Chapter 6: Sequences and Series of Functions

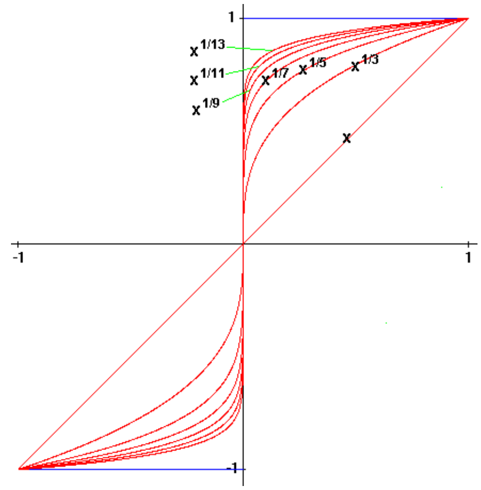
Definition. A sequence $(f_n)_{n=1}^{\infty}$ of functions on a subset A of $\mathbb{R}$ into $\mathbb{R}$:

- *converges pointwise* to $f : A \to \mathbb{R}$ iff, $\forall\, x \in A$, $(f_n(x)) \to f(x)$; i.e., $\forall\, x \in A$ and $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t., if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$.
- *converges uniformly* to $f : A \to \mathbb{R}$ iff $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t., if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ $\forall\, x \in A$. (In other words, the same N works for all x-values.)

Example. $A = [-1, 1]$, $f_n(x) = x^{1/(2n-1)}$,

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Notice that $f_n \to f$ pointwise, but not uniformly: For x near 1 or -1, a much smaller N is needed to make $f_N(x)$ close to $f(x)$ than is true for x near 0.



Theorem. *If each f_n is continuous and $f_n \to f$ uniformly, then f is also continuous.*

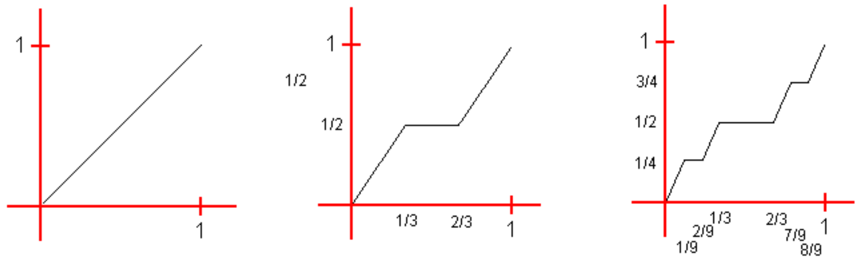
Proof. Let $\varepsilon > 0$ be given, pick $N \in \mathbb{N}$ s.t., $\forall\, n \geq N$, $|f_n(x) - f(x)| < \varepsilon/3$ $\forall\, x \in A$. Fix c in A, and let $\delta > 0$ be such that $|x - c| < \delta$ (and $x \in A$) implies $|f_N(x) - f_N(c)| < \varepsilon/3$. Then $|x - c| < \delta$ implies

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_n(x) - f_N(c)| + |f_N(c) - f(c)| < \varepsilon\ .$$

So f is continuous at c. □

The Cauchy criterion for convergence of a sequence of numbers translates to a sequence of functions, for either kind of convergence: A sequence (f_n) of functions on A is *pointwise Cauchy* iff, $\forall\, x \in A$ and $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t., if $m, n \geq N$, then $|f_m(x) - f_n(x)| < \varepsilon$. The sequence is (of course) *uniformly Cauchy* iff, in this definition, we can move the "$\forall\, x \in A$" to the end. Because a Cauchy sequence of real numbers is convergent, a sequence of functions that is Cauchy in either sense has a limit function; and if the sequence is uniformly Cauchy, the convergence is uniform. So by the theorem, a uniformly Cauchy sequence of continuous functions has a continuous limit.

Example. A sequence of functions is defined as follows: The function f_0 is just the function x. The function f_1 is the function that is constant $1/2$ on the interval $(1/3, 2/3)$ (i.e., the first interval removed in constructing the Cantor set), and linear from the point $(0, 0)$ to the point $(1/3, 1/2)$ and from the point $(2/3, 1/2)$ to the point $(1, 1)$. The function f_2 is constantly $1/4$ on the interval $(1/9, 2/9)$, $1/2$ on the interval $(1/3, 2/3) = (3/9, 6/9)$ and $3/4$ on the interval $(7/9, 8/9)$ (i.e., the first three intervals removed in constructing the Cantor set), and linear on each remaining piece of $[0, 1]$ to make the function continuous. Each new f_n is defined in this way: f_{n+1} agrees with f_{n-1} on the intervals removed by the n-th stage of forming the Cantor set, and on the remaining subintervals of $[0, 1]$ it is made constantly $k/2^{n+1}$ on the interval removed at the $(n+1)$-th stage that is k-th from the left among all the pieces removed to this point, and linear (but steeper than the linear pieces of f_n) on the what is left of $[0,1]$, to make it continuous.



