ \frac{1}{n} & \text{if } x = \frac{1}{n} \end{cases} .]$$

Question. For $A \subseteq \mathbb{R}$, what is the set of points at which χ_A is continuous? [$A^\circ \cup (A^c)^\circ$]

[At this point students are ready to do the 7th problem set.]

Challenge. Prove or give a counterexample in each case: For a continuous function from a subset of \mathbb{R} into \mathbb{R} the \mathcal{Q} of a \mathcal{P} set is a \mathcal{P} set (in the domain of the function).

$\mathcal{P} \setminus \mathcal{Q}$	image	inverse image
open		T: equivalent to “continuous”
closed		T: equivalent to “continuous” [2]
bounded	F: [3]	
compact		
connected [1]	T: [4]	

Notes:

- [1] For a subset A of \mathbb{R} , the following are equivalent:
- A is *connected*.
 - A is an interval (of some kind: bounded or unbounded; open, closed or half-open).
 - For all $r, s, t \in \mathbb{R}$ with $r < s < t$ and $r, t \in A$, we have $s \in A$.
- [2] For any function $f : A \rightarrow B$ and any subset C of B , we have $A \setminus f^{-1}(C) = f^{-1}(B \setminus C)$; so “inverse images respect complements”. Verify this equality of sets, and conclude that the conditions “the inverse image of every open set is open” and “the inverse image of every closed set is closed” are equivalent.
- [3] This one is false, and it is the only one where you can’t get a function f coming from all of \mathbb{R} .
- [4] This is the Intermediate Value Theorem — we’ll do it in class.

[Answers: The image of an open set need not be open; e.g., the image of an open interval under a constant function is a single point. The image of a closed set need not be closed; e.g., the image of $[0, \infty)$ under $x \mapsto x/\sqrt{x^2 + 1}$ is $[0, 1)$. The image of a bounded set need not be bounded; e.g., the image of $[0, \pi/2)$ under $x \mapsto \tan x$ is $[0, \infty)$. The inverse image of a bounded set need not be bounded; e.g., the inverse image of the range (one point) of a constant function is \mathbb{R} . The image of a compact set is compact; we’ll prove it, and use it, shortly. The inverse image of a compact set need not be compact; again, think of a constant function. And finally, the inverse image of a connected set need not be connected; for example, the inverse image of $(-1/2, 1/2)$ under the sine function is the union of infinitely many open intervals, among them $(-\pi/6, \pi/6)$ and $(5\pi/6, 7\pi/6)$.]

Proposition. *Let S be a compact subset of \mathbb{R} and $f : S \rightarrow \mathbb{R}$ be a continuous function. Then $f(S)$ is compact.*

Proof. Let \mathcal{U} be an open cover of $f(S)$, and consider $\mathcal{U}' = \{f^{-1}(U) : U \in \mathcal{U}\}$. Because the U ’s are open in \mathbb{R} , the $f^{-1}(U)$ ’s are open in S ; and $\forall x \in S, f(x) \in f(S)$, so $f(x)$ is in some U in the open cover \mathcal{U} , and hence $x \in f^{-1}(U)$. Therefore, \mathcal{U}' is an open cover of S . By hypothesis, \mathcal{U}' has a finite subcover $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$. Now every element of $f(S)$ is of the form $f(x)$ for some x in S , and each x is in some $f^{-1}(U_j)$, so each $f(x)$ is in some U_j . It follows that $\{U_1, \dots, U_n\}$ is a finite subcover of $f(S)$, and the proof is complete. \square

Now we finally get to some of the basic theorems of calculus, the proving of which is a major objective of this course:

Corollary. (Extreme Value Theorem) Let $S \subseteq \mathbb{R}$ be compact and $f : S \rightarrow \mathbb{R}$ be continuous. Then there exist $a, b \in S$ s.t., $\forall s \in S, f(a) \leq f(s) \leq f(b)$.

Proof. $f(S)$ is compact, so it includes its sup and inf. □

Theorem. (Intermediate Value Theorem) Let I be an interval in \mathbb{R} , $f : I \rightarrow \mathbb{R}$ be continuous, $a, b \in I$ and $r \in \mathbb{R}$ be between $f(a)$ and $f(b)$. Then $\exists c \in [a, b] \subseteq I$ s.t. $f(c) = r$.

