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3.2.1 (a) The finiteness of the family of open sets is used when the ε is chosen to be the minimum of the set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$. If the family were infinite, then the set of ε's would be infinite as well, and the minimum might not exist; the infimum might be 0.

(b) $O_n = (-\frac{n+1}{n}, \frac{n+1}{n})$; the intersection of these sets is $[-1, 1]$.

3.2.2 (a) The limit points are 1 and -1.

(b) No, B is not closed, because it does not contain its limit (or accumulation) points -1 and 1.

(c) No, B is not open, because it does not contain any intervals, much less one around every point in it.

(d) All of the points of B are isolated points.

(e) $\overline{B} = B \cup \{-1, 1\}$.

3.2.3 (a) $\mathbb{Q}$ is neither open (it contains no intervals at all, much less an interval around, say, 0) nor closed (any irrational number is a limit (or accumulation) point of $\mathbb{Q}$ that is not in $\mathbb{Q}$).

(b) $\mathbb{N}$ is closed but not open, because it includes no intervals.

(c) $\{x \in \mathbb{R} : x > 0\} = (0, \infty)$ is open but not closed: 0 is a limit (or accumulation) point of the set that is not in it.

(d) $(0, 1]$ is neither open (1 has no interval around it that is in the set) nor open (0 is a limit point that is not in it).

(e) This is the set of partial sums in the convergent series $\sum(1/n^2)$; it is not open because it contains no intervals, and it is not closed because its limit, i.e., the sum of the series, is not one of the numbers in the set.

3.2.7 Because $x \in O$, there is an $\varepsilon > 0$ for which $V_\varepsilon(x) \subseteq O$. Now by the definition of convergence, there is an $N \in \mathbb{N}$ such that, for all $n > N$, $|x_n - x| < \varepsilon$, i.e., $x_n \in V_\varepsilon(x) \subseteq O$. Thus, the only x_n's that might not be in O are the finitely many with $n < N$.

3.2.8 This problem was not assigned; I include the solution because it is a useful method.

(a) Let x be a limit (or accumulation) point of L, and let (x_n) be a sequence in $L\backslash\{x\}$ that converges to x. Now by hypothesis each x_n is a limit of a sequence $(x_{n,k})_{k=1}^{\infty}$ of elements of $A\backslash\{x_n\}$. We may assume these sequences are in $A\backslash\{x\}$, because each of them has infinitely many distinct terms, and we can drop out any occurrences of x without affecting the limit. For each $n \in \mathbb{N}$, pick $k_n \in \mathbb{N}$ such that $k > k_n$ implies $|x_{n,k} - x_n| < 1/(2n)$. We claim that the "diagonal" sequence $(x_{n,k_n})_{N=1}^{\infty}$ has limit x; if we do, then we will have shown x is a limit point of A, so it is in L, and it will follow that L is closed. Let $\varepsilon > 0$ be given, and pick $N \in \mathbb{N}$ such that $1/N < \varepsilon$ and for all $n \geq N$, $|x_n - x| < \varepsilon/2$. Then for all $n \geq N$ we have

$$|x_{n,k_n} - x| \leq |x_{n,k_n} - x_n| + |x_n - x| < \frac{1}{2n} + \frac{\varepsilon}{2} \leq \frac{1}{2N} + \frac{\varepsilon}{2} < \varepsilon\ .$$

Therefore, x is a limit of a sequence in $A\backslash\{x\}$, so it is in L.

(b) The proof just given shows that, if x is a limit point of L, then it is also a limit point of A. And in general (see the next exercise) a limit point of the union of two sets is a limit point of at least one of them. (For, a sequence in $(A \cup B)\backslash\{x\}$ that converges to x has infinitely many of its terms in at least one of $A\backslash\{x\}$ or $B\backslash\{x\}$, and so at least one of them has a sequence converging to x.) So if x is a limit point of $A \cup L$, then it is a limit point of A. Therefore, it is in L, so $A \cup L$ contains all of its limit points, so it is closed. On the other hand, any closed set that contains A also contains all of the limit points of A, so it contains L as well. Thus, $A \cup L$ is contained in any closed set containing A, so $A \cup L$ is the smallest closed set containing A.

3.2.9 (a) A sequence in $(A \cup B)\backslash\{y\}$ that converges to y has infinitely many of its terms in at least one of $A\backslash\{y\}$ or $B\backslash\{y\}$, so at least one of them contains a sequence converging to y.

(b) $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. On the other hand, $\overline{A \cup B}$ is a closed set containing A and B, so it also contains $\overline{A}$ and $\overline{B}$ and hence also their union. Therefore $\overline{A \cup B} = \overline{A} \cup \overline{B}$. (Another way to see that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$: By (a) any limit point of $A \cup B$ is in either $\overline{A}$ or $\overline{B}$.)

(c) No: Let $F_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$. Then

$$\bigcup_{n=1}^{\infty} \overline{F_n} = \bigcup_{n=1}^{\infty} F_n = (-1, 1) \qquad \text{but} \qquad \overline{\bigcup_{n=1}^{\infty} F_n} = \overline{(-1, 1)} = [-1, 1] \ .$$

3.2.12 (a) True: The complement of a closed set is open.

(b) True: That isolated point has no ε-neighborhood contained in the set.

(c) True: If A is closed, then A is surely the smallest closed set containing A. And conversely, $\overline{A}$ is closed, so if $A = \overline{A}$, then A is closed.

(d) False, using the book's definition of limit (accumulation) point, because $\sup A$ may be an isolated point of A.

(e) True: Its set of limit (accumulation) points is empty, so its closure is itself.

(f) False: $\mathbb{R}\backslash\{\pi\}$ is open and contains every rational.

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3.3.1 Because K is bounded, it has both a supremum and an infimum, which are either in K or limit points of K — and hence, because K is closed, they are still in K.

3.3.3 The Cantor set is bounded: above by 1 and below by 0. And it is closed, because its complement is a union of open intervals. So it is compact.

3.3.4 $K \cap F$ is closed because the intersection of closed sets is closed; and it is bounded by the same bounds as K. So it is compact.

3.3.5 (a) Not compact: Any sequence of rationals that converges to an irrational (say $\sqrt{2}$) works.

(b) Again, not compact, but the irrational limit must be between 0 and 1. How about $\sqrt{2}/2$?

(c) Not compact: (n) is a sequence with no convergent subsequence.

(d) This is just $[0, 1]$, which is compact.

(e) Not compact, because the limit, 0, of $(1/n)$ is not in the set.

(f) Compact: Most of the elements are the terms of a sequence that approaches 1, so the only limit point of the set is 1, which is already in the set.

3.3.7 (a) True: It is closed because any intersection of closed sets is closed; and it is bounded by the bounds on any of the sets that go into the intersection.

(b) False: $\mathbb{Q} \cap [0, 1]$ is not compact.

(c) False: $F_n = [n, \infty)$.

(d) True: It is closed and bounded. (Or, any open cover has a finite subcover: If $\mathcal{U}$ is an open cover of $\{x_1, x_2, \ldots, x_n\}$, then for each x_i pick an element U_i in $\mathcal{U}$ that contains it; then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover.)

(e) False: Neither $\mathbb{N}$ nor $\mathbb{Q}$ is compact.

3.3.9 (a) $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where $U_n = \mathbb{R} \backslash [(\sqrt{2}/2) - (1/n), (\sqrt{2}/2) + (1/n)]$.

(b) Same as (a).

(c) $\{(-n, n) : n \in \mathbb{N}\}$.

(e) $\{(1/n, 2) : n \in \mathbb{N}\}$.