It can be shown that this sequence of functions is uniformly Cauchy, so its limit f is a continuous function, constant on all the pieces of $[0,1]$ that were removed in the construction of the Cantor set. Thus, we have a continuous monotone increasing function from $[0,1]$ <u>onto</u> itself that has derivative 0 everywhere except on the Cantor set (i.e., derivative 0 on a subset of measure 1).

The next proposition justifies the process of differentiating a power series term-by-term to get the derivative of the sum function.

Proposition. *If $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a,b]$, then f is differentiable and $f' = g$ on $[a,b]$.*

Proof. Fix c in $[a,b]$. We want to show that $f'(c)$ (which is a limit, by definition) is equal to $g(c)$, so let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ for which $0 < |x-c| < \delta$ (and $x \in [a,b]$) implies

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \varepsilon\ .$$

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| \le \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| + \left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| + |f_n'(c)-g(c)|\ ,$$

we want to show that we can find an n in $\mathbb{N}$ and a $\delta > 0$ for which each of these terms is $\le \varepsilon/3$, with at least one inequality strict. The last one is easy: There is an N_2 in $\mathbb{N}$ for which $n \ge N_2$ implies $|f_n'(c) - g(c)| < \varepsilon/3$. And once we have fixed on a subscript n that works in the first and third terms, then we can pick a $\delta > 0$ that works in the second term for that f_n. So it remains to make the first term work: Pick N_1 in $\mathbb{N}$ so that $m, n \ge N_1$ implies

$$|f_m'(t) - f_n'(t)| \le \frac{\varepsilon}{3}$$

for all t in $[a,b]$. Then by the MVT applied to $f_m - f_n$, for all x, c in $[a,b]$, there is a t between x, c for which

$$\begin{aligned}\left|\frac{f_m(x)-f_n(c)}{x-c} - \frac{f_m(x)-f_n(c)}{x-c}\right| &= \left|\frac{(f_m-f_n)(x)-(f_m-f_n)(c)}{x-c}\right| \\ &= |(f_m-f_n)'(t)| = |f_m'(t)-f_n'(t)| < \frac{\varepsilon}{3}\ .\end{aligned}$$

And because $f_m(x) \to f(x)$ and $f_m(c) \to f(c)$ as $m \to \infty$, we see that for $n \ge N_1$,

$$\left|\frac{f(x)-f_n(c)}{x-c} - \frac{f(x)-f_n(c)}{x-c}\right| \le \frac{\varepsilon}{3}\ .$$

(In the limit, the inequality isn't strict anymore.) So picking n to the the larger of N_1, N_2 and a $\delta > 0$ that works in the second term for f_n, we get the inequality we wanted. $\square$

So now we have our basic objective:

Definition. A *power series centered at* c is an indicated sum

$$\sum_{n=0}^{\infty} a_n(x-c)^n \ .$$

The partial sums

$$s_n(x) = \sum_{j=0}^{n} a_j(x-c)^j$$

are polynomials that form a sequence of functions; if they converge (in any sense, on any set), to a function $f(x)$, then we say the power series converges to $f(x)$ (in that sense, on that set). Of course, $s_n(c) = a_0$, so the power series does converge at c.

Proposition. *If, for a given x-value x_0, the series of constants $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converges absolutely, then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges uniformly on $[c-r, c+r]$ where $r = |x_0 - c|$.*

Proof. Let $\varepsilon > 0$ be given. We will use the Cauchy criterion for convergence of a series of constants, applied to $\sum_{n=0}^{\infty} |a_n(x_0-c)^n|$: There is an N in $\mathbb{N}$ for which, if $m > n \geq N$, then $\sum_{j=n+1}^{m} |a_j(x_0 - c)^j| < \varepsilon$. So for all x in $[c-r, c+r]$, we have $|x-c| \leq |x_0 - c| = r$, so

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{j=n+1}^{m} a_j(x-c)^j \right| \\ &\leq \sum_{j=n+1}^{m} |a_j||(x-c)^j| \leq \sum_{j=n+1}^{m} |a_j||(x_0-c)^j| < \varepsilon \ . \end{aligned}$$

□

Corollary. *If $\sum_{n=0}^{\infty} a_n(x-c)^n$ and $\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$ $(= \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-c)^n)$ converge, the latter uniformly, on $[c-r, c+r]$, then (the limit function of) $\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-c)^n$ is the derivative of (the limit function of $\sum_{n=0}^{\infty} a_n(x-c)^n$ on $[c-r, c+r]$.*

Corollary. *If all the term-by-term "formal" derivatives of $\sum_{n=0}^{\infty} a_n(x-c)^n$ converge uniformly on $[c-r, c+r]$ for some $r > 0$, then $a_n = (1/n!)f^{(n)}(c)$.*