Proof. WLOG, $f(a) < r < f(b)$ (because if $r = f(a)$ or $r = f(b)$, then we could take $c = a$ or $c = b$; and if $f(a) > r > f(b)$, it would be simple to rewrite the proof). Let $c = \sup\{x \in [a, b] : f(x) \leq r\}$; we claim that $f(c) = r$.

Assume BWO that $f(c) > r$, and pick $\delta > 0$ s.t., if $|x - c| < \delta$ and $x \in [a, b]$, then $|f(x) - f(c)| < f(c) - r$. Now $c - \delta$ is not an upper bound for $\{x \in [a, b] : f(x) \leq r\}$, so $\exists x^* \in [a, b]$ s.t. $x^* > c - \delta$ and $f(x^*) \leq r$. We have $c - \delta < x^* < c$, so $|x^* - c| < \delta$, but also $f(c) - r \leq f(c) - f(x^*) = |f(x^*) - f(c)|$, ~~✗~~.

Assume BWO that $f(c) < r$; note $c < b$ because $f(b) > r$. Pick $\delta < 0$ s.t. $|x - c| < \delta$ and $x \in [a, b]$ implies $|f(x) - f(c)| < r - f(c)$. Then $x^* = c + \min\{\delta/2, b - c\}$; then $x^* \in [a, b]$ but $x^* > c$, so $x^* \notin \{x \in [a, b] : f(x) \leq r\}$, i.e., $f(x^*) > r$. So we have $|x^* - c| < \delta$ and $r - f(c) < f(x^*) - f(c) = |f(x^*) - f(c)|$, ~~✗~~. □

Definition. Let $S \subseteq \mathbb{R}$ and $f : S \rightarrow \mathbb{R}$. Then f is *uniformly continuous* on S iff, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x_1, x_2 \in S$ and $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \varepsilon$.

The point is that, given $\varepsilon > 0$, the same $\delta > 0$ “works” at every point of S .

[This is a good time to look at the graphic "Given epsilon, find delta," to which there is a link on the course home page.]

Example. (A continuous, but not uniformly continuous, function on a closed, unbounded set) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$, and let's restrict to $x_1 > 0$ and $\varepsilon = 1$. The horizontal stripe around $y = x_1^2$ with radius 1 cuts off the same piece of the graph of $y = x^2$ as the vertical stripe from $x = \sqrt{x_1^2 - 1}$ to $x = \sqrt{x_1^2 + 1}$. Because the graph is concave up, $x_1 - \sqrt{x_1^2 + 1} < \sqrt{x_1^2 - 1} - x_1$, so the largest δ that works for $\varepsilon = 1$ at x_1 is $x_1 - \sqrt{x_1^2 + 1}$. As $x_1 \rightarrow \infty$, this largest δ approaches 0. So there is no $\delta > 0$ that works at every x_1 .

Example. (A continuous, but not uniformly continuous, function on a non-closed, bounded set) Let $f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ be given by $f(x) = \tan x$. For $\varepsilon = 1$, the largest $\delta > 0$ at $x_1 > 0$ is $\arctan(\tan x_1 + 1) - x_1$, which approaches 0 as $x_1 \rightarrow (\pi/2)^-$, so f is not uniformly continuous.

Theorem. If $S \subseteq \mathbb{R}$ is compact and $f : S \rightarrow \mathbb{R}$ is continuous, then f is uniformly continuous on S .

Proof. Let $\varepsilon > 0$ be given. Then $\forall s \in S, \exists \delta(s) > 0$ (varying with s as far as we know, so we put s into the notation for δ — the point is to show that we can find a δ that does not depend on s) s.t., if $|x - s| < \delta(s)$ and $x \in S$, then $|f(x) - f(s)| < \varepsilon/2$. Consider $\mathcal{U} = \{V_{\delta(s)/2}(s) : s \in S\}$; it is an open cover of S , because each s is at least in its own neighborhood, indeed, at the center of it. So by hypothesis, \mathcal{U} has a finite subcover $\mathcal{U}' = \{V_{\delta(s_1)/2}(s_1), \dots, V_{\delta(s_n)/2}(s_n)\}$. Set

$$\delta = \min\{\delta(s_1)/2, \dots, \delta(s_n)/2\},$$

and suppose $x_1, x_2 \in S$ s.t. $|x_1 - x_2| < \delta$. Then because \mathcal{U}' is a cover of S , $x_1 \in V_{\delta(s_j)/2}(s_j)$ for some $j = 1, \dots, n$, so

$$|x_1 - s_j| < \delta(s_j)/2 < \delta(s_j) \quad \text{and hence} \quad |f(x_1) - f(s_j)| < \varepsilon/2 .$$

