

**Page 108, Exercises 4.2**

- 4.2.1 (a) (We remark first that the function  $2x+4$  is defined on all of  $\mathbb{R}$ , so its domain is  $\mathbb{R}$ ; and that 2 is an accumulation point of  $\mathbb{R}$ . Normally we don't bother to check or mention this, but because this is the first limit proof, I thought I should.) Let  $\varepsilon > 0$  be given, and let  $\delta = \varepsilon/2$ . Then  $0 < |x-2| < \delta$  (and  $x \in \mathbb{R}$ , of course) implies  $|(2x+4) - 8| = |2x-4| = 2|x-2| < \varepsilon$ ; so  $\lim_{x \rightarrow 2} (2x+4) = 8$ .
- 4.2.6 Let  $\varepsilon > 0$  be given, and pick  $\delta > 0$  so that  $0 < |x-c| < \delta$  and  $x \in A$  implies  $|g(x)| = |g(x) - 0| < \varepsilon/M$ . Then  $0 < |x-c| < \delta$  and  $x \in A$  implies  $|g(x)f(x) - 0| = |g(x)||f(x)| < (\varepsilon/M)M = \varepsilon$ ; so  $\lim_{x \rightarrow c} (g(x)f(x)) = 0$ .
- 4.2.9 Let  $\varepsilon > 0$  be given, and pick  $\delta > 0$  so that  $0 < |x-c| < \delta$  and  $x \in A$  implies  $|f(x) - L| < \varepsilon$  and  $|h(x) - L| < \varepsilon$ . Then because  $f(x) \leq g(x) \leq h(x)$ , we have  $f(x) - L \leq g(x) - L \leq h(x) - L$ , so  $|g(x) - L| \leq \max\{|f(x) - L|, |h(x) - L|\} < \varepsilon$ ; so  $\lim_{x \rightarrow c} g(x) = L$ . [Note that, just as in the sequence case, we cannot simply appeal to some "Order Limit Theorem" — which the book does not include anyway, at least not in the function limit chapter — because it would have the hypothesis that  $\lim_{x \rightarrow c} g(x)$  exists, and our problem does not provide that hypothesis.]

**Page 113, Exercises 4.3**

- 4.3.2 (a) Let  $\varepsilon > 0$  be given. Because  $f(c) \in B$  and  $g$  is continuous at  $f(c)$ , we can pick  $\delta > 0$  so that, if  $|y - f(c)| < \delta$  and  $y \in B$ , then  $|g(y) - g(f(c))| < \varepsilon$ ; and because  $f$  is continuous at  $c$ , we can pick  $\gamma > 0$  so that, if  $|x - c| < \gamma$  and  $x \in A$ , then  $|f(x) - f(c)| < \delta$ . Then if  $|x - c| < \gamma$  and  $x \in A$ , we have  $|g(f(x)) - g(f(c))| < \varepsilon$ . Therefore,  $g \circ f$  is continuous at  $c$ .
- (b) Take any sequence  $(x_n)$  in  $A$  that converges to  $c$ . Then by the continuity of  $f$  at  $c$ , the sequence  $(f(x_n))$  converges to  $f(c)$ ; and so by the continuity of  $g$  at  $f(c)$ , the sequence  $(g(f(x_n)))$  converges to  $g(f(c))$ . Therefore,  $g \circ f$  is continuous at  $c$ .
- 4.3.3 Take any  $c$  in  $\mathbb{R}$ , and let  $\varepsilon > 0$  be given. Pick  $\delta$  to be 1 (or any other positive number) if  $a = 0$ , and otherwise pick  $\delta = \varepsilon/|a|$ . Then if  $|x - c| < \delta$ , we have  $|(ax+b) - (ac+b)| = |a||x-c| < \varepsilon$ . Therefore, the given function is continuous at  $c$ .
- 4.3.5 Because  $g$  is continuous at  $c$  and  $g(c) \neq 0$ , we can take  $\varepsilon = |g(c)|$  and find a  $\delta > 0$  so that if  $|x - c| < \delta$  (and  $x \in A$ ), we have  $|g(x) - g(c)| < |g(c)|$ . Thus, for all  $x$  in  $V_\delta(c)$  (and also in  $A$ , which are the only  $x$ 's we care about), we have  $|g(c) - |g(x)|| < |g(c)|$ , so  $|g(x)| > 0$ , i.e.,  $g(x) \neq 0$ ; and hence  $f/g$  is defined on  $V_\delta(c) \cap A$ .
- 4.3.9 (a) Take  $x \in \mathbb{R}$ , and let  $\varepsilon > 0$  be given. Take  $\delta = \varepsilon/c$ ; then  $|x - y| < \delta$  (and  $y \in \mathbb{R}$ , of course) implies  $|f(x) - f(y)| \leq c|x - y| < \varepsilon$ . So  $f$  is continuous at  $x$ .
- (b)  $|y_{n+1} - y_n| = |f(y_n) - f(y_{n-1})| \leq c|y_n - y_{n-1}|$ , so (by induction, if necessary)  $0 < |y_{n+1} - y_n| \leq c^{n-1}|y_2 - y_1|$ . Thus, for  $m > n$  we have:

$$\begin{aligned} |y_m - y_n| &\leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + \cdots + |y_{n+1} - y_n| \\ &\leq (c^{m-2} + c^{m-3} + \cdots + c^{n-1})|y_2 - y_1| \\ &\leq |y_2 - y_1|(c^{n-1} + c^{n-2} + \cdots) \\ &= |y_2 - y_1|(c^{n-1}/(1-c)) , \end{aligned}$$

Because  $|y_2 - y_1|(1/(1-c))$  is constant and  $0 < c < 1$ , the last expression is smaller than any given  $\varepsilon > 0$  for sufficiently large  $n$ , so the sequence is Cauchy.

- (c) Assume BWOC that  $f(y) \neq y$ , and take  $N \in \mathbb{N}$  so that, for  $n \geq N$ , we have  $|y_n - y| < \frac{1}{2}|f(y) - y|$ . Then

$$|f(y) - y| \leq |f(y) - f(y_N)| + |f(y_N) - y| \leq c|y - y_N| + |y_{N+1} - y| < |f(y) - y| .$$

This contradiction shows that  $y$  is a fixed point of  $f$ . And if we assume that  $x$  is a different fixed point of  $f$ , then we have

$$|x - y| = |f(x) - f(y)| \leq c|x - y| < |x - y| ,$$

another contradiction; so  $y$  is the only fixed point of  $f$ .

- (d) The result of (b) and (c) is that any such sequence converges to a fixed point of  $f$ , and that there is only one fixed point of  $f$ , so all such sequences converge to the same point.