Proof. By induction,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)a_n(x-c)^{n-k} \ ,$$

so $f^{(k)}(c) = k(k-1)\ldots(1)a_k$. □

Example. Suppose that we have already defined (or somehow shown) that $e^x = \sum_{n=0}^{\infty} x^n/n!$, but we want to find the coefficients a_n of the power series of this function centered at 1: $e^x = \sum_{n=0}^{\infty} a_n(x-1)^n$. Because the n-th derivative of e^x is e^x, we have $a_n = (1/n!)e^1 = e/(n!)$; so

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n (= e \cdot e^{(x-1)} \ .$$

We saw above that, if a power series converges uniformly at some point other than its center c, then it converges on the whole interval out to that point from c. A natural question is: Just how wide is the interval on which a series converges? Here is the answer:

Proposition. *Set* $R = (\limsup \sqrt[n]{|a_n|})^{-1}$ *(or* ∞ *if the lim sup is 0). Then* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *converges absolutely for* x *in* $(c-R, c+R)$ *and diverges for* $x < c-R$ *or* $x > c+R$. *(At* $x = c-R$ *and* $x = c+R$*, the series may converge or diverge.)*

Proof. (partial) Suppose $x \in (c-R, c+R)$, and pick $s > 1$ s.t. $|x-c|s^2 < R$. Then pick N in $\mathbb{N}$ for which $n \geq N$ implies $\sqrt[n]{|a_n|} < s/R$. Then for $n \geq N$

$$\sqrt[n]{|a_n(x-c)^n} = \sqrt[n]{|a_n|}|x-c| < \frac{1}{s} < 1 \ ,$$

so by the Root Test, $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely.

If $x < c-R$ or $x > c+R$, then the terms of the series don't approach 0. $\square$

Definition. $R = (\limsup \sqrt[n]{|a_n|})^{-1}$ is the *radius of convergence* of $\sum_{n=0}^{\infty} a_n(x-c)^n$.

Note:

$$\sqrt[n]{|(n+1)a_{n+1}|} = \sqrt[n]{n+1}\left(\sqrt[n+1]{|a_{n+1}|}\right)^{(n+1)/n} \ ,$$

and as $n \to \infty$, $\sqrt[n]{n+1}$ and $(n+1)/n$ approach 1, so

$$\limsup \sqrt[n]{|(n+1)a_{n+1}|} = \limsup \sqrt[n+1]{|a_{n+1}|} = \limsup \sqrt[n]{|a_n|} \ .$$

Thus, a power series and its term-by-term first derivative (and by induction all the higher term-by-term derivatives) have the same radius of convergence.

Example. Consider $\sum_{n=0}^{\infty} x^n/n!$. Because $\sqrt[n]{n!}$ is not bounded, $\limsup \sqrt[n]{1/n!} = 0$, so the radius of convergence for the power series for e^x is ∞. (To see that $\sqrt[n]{n!}$ is not bounded, assume BWOC that it is bounded, by B. Then $n!/B^n < 1$ for all n. But if we take $n > B^{[B]+1}/[B]!$ (where the brackets mean the "greatest integer" or "floor" function: $[B]$ is the greatest integer less than or equal to B, so that its factorial makes sense), then

$$\frac{n!}{B^n} = \frac{n}{B} \cdot \frac{n-1}{B} \cdot \ldots \cdot \frac{[B]+1}{B} \cdot \frac{[B]!}{B^{[B]}} > n \cdot \frac{[B]!}{B^{[B]+1}}$$

because $k/B > 1$ for $k = [B]+1, [B]+2, \ldots, n-1$. (It is also true for $k = n$, but we want to keep that factor.) But the last expression is > 1, so we have a contradiction.)

Example. Consider

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}5^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{\sin n\pi}{n!}x^n \ .$$

Because

$$\limsup \sqrt[n]{\left|\frac{\sin n\pi}{n!}\right|} = 0 \ ,$$

so the radius of convergence is ∞.

Example. Consider

$$\frac{1}{2\cot 1}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3} + \cdots = \sum_{n=0}\infty \frac{1}{(n+2)(n+1)}x^{n+2} \ .$$

Because

$$\limsup \sqrt[n]{\frac{1}{(n+2)(n+1)}} = \limsup \frac{1}{\sqrt[n]{n+2}\,\sqrt[n]{n+1}} = 1 \ ,$$

so the radius of convergence is 1. The series converges (absolutely) at $x = 1$ and $x = -1$, because $1/((n+2)(n+1)) < 1/n^2$. The derivative,

$$\sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1} \ ,$$

diverges at $x = 1$ and converges (conditionally) at $x = -1$. The second derivative

$$\sum_{n=0}^{\infty} x^n$$

diverges at both $x = 1$ and $x = -1$.

Example. By long division $1/(1+x) = 1 - x + x^2 - x^3 + \ldots$, and because the right side is a geometric series, it converges iff $|x| < 1$. Integrating (term by term) gives

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

The latter diverges at $x = -1$ and converges conditionally at $x = 1$.