HW #3 Solutions (Math 323)

8.1) a) Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Then $n > N$ (i.e., $\frac{1}{n} < \epsilon$) implies that $\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \epsilon$.

b) Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon^3}$. Then $n > N$ (i.e., $\frac{1}{n^{1/3}} < \epsilon$) implies that $\left|\frac{1}{n^{1/3}} - 0\right| = \frac{1}{n^{1/3}} < \epsilon$.

c) Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Then $n > N$ (i.e., $\frac{1}{n} < \epsilon$) implies that $\left|\frac{2n-1}{3n-2} - \frac{2}{3}\right| = \left|\frac{3(2n-1) - (3n-2)(2)}{3(3n-2)}\right| = \frac{1}{3(3n-2)} \leq \frac{1}{3n-2} \leq \frac{1}{n} < \epsilon$.

d) Let $\epsilon > 0$. Let $N = \max(6, \frac{4}{\epsilon})$. Then $n > N$ (i.e., $\frac{4}{n} < \epsilon$) gives $\left|\frac{n+6}{n^2-6} - 0\right| \underset{(n\geq 6)}{\leq} \frac{2n}{n^2-6} \underset{(n\geq 4)}{\leq} \frac{2n}{n^2/2} = \frac{4}{n} < \epsilon$.

8.2) a) The limit is 0: Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Then $n > N$ (i.e., $\frac{1}{n} < \epsilon$) implies that $\left|\frac{n}{n^2+1} - 0\right| \leq \frac{n}{n^2} = \frac{1}{n} < \epsilon$.

b) The limit is $\frac{7}{3}$: Let $N = \frac{12}{\epsilon}$. Then $n > N$ (i.e., $\frac{12}{n} < \epsilon$) implies that $\left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| = \frac{106}{3(3n+7)} \leq \frac{106}{9n} \leq \frac{12}{n} < \epsilon$.

c) The limit is $\frac{4}{7}$. Let $N = \max(5, \frac{1}{\epsilon})$. Then $n > N$ gives $\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| = \frac{41}{7(7n-5)} \underset{(n\geq 5)}{\leq} \frac{41}{7(7n-n)} \leq \frac{1}{n} < \epsilon$.

d) The limit is $\frac{2}{5}$: Let $N = \frac{1}{\epsilon}$. Then $n > N$ (i.e., $\frac{1}{n} < \epsilon$) implies that $\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| = \frac{16}{5(5n+2)} \leq \frac{16}{25n} \leq \frac{1}{n} < \epsilon$.

e) The limit is 0: Let $N = \frac{1}{\epsilon}$. Then $n > N$ (i.e., $\frac{1}{n} < \epsilon$) implies that $\left|\frac{\sin(n)}{n} - 0\right| = \frac{|\sin(n)|}{n} \leq \frac{1}{n} < \epsilon$.

8.3) We have $(s_n) \to 0$ and will show $(\sqrt{s_n}) \to 0$. Given $\epsilon > 0$, there exists N such that for all $n > N$, $|s_n| = s_n < \epsilon$. Let ϵ be given and choose $\hat{N}$ so that for all $n > \hat{N}$ we have $s_n < \epsilon^2$. This gives us $\sqrt{s_n} < \epsilