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1.3.3 (a) If $A = \emptyset$, then $B = \mathbb{R}$, so $\sup B = +\infty = \inf A$. Suppose $A \neq \emptyset$; then B is bounded above by any element of A. Let $b^* = \sup B$. First we need to show that b^* is a lower bound for A, so assume not, BWOC. Then there is an element a of A for which $a < b^*$. And because $b^* = \sup B$, any number less than b^* is not an upper bound for B, so a is not an upper bound for B; i.e., there is an element b of B for which $a < b$. But B was the set of lower bounds for A, so b is a lower bound for A, so $a < b$ is impossible. So we have a contradiction, and b^* is a lower bound of A. Now take c a lower bound of A; we must show that $c \leq b^*$. But by definition of B, $c \in B$, and b^* is an upper bound for B, so $c \leq b^*$. Therefore, $b^* = \inf A$. [Here is another way: Let $b^* = \inf A$; then by definition b^* is the largest element in B, so as in Exercise 1.3.7 below, $b^* = \sup B$.]

1.3.4 Let $a^* = \sup A$. Then for every b in B, $b \in A$, so $b \leq a^*$; so a^* is an upper bound for B. Therefore $a^* \geq \sup B$.

1.3.5 (a) Let $a^* = \sup A$. Then for all a in A, $a \leq a^*$, so $c + a \leq c + a^*$; so $c + a^*$ is an upper bound for $c + A$. And if b is an upper bound for $c + A$, i.e., $c + a \leq b$ for all a in A, then $a \leq b - c$ for all a in A, so $b - c$ is an upper bound for A, so $a^* \leq b - c$, so $c + a^* \leq b$. Therefore $c + a^* = \sup(c + A)$.

(b) Again, let $a^* = \sup A$. If $c = 0$, then $cA = \{0\}$, so $\sup(cA) = 0 = ca^*$, and we are finished; so assume $c > 0$. Then for all a in A, $a \leq a^*$, so $ca \leq ca^*$ (because $c > 0$); so ca^* is an upper bound for cA. And if b is an upper bound for cA, i.e., $ca \leq b$ for all a in A, then $a \leq b/c$ for all a in A (because $1/c > 0$ also), so b/c is an upper bound for A, so $a^* \leq b/c$, so $ca^* \leq b$. Therefore $ca^* = \sup(cA)$.

(c) If $c < 0$ and $A \subset \mathbb{R}$ is bounded below, then $\sup(cA) = c(\inf A)$.

1.3.6 (a) The sup is 3 (because we are only looking at natural numbers), and the inf is 1.

(b) The sup is 1 (fix m and let n get large), and the inf is 0 (fix n and let m get large).

(c) As n increases, so does $n/(2n+1)$: A little algebra of inequalities shows that $(n+1)/(2(n+1)+1) > n/(2n+1)$. So the sup is $1/2$ (let n get large), and the inf is $1/3$ (take $n = 1$).

(d) The sup is 9 (let $n = 9$ and $m = 1$), and the inf is $1/9$ (reverse m and n).

1.3.7 To see that a is the least upper bound of A, we must show that it is less than or equal to any upper bound b of A; but because a is one of the elements of A, we must have $a \leq b$. So $a = \sup A$.

1.3.8 Because $\sup A < \sup B$, $\sup A$ is not an upper bound for B, so there is an element b of B for which $b > \sup A$, so for all $a \in A$ we have $b > a$, i.e., b is an upper bound for A.

1.3.9 (a) True.

(b) False: Take $A = \{n/(n+1) : n \in \mathbb{N}\}$ and $L = 1$.

(c) False: For the same A as in (b), take $B = \{1\}$.

(d) True.

(e) False: Take $A = \{1\}$ and $B = \{n/(n+1) : n \in \mathbb{N}\}$.

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1.4.2 (c) $\mathbf{I}$ is not closed under either addition or multiplication: $\sqrt{2}, -\sqrt{2} \in \mathbf{I}$, but $\sqrt{2} + (-\sqrt{2}) = 0 \notin \mathbf{I}$ and $\sqrt{2}(-\sqrt{2}) = -2 \notin \mathbf{I}$. On the other hand, $\sqrt{2} + \sqrt{2} = 2\sqrt{2} \in \mathbf{I}$ and $\sqrt{2}(\sqrt{3}) = \sqrt{6} \in \mathbf{I}$, so we can't say much of anything about the sum and product of two irrationals.

1.4.4 Because $n > 0$ for all n in $\mathbb{N}$, also $1/n > 0$, so 0 is a lower bound for $\{1/n : n \in \mathbb{N}\}$. Take any lower bound ε for this set. If $\varepsilon > 0$, then by the Archimedean property, there is an n in $\mathbb{N}$ for which $1/n < \varepsilon$; so ε is not a lower bound for this set, a contradiction. Therefore, $\varepsilon \leq 0$, so $0 = \inf\{1/n : n \in \mathbb{N}\}$.