But we also have

$$|x_2 - s_j| = |(x_2 - x_1) + (x_1 - s_j)| \leq |x_2 - x_1| + |x_1 - s_j| < \delta + \delta(s_j)/2 \leq \delta(s_j) ,$$

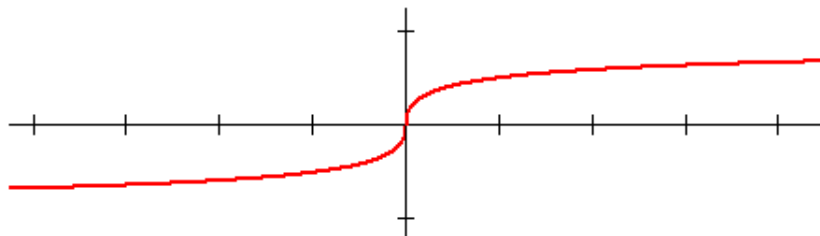
so we have $|f(x_2) - f(s_j)| < \varepsilon/2$, and hence

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(s_j)| + |f(s_j) - f(x_2)| < \varepsilon .$$

So f is uniformly continuous. □

In the two examples before the theorem, it may have seemed that the reason that the function was not uniformly continuous was that the tangent became vertical. This is almost true, as we can see by looking ahead: If the derivative of f defined on some interval I is bounded, say $|f'| \leq M$ on I , then for x_1, x_2 in I , $f(x_2) - f(x_1) = f'(z)(x_2 - x_1)$ for some z between x_1, x_2 by the Mean Value Theorem, so $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$. Thus, given an $\varepsilon > 0$, the value $\delta = \varepsilon/M$ will “work” at every point of I . But:

Example. (A function with a vertical tangent that is uniformly continuous) Consider $f(x) = \operatorname{sgn}(x)\sqrt{|x|}/(\sqrt{|x|} + 1)$ on \mathbb{R} , where sgn denotes the “signum” function: 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$.



The tangent line to the graph is vertical at $x = 0$, but because the function is continuous on the closed interval $[-1, 1]$, it is uniformly continuous on the whole interval. Also, because the derivative is bounded on $[1, \infty)$ and on $(-\infty, -1]$, f is on these intervals as well, so f is uniformly continuous on all of \mathbb{R} .

Here is one use for uniform continuity: Let us define $3^{\sqrt{2}}$. We know what 3^q means if $q \in \mathbb{Q}$: If $q = m/n$, then $3^q = \sqrt[n]{3^m} = (\sqrt[n]{3})^m$. So the natural way to extend the definition to $3^{\sqrt{2}}$ (and similarly for other exponentials with irrational exponents) is “by continuity”: take a sequence of rationals (q_n) that converges to $\sqrt{2}$ and set $3^{\sqrt{2}} = \lim 3^{q_n}$. This is the natural way, but does it really work — for example, does it matter whether which sequence of rationals you take to approach $\sqrt{2}$? It turns out that continuity isn’t quite enough:

Example. The function $f(x) = 1/(x - \pi)$ is continuous on \mathbb{Q} , even on $\mathbb{R}/\{\pi\}$, but it is not uniformly continuous, and there is no function $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ that extends f .

Proposition. If $S \subseteq \mathbb{R}$ and $f : S \rightarrow \mathbb{R}$ is uniformly continuous, then there is a unique continuous (in fact, uniformly continuous) $\bar{f} : \bar{S} \rightarrow \mathbb{R}$ for which $\bar{f}(x) = f(x) \forall x \in S$ (i.e., \bar{f} is an “extension” of f to the closure \bar{S} of its domain S).

Proof. We want to define \bar{f} as follows: For $x \in S$, $\bar{f}(x) = f(x)$; and for $x \in \bar{S} \setminus S$, pick a sequence (x_n) in S with limit x , and set $\bar{f}(x) = \lim f(x_n)$. To see that this limit exists, let $\varepsilon > 0$, and pick $\delta > 0$ s.t., $\forall s, t \in S$ with $|s - t| < \delta$, we have $|f(s) - f(t)| < \varepsilon$. Because (x_n) is Cauchy, $\exists N \in \mathbb{N}$ s.t. $m, n \geq N$ implies $|x_m - x_n| < \delta$ and hence $|f(x_m) - f(x_n)| < \varepsilon$. Therefore $(f(x_n))$ is Cauchy, so it has a limit in \mathbb{R} . We define $\bar{f}(x)$ to be that limit.

