Page 61, Exercises 2.6

2.6.1 (a) $((-1)^n(1/n))$.

(b) (n).

(c) Impossible: A Cauchy sequence (in $\mathbb{R}$) is convergent, so all of its subsequences are also convergent (to the same limit).

(d) $(n-(-1)^n n) = (2, 0, 6, 0, 10, 0, 14, 0, \ldots)$

2.6.3 (a) Pseudo-Cauchy apparently only requires that each pair of successive terms be close together (with "close" being specified by some positive ε, as usual), not that all the terms after some point be close together.

(b) The sequence of partial sums of the harmonic series is pseudo-Cauchy but divergent: Let $s_n = \sum_{k=1}^{n}(1/k)$; then $|s_{n+1} - s_n| = 1/(n+1)$, so the differences between successive terms can be made as small as desired; but we know that the sequence diverges because the harmonic series diverges.

2.6.4 Let $\varepsilon > 0$ be given, and pick $N \in \mathbb{N}$ such that, for all $m, n \geq N$, $|a_m - a_n| < \varepsilon/2$ and $|b_m - b_n| < \varepsilon/2$. Then for all $m, n \geq N$, we have:

$$\begin{aligned} |c_m - c_n| &= ||a_m - b_m| - |a_n - b_n|| \leq |(a_m - b_m) - (a_n - b_n)| \\ &= |(a_m - a_n) - (b_m - b_n)| \leq |a_m - a_n| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ . \end{aligned}$$

Thus, (c_n) is Cauchy.

Page 68, Exercises 2.7

2.7.4 $x_n = y_n = 1/n$ works.

2.7.5 (a) Because $\sum |a_n|$ converges, $|a_n|$ has limit 0, so $|a_n| < 1$ for n sufficiently large. Thus, $0 < a_n^2 = |a_n|^2 \leq |a_n|$ for all such N, and hence by the Comparison Test, $\sum a_n^2$ converges because $\sum |a_n|$ converges.

The result does not hold if we assume only that $\sum a_n$ converges. A counterexample is the series $\sum(-1)^{n-1}(1/\sqrt{n})$: This series converges by the Alternating Series Test (the terms alternate in sign, and their absolute values decrease to 0), but the sum of squares of the terms is the harmonic series $\sum(1/n)$, which diverges.

(b) Knowing only that $\sum a_n$ converges and that $a_n \geq 0$ for all n, we cannot conclude either that $\sum \sqrt{a_n}$ converges or that it diverges. Examples are given by $a_n = 1/n^4$ (for which $\sum \sqrt{a_n}$ converges) and $a_n = 1/n^2$ (for which $\sum \sqrt{a_n}$ diverges).

2.7.9 (a) Because we are assuming that $r = \lim |a_{n+1}/a_n| < 1$, there is a real number r' strictly between r and 1; for instance, $r' = (r+1)/2$ works. Let $\varepsilon = r' - r > 0$. Then there is an $N \in \mathbb{N}$ such that, for all $n \geq N$, $||a_{n+1}/a_n| - r| < \varepsilon$, i.e., $r - \varepsilon < |a_{n+1}/a_n| < r + \varepsilon = r + (r' - r) = r'$. Dropping the first inequality (which we don't need) and multiplying the ends of the remaining string of relations by the positive number $|a_n|$, we get $|a_{n+1}| < |a_n|r'$ for all $n \geq N$.

(b) This series converges because it is geometric with common ration r' which is (in absolute value) less than 1.

(c) With N as in part (a), we have, for all $n \geq N$,

$$|a_n| < |a_{n-1}|r' < |a_{n-2}|(r')^2 < \cdots < |a_N|(r')^{n-N} = (|a_N|/(r')^N)(r')^n\ .$$

Because $\sum(|a_N|/(r')^N)(r')^n$ is a geometric series with common ration r', it converges, so $\sum |a_n|$ also converges, by the Comparison Test.

2.7.10 (a) Choose ℓ' strictly between ℓ and 0; then for all n sufficiently large, $a_n > \ell'/n$; because $\ell' \sum(1/n)$ diverges, so does $\sum a_n$, by the Comparison Test.

(b) Choose r strictly greater than $\lim(n^2 a_n)$. Then for n sufficiently large, $n^2 a_n < r$, so $0 < a_n < r/n^2$. Because $r \sum(1/n^2)$ converges, the Comparison Test shows that $\sum a_n$ converges also.