Math 329 Unit 4 Fall 2014

Due: Wednesday, October 22

1. Derive the five point central difference scheme for the first derivative of a function. Derive the five point scheme for the first derivative at a point on the (left) boundary of the known points. Using symmetry, argue for what the five point scheme is for the first derivative on the right boundary. Find a five point scheme for the first derivative of a function one step away from the (left) boundary. You probably will want to set up these problems by hand, but use a Matlab script to solve the systems.

2. Create a script that will make a table of step size, estimate and error for computing the standard (3-point) central differencing scheme for the derivative of the function $f(x) = x^3 * \sin(x)$ at the point $x = \frac{\pi}{2}$. The step sizes should start at $h = 0.1$ and decrease by a factor of 10 until $h = 10^{-18}$. For each step size, compute the approximate derivative and compare to the actual derivative's value to get the error. Printing the table can be done with code something like:

   ```
   disp(sprintf('h=%18.12e Deriv=%18.12e Error=%18.12e',h,deriv,err))
   ```

   Display in your script (or write by hand and hand in) the answer to these questions:

   What power of h does the truncation error appear to be for this scheme? At what point does the error start to level off or increase? Truncation error gets small as h gets small. What kind of error is interfering? What is the best value of h to use?

3. Now we will evaluate the integral

   $$\int_0^{\pi} \sin(x)dx$$

   for a specified step size h using the (composite) trapezoidal rule. Write some Matlab code that creates a table of estimates and errors by step size for $h = \frac{\pi}{4}, \frac{\pi}{8}, \ldots \frac{\pi}{2^{25}}$. Use a loop to evaluate the integral for each of these step sizes, comparing the estimate to the actual answer. The printing line can be something like:

   ```
   disp(sprintf('h=%18.12e I=%18.12e Error=%18.12e',h,I,err))
   ```

   Display in your script (or write by hand and hand in) the answer to these questions:

   What power of h does the error appear to be for this scheme? At what point does the error start to level off or increase? Truncation error (by design) gets small as h gets small. What kind of error is interfering? Compare and contrast to the similar table for derivatives.

4. Write a Matlab function (romberg.m) to calculate $\int_a^b f(t)dt$ using the trapezoidal rule and Romberg adaptive step size. Inputs should be f, a, b, and an error tolerance level as well as optional arguments for the minimum number of intervals and for whether to print intermediate calculations. Use the Romberg integration algorithm that adaptively doubles the number of points until the estimate of the integral changes by less than the tolerance level. Once the estimate of the error is less than your tolerance, use Richardson's Extrapolation to return the best answer combining the last two estimates you got.

   Make sure to use comments efficiently including a set of comments just after the function command that will be displayed by the "help" facility. Include as an optional argument a minimum number of intervals to use (with a small default like four). This will be useful for integrals over large domains where the integrand gets small for large values of x. Include an option to print each approximation along with its step size. Make the default to print these approximations so you can see what is happening. Something like:

   ```
   disp(sprintf('%2i: h=%18.12e N=%6i I=%18.12e',count,h,N,I))
   ```

   Write a test script to test your function initially by integrating something you can do by hand (like $f(t) = t$). Include a second test computing the integral $\int_0^1 e^{-t^2} dt$.

5. Use your romberg routine and the bisection routine you wrote in unit 1 to solve for the points where $\frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt = \frac{1}{4}$. (For those of you who have had statistics, the answer is the 75^{th} percentile of the standard normal distribution.) Have printing options turned on for both the integration routine and the bisection routine.

6. The Gamma function is a generalization of the factorial function.

   $$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

   Show numerically that $\Gamma(6) = 5!$ to six decimal places. Note: This integrates over an infinite domain. We can't do that, so use your romberg.m function on finite domains [0,1]; [0,10]; [0,100]; [0,1000]. With this info you can get an idea for the size of the error due to cutting off the right tail of the domain over which we integrate. (Be careful though. If your domain is too big, the integrand is very small and using 2 intervals will show no difference from using 4 intervals. The code may stop due to lack of change simply because all points are in the region where f is very small. If so, increase the minimum number of intervals to use.)

7. It can be shown that $\Gamma(n) = (n-1)!$ for integers $n > 0$. But what about factorials of fractions? Find $\frac{1}{2}!$ accurate to 6 decimal places. ($\frac{1}{2}! \equiv \Gamma(\frac{3}{2})$). Check that it is $\frac{1}{2}\sqrt{\pi}$. What is the error? (You probably never thought that π was related to factorials.)