Page 49, Exercises 2.3

2.3.1 Let $\varepsilon > 0$ be given, and let $N = 1$. Then for all $n \geq N$, we have $|a - a| = 0 < \varepsilon$; so the limit of the constant sequence $(a, a, a, \ldots)$ is a.

2.3.2 (a) Let $\varepsilon > 0$ be given. [We want $\sqrt{x_n} = |\sqrt{x_n} - 0| < \varepsilon$, i.e., $x_n < \varepsilon^2$, and that we can arrange.] Because $x_n \to 0$, we can find an $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n = |x_n - 0| < \varepsilon^2$; and hence $|\sqrt{x_n} - 0| = \sqrt{x_n} < \varepsilon$. Therefore, $\sqrt{x_n} \to 0$.

(b) Because the case $x = 0$ was done in (a), we may assume that $x > 0$. Let $\varepsilon > 0$ be given. [We want to end up with $|\sqrt{x_n} - \sqrt{x}| < \varepsilon$; and we know that we can get $|x_n - x| < \sigma$, where σ means any small quantity, probably an expression in ε. Now in exploring possibilities, we might realize that $|x_n - x| = |\sqrt{x_n} - \sqrt{x}|(\sqrt{x_n} + \sqrt{x})$, so when we make $|x_n - x|$ approach 0, one of these factors must approach 0 also. Can we be sure that it's not the second factor that is approaching 0? Yes, because it's at least as large as $\sqrt{x}$, and that's not 0 because we have assumed $x > 0$. Now again, we want to end up with $|\sqrt{x_n} - \sqrt{x}| < \varepsilon$, which is equivalent to $|x_n - x| < \varepsilon(\sqrt{x_n} + \sqrt{x})$. If we arrange to have the stronger statement $|x_n - x| < \varepsilon\sqrt{x}$ hold, we'll have what we need.] Because $x_n \to x$, we can find an $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < \varepsilon\sqrt{x}$; and hence, because $\sqrt{x_n} + \sqrt{x_n} \geq \sqrt{x_n}$, we have

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{\varepsilon\sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \leq \frac{\varepsilon\sqrt{x}}{\sqrt{x}} = \varepsilon\ .$$

Therefore, $\sqrt{x_n} \to \sqrt{x}$.

2.3.3 Let $\varepsilon > 0$ be given. [We know we can make $|x_n - l|$ and $|z_n - l|$ small; how do we show that that forces $|y_n - l|$ to be small? We have $x_n - l \leq y_n - l \leq z_n - l$, but what inequalities hold with their absolute values? We don't know which of these quantities are positive and which are negative; but we can be sure the middle one is no further away from 0 than the larger of the top and the bottom; i.e., $|y_n - l| \leq \max\{|x_n - l|, |z_n - l|\}$.] Note that, because $x_n \leq y_n \leq z_n$, we have $x_n - l \leq y_n - l \leq z_n - l$, and hence $|y_n - l| \leq \max\{|x_n - l|, |z_n - l|\}$. Now there is an $N_x \in \mathbb{N}$ such that, if $n \geq N_x$, then $|x_n - l| < \varepsilon$; and there is an $N_z \in \mathbb{N}$ such that, if $n \geq N_z$, then $|z_n - l| < \varepsilon$. Let N be the larger of N_x and N_z; then if $n \geq N$, we have $|y_n - l| \leq \max\{|x_n - l|, |z_n - l|\} < \varepsilon$. Therefore, $\lim y_n = l$ also.

2.3.4 Assume BWOC that $l_1 \neq l_2$. [What happens when we pick ε so small that the ε-neighborhoods around l_1 and l_2 don't overlap? Then there is no place for the a_n's to be, because they are supposed to be in both neighborhoods. The largest ε for which there is no overlap is half the distance between the l's.] Note that $|l_1 - l_2| > 0$. Let $N_1 \in \mathbb{N}$ be such that, if $n \geq N_1$, then $|a_n - l_1| < \frac{1}{2}|l_1 - l_2|$; let $N_2 \in \mathbb{N}$ be such that, if $n \geq N_2$, then $|a_n - l_2| < \frac{1}{2}|l_1 - l_2|$; and let N be the larger of N_1 and N_2. Then for all $n \geq N$, we have, by the triangle inequality,

$$|l_1 - l_2| \leq |a_n - l_1| + |a_n - l_2| < \frac{1}{2}|l_1 - l_2| + \frac{1}{2}|l_1 - l_2| = |l_1 - l_2|\ .$$

We have just proved that a number is less than itself, so we have the desired contradiction.

2.3.6 (a) Let $\varepsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that, if $n \geq N$, then $|b_n - b| < \varepsilon$. Then if $n \geq N$, we saw in class that $||b_n| - |b|| \leq |b_n - b| < \varepsilon$. Therefore, $|b_n| \to |b|$.

(b) No: Let $b_n = (-1)^n$.

2.3.8 (a) $x_n = (-1)^n$, $y_n = (-1)^{n+1}$

(b) Impossible: $y_n = (x_n + y_n) - x_n$, so by the Algebraic Limit Theorem (2.3.3), if both $(x_n + y_n)$ and (x_n) both converge, then so does (y_n).

(c) $b_n = 1/n$

(d) Impossible: (b_n) is bounded by Theorem 2.3.2, and $(a_n - b_n)$ is bounded by hypothesis, so their sum (a_n) is bounded (by the sum of the bounds on the summands).

(e) $a_n = 1/n$ and $b_n = n$.

2.3.9 No, the statement of the Order Limit Theorem does not remain true with strict inequalities. For example, if (x_n) is convergent and $x_n > 0$ for all $n \in \mathbb{N}$, all we know is that the limit is greater than <u>or equal to</u> 0. Example: $x_n = 1/n$.

2.3.10 Let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ be such that, if $n \geq N$, then $|a_n - 0| \leq \varepsilon$. Now we are told that $|b_n - b| \leq a_n$, so $a_n \geq 0$, so $|a_n - 0| = a_n$. Hence, if $n \geq N$, then $|b_n - b| \leq a_n < \varepsilon$. Therefore, $(b_n) \to b$.

2.3.11 Denote the limit of (x_n) by x. Let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ be such that, if $n \geq N$, then $|x_n - x| < \varepsilon/2$, and then (by the Archimedean property) choose $N_1 > N$ such that $(|x_1 - x| + |x_2 - x| + \cdots + |x_N - x|)/N_1 < \varepsilon/2$. Then, using the triangle inequality and the fact that decreasing the denominator increases the fraction, we have for all $n \geq N_1$,

$$\begin{aligned}
|y_n - x| &= \left|\frac{x_1 + x_2 + \cdots + x_n}{n} - x\right| = \left|\frac{x_1 + x_2 + \cdots + x_n}{n} - \frac{nx}{n}\right| \\
&= \left|\frac{(x_1 - x) + (x_2 - x) + \cdots + (x_n - x)}{n}\right| \leq \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|}{n} \\
&= \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_N - x|}{n} + \frac{|x_{N+1} - x| + |x_2 - x| + \cdots + |x_n - x|}{n} \\
&\leq \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_N - x|}{N_1} + \frac{|x_{N+1} - x| + |x_2 - x| + \cdots + |x_n - x|}{n - N} \\
&< \frac{\varepsilon}{2} + \frac{(n - N)(\varepsilon/2)}{n - N} = \varepsilon\ .
\end{aligned}$$

Therefore, $(y_n) \to x$ also.

If $x_n = (-1)^n$, then the y_n's are alternately $-1/n$ and 0, so (y_n) converges (to 0) even though (x_n) does not converge.