Page 43, Exercises 2.2

2.2.1 (a) Let $\varepsilon > 0$ be given. [We want

$$\left|\frac{1}{6n^2+1} - 0\right| < \varepsilon , \qquad \text{i.e.,} \qquad 6n^2+1 > \frac{1}{\varepsilon} , \qquad \text{i.e.,} \qquad n > \sqrt{\frac{1}{6}\left(\frac{1}{\varepsilon} - 1\right)} .$$

But what if $\frac{1}{\varepsilon} - 1 < 0$? That would mean $\frac{1}{\varepsilon} < 1$, or $\varepsilon >1$, and big ε's are no problem — we just have to rule out taking the square root of a negative number.] Let $N \in \mathbb{N}$ be such that $N > \sqrt{\max\{0, (1/\varepsilon) - 1\}/6}$. Then for all $n \geq N$, we have (using the fact that squaring positive numbers does not change their relative magnitudes):

$$\begin{aligned} n &> \sqrt{\frac{\max\{0, (1/\varepsilon) - 1\}}{6}} \\ 6n^2 &> \max\{0, \frac{1}{\varepsilon} - 1\} \ \geq \ \frac{1}{\varepsilon} - 1 \\ 6n^2 + 1 &> \frac{1}{\varepsilon} \\ \left|\frac{1}{6n^2+1} - 0\right| = \frac{1}{6n^2+1} &< \varepsilon ; \end{aligned}$$

so $\lim(1/(6n^2+1)) = 0$.

(b) Let $\varepsilon > 0$ be given. [We want $|(3n+1)/(2n+5) - 3/2| < \varepsilon$. Now

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{2(3n+1) - 3(2n+5)}{2(2n+5)}\right| = \left|\frac{-13}{4n+10}\right| ,$$

so we want $13/(4n+10) < \varepsilon$, i.e., $4n+10 > 13/\varepsilon$, i.e., $n > ((13/\varepsilon) - 10)/4$.] Let $N \in \mathbb{N}$ be such that $N > ((13/\varepsilon) - 10)/4$. Then for all $n \geq N$, we have:

$$\begin{aligned} n &> \frac{1}{4}\left(\frac{13}{\varepsilon} - 10\right) \\ 4n + 10 &> \frac{13}{\varepsilon} \\ \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{13}{4n+10} &< \varepsilon \end{aligned}$$

so $\lim((3n+1)/(2n+5)) = 3/2$.

(c) Let $\varepsilon > 0$ be given. [We want

$$\left|\frac{2}{\sqrt{n+3}} - 0\right| < \varepsilon , \qquad \text{i.e.,} \qquad \frac{\sqrt{n+3}}{2} > \frac{1}{\varepsilon} , \qquad \text{i.e.,} \qquad n > \left(\frac{2}{\varepsilon}\right)^2 - 3 .]$$

Let $N \in \mathbb{N}$ be such that $N > (2/\varepsilon)^2 - 3$. Then for all $n \geq N$, we have (because taking square roots of positive quantities doesn't change inequalities):

$$\begin{aligned} n &> \left(\frac{2}{\varepsilon}\right)^2 - 3 \\ \sqrt{n+3} &> \frac{2}{\varepsilon} \\ \left|\frac{2}{\sqrt{n+3}} - 0\right| = \frac{2}{\sqrt{n+3}} &< \varepsilon , \end{aligned}$$

so $\lim(2/\sqrt{n+3}) = 0$.

2.2.2 All convergent sequences are vercongent. One vercongent sequence that is divergent is $((-1)^n)_{n=1}^{\infty} = (-1, 1, -1, 1, \ldots)$. For this sequence, one possible x is 0, with corresponding $\varepsilon = 2$: For any $N \in \mathbb{N}$, $n \geq N$ implies $|(-1)^n - 0| < 2$.

In fact, a vercongent sequence is just a sequence the set of whose terms is bounded above and below: The definition says that all of the x_n's are between $x-\varepsilon$ and $x+\varepsilon$; so a vercongent sequence is bounded. For the converse, suppose all the terms of the sequence $(x_n)_{n=1}^{\infty}$ are bounded below by A and above by B. Then the midpoint $(A+B)/2$ will work for x and anything greater than half the distance between A and B, for example $((B-A)/2)+1$, will work for ε to show that $(x_n)_{n=1}^{\infty}$ is vercongent: Because $A \leq x_n \leq B$, we have

$$-\frac{B-A}{2} = A - \frac{A+B}{2} \leq x_n - \frac{A+B}{2} \leq B - \frac{A+B}{2} = \frac{B-A}{2} \ ,$$

so $|x_n - (A+B)/2| \leq (B-A)/2 < ((B-A)/2)+1$.

2.2.6 (a) ... larger N ... (b) ... larger value of ε. Because: Suppose that we already have an N that "works" for ε, and that $\varepsilon_1 > \varepsilon$ and $N_1 > N$. Then for all $n \geq N_1$, we have $n \geq N$ also, so $|x_n - x| < \varepsilon$, so $|x_n - x| < \varepsilon_1$; i.e., N_1 "works" for both ε and ε_1.

2.2.7 (a) The part of the definition of limit that is hard to generalize is the part about $|x_n - x| < \varepsilon$, because it says that x_n "gets close to x"; how do we give a specific mathematical meaning to "gets close to ∞"? Clearly, we can only "get close to ∞" from below, and getting "within ε of ∞" is meaningless. The best we can do in saying "close to ∞" is to say "greater than some number", and we think of that number as big (close to ∞) rather than small (close to 0, like ε). So that phrase is what replaces "within ε of x" in the definition: "$\lim x_n = \infty$ means that, for any $B > 0$, there is an $N \in \mathbb{N}$ such that, for all $n > N$, $x_n > B$."

To prove $\lim \sqrt{n} = \infty$: Let $B > 0$ be given, and choose $N \in \mathbb{N}$ so that $N > B^2$. Then for all $n \geq N$, we have $\sqrt{n} > B$, which is what we needed to show.

(b) It should say (even if you didn't phrase your definition exactly as I did) that the sequence has no limit at all. It does not have limit ∞ because not all the terms after some point are larger than, say, 1, because 0 keeps reappearing throughout the sequence. And the other terms, as they increase without bound, do not get close to any finite real number.

2.2.8 (a) The sequence $(-1)^n$ is frequently in the set $\{1\}$, because every even-numbered term is there; but not eventually, because every odd-numbered term is not there.

(b) Eventually implies frequently, but not conversely.

(c) "$\lim a_n = a$ means that, for every $\varepsilon > 0$, the sequence $(a_n)_{n=1}^{\infty}$ is eventually in the interval $(a-\varepsilon, a+\varepsilon)$."

(d) It is not (necessarily) eventually in that interval; a counterexample would be $(2(-1)^n)_{n=1}^{\infty}$. But it is frequently in that interval, because the first N terms can only account for a finite number of the terms in the sequence; there are always more (in fact, infinitely many) further along than the N-th one that are equal to 2 and hence in the interval $(1.9, 2.1)$.