This is the only way we could hope to define a continuous function extending f to \bar{S} , because continuity must respect sequential limits. Let's check that this $\bar{f} : \bar{S} \rightarrow \mathbb{R}$ is uniformly continuous: Let $\varepsilon > 0$ be given, and pick $\delta > 0$ s.t., if $|u - v| < \delta$ and $u, v \in S$, then $|f(u) - f(v)| < \varepsilon/3$. Take $x, y \in \bar{S}$, and pick sequences $(x_n), (y_n)$ of elements of S with limits x, y respectively; then by the definition of \bar{f} or just the continuity of f , $\lim f(x_n) = \bar{f}(x)$ and $\lim f(y_n) = \bar{f}(y)$. Pick $N \in \mathbb{N}$ s.t. $\forall n \geq N$, we have all four of

$$\begin{aligned} |x - x_n| &< \frac{\delta}{3}, & |y - y_n| &< \frac{\delta}{3}, \\ |f(x_n) - \bar{f}(x)| &< \varepsilon/3, & |f(y_n) - \bar{f}(y)| &< \varepsilon/3. \end{aligned}$$

Then if $|x - y| < \delta/3$, we have for $n \geq N$

$$|x_n - y_n| \leq |x_n - x| + |x - y| + |y - y_n| < 3 \cdot \frac{\delta}{3} = \delta,$$

so $|f(x_n) - f(y_n)| < \varepsilon/3$, and hence

$$|\bar{f}(x) - \bar{f}(y)| \leq |\bar{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \bar{f}(y)| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $\delta/3$ “works” for ε at every element of \bar{S} ; so \bar{f} is uniformly continuous. \square

Corollary. For $a > 0$, there is a unique continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ for which $f(q) = a^q$ for all $q \in \mathbb{Q}$.

Proof. The function $f : \mathbb{Q} \rightarrow \mathbb{R}$ given by $f(q) = a^q$ isn't uniformly continuous on all of \mathbb{Q} , but it is enough to show that it is uniformly continuous on each $[-N, N] \cap \mathbb{Q}$ for some N in \mathbb{N} , because then we can extend f to the whole interval $[-N, N]$; then we let $N \rightarrow \infty$. But even though we can't take derivatives yet (and even if we could, the domain here is only rational numbers), we can see that any difference quotient $(a^{q_2} - a^{q_1})/(q_2 - q_1)$ with q_1, q_2 in $[-N, N] \cap \mathbb{Q}$ is no larger than the slope of the tangent line to the graph of $f(q) = a^q$ at the right end, $q = N$ [where the derivative is $a^N \ln a$ — unless $0 < a < 1$, in which case the function is decreasing and the absolute value of the slope at the left end, $a^{-N} \ln a$, is the bound]. So the difference quotient is bounded, say by M , and that means that $\delta = \varepsilon/M$ works to show f is uniformly continuous on $[-N, N] \cap \mathbb{Q}$. \square

Here are two other (ultimately equivalent) ways that exponentials with irrational exponents might be defined. In both cases, the starting point is more advanced than uniform continuity, and the derivation of the properties of the exponential function might be easier, but they both have the problem of seeing that they agree with the basic definition of exponential when the exponent is rational.

- Define a^x (after we have studied “power series,” which we haven't discussed yet) as follows: Verify that the series $\sum_{n=0}^{\infty} x^n/n!$ converges for all x in \mathbb{R} , and define the function

$$\exp(x) = \sum_{n=0}^{\infty} x^n/n! .$$

Verify that \exp is strictly increasing and has range $(0, \infty)$, so it has an inverse function $\ln : (0, \infty) \rightarrow \mathbb{R}$. Then we set

$$a^x = \exp(x \ln a) .$$

- Define a^x (after studying derivatives, integrals and differential equations) by verifying that “initial value problems” like $y' = y$, $y(0) = 1$ have unique solutions, and define the solution to this one to be \exp . Then define a^x as above.

[At this point students are ready to do the 8th problem set.]