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2.4.2 (a) [The first few terms in the sequence are 3, 1, $\frac{1}{3}$, which are clearly getting smaller; so we would like to show that the sequence is decreasing and bounded below, because then the Monotone Convergence Theorem would show that it has a limit. So we would like to show that $x_{n+1} < x_n$, i.e., $1/(4 - x_n) < x_n$. Now if $4 - x_n > 0$, then we could multiply or divide both sides by it without changing the sense of the inequality; and because the terms seem to be decreasing and start below 4, at 3, we shouldn't have any trouble with that. So, is it true that $x_n^2 - 4x_n + 1 < 0$? A standard trick is to complete the square, so let's try adding 3 to both sides: Is it true that $x_n^2 - 4x_n + 4 = (x_n - 2)^2 < 3$? It is true for $x_1 = 3$; in fact, it is true as long as $2 - \sqrt{3} < x_n < 2 + \sqrt{3}$; so we want to show that none of the x_n's is less that $2 - \sqrt{3}$. That will give us both the decreasing property and the lower bound. But will we be able to show that, if $x_n > 2 - \sqrt{3}$, then the same is true of x_{n+1}? Well, let's blunder ahead and see:] We claim that, for each n in $\mathbb{N}$, we have $2 - \sqrt{3} < x_{n+1} < x_n$: We have $x_1 = 3$ and $x_2 = 1$, so the claim is true for $n = 1$. Assume that it is true for x_n. Then because $2 - \sqrt{3} < x_n < x_{n-1} < \ldots < x_1 = 3 < 2 + \sqrt{3} < 4$, we have

$$\begin{aligned} x_n^2 - 4x_n + 4 = (x_n - 2)^2 &< 3 \\ 1 &< -x_n^2 + 4x_n = x_n(4 - x_n) \\ x_{n+1} = \frac{1}{4 - x_n} &< x_n \end{aligned}$$

It remains to show that $x_{n+1} > 2 - \sqrt{3}$: We have $x_n > 2 - \sqrt{3}$, so $4 - x_n < 4 - (2 - \sqrt{3}) = 2 + \sqrt{3}$, so

$$x_{n+1} = \frac{1}{4 - x_n} > \frac{1}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{(2 + \sqrt{3})(2 - \sqrt{3})} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}\ .$$

[Wow! It worked!] So the sequence $(x_n)_{n=1}^{\infty}$ is decreasing and bounded below by $2 - \sqrt{3}$, so by the Monotone Convergence Theorem, it converges to a limit.

(b) To say that the x_n's converge to a limit L is to say that for all sufficiently large n, the x_n's are close to L; so the x_{n+1}'s are also close to the same L.

(c) Letting $n \to \infty$ and letting L denote the limit of the x_n's (and hence of the x_{n+1}'s), the defining equation becomes $L = 1/(4 - L)$, i.e., $-L^2 + 4L - 1 = 0$, and using the quadratic formula to solve for L, we get $L = (-4 \pm \sqrt{16 - 4(-1)(-1)})/(2(-1)) = 2 \pm \sqrt{3}$. [So that's where that bound in (a) came from!] Because the x_n's are decreasing from 3, they cannot approach $2 + \sqrt{3}$; so the limit must be $2 - \sqrt{3}$. [An interesting question might be: If x_1 had been given a different value, could the limit have been $2 + \sqrt{3}$?]

2.4.4 The sequence is defined ("recursively" or "by induction") as follows: Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$. Thus, if $x_n < 2$, then $x_{n+1} = \sqrt{2x_n} = \sqrt{2}\sqrt{x_n} > \sqrt{x_n}\sqrt{x_n} = x_n$, so the sequence is increasing, and $x_{n+1} = \sqrt{2x_n} = \sqrt{2}\sqrt{x_n} < \sqrt{2}\sqrt{2} = 2$; so 2 is an upper bound for the sequence (by an induction argument, if necessary). Therefore by the Monotone Convergence Theorem, the sequence has a limit. And by passing to the limit L as in Exercise 2.4.2, we get $L = 2^{1/2}L^{1/2}$, so by algebra $L = 2$.

2.4.5 (a) Note first that if $x_n > 0$, then it is clear from the definition that $x_{n+1} > 0$ also; so all the x_n's are positive. We will use this below. Now we proceed by induction to show $x_n > \sqrt{2}$: Clearly $x_1 = 2 > \sqrt{2}$, so assume $x_n > \sqrt{2}$. Then $x_n^2 - 2\sqrt{2}x_n + 2 = (x_n - \sqrt{2})^2 > 0$, so $x_n^2 + 2 > 2\sqrt{2}x_n$, and dividing both sides by the positive number $2x_n$ shows that $x_{n+1} = \frac{1}{2}(x_n + 2/x_n) > \sqrt{2}$; and the induction is complete. Thus,

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2x_n}(x_n^2 - 2) > 0\ .$$

We conclude that the sequence (x_n) is decreasing, and it is bounded below by 0, so it converges. Moreover, because the differences $x_n - x_{n+1}$ must approach 0 (because the sequence is convergent), it follows that the numerator in the last display, $x_n^2 - 2$, also approaches 0 (because the denominator, $2x_n$, is bounded). Thus, x_n approaches $\sqrt{2}$.

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2.5.3 (a) One example: $(\frac{1}{2} + \frac{1}{2}(-1)^n(\frac{1}{n}))$

(b) Impossible. If the sequence is monotone — say increasing — and divergent, then it cannot be bounded, i.e., the terms are "frequently" (to use the terminology of Exercise 2.2.8) larger than any given bound; but because it is increasing, the terms are "eventually" <u>all</u> larger than any given bound; so they cannot converge (to anything except ∞).

(c) This one is tricky; here's one way to do it: Consider the array

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then form a sequence by starting at the upper left and repeatedly follwing longer and longer upper right to lower-left diagonals:

$$(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots) \; .$$

(d) One simple example is $(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \ldots)$

(e) Impossible: The bounded subsequence has a convergent subsequence, which is also a subsequence of the original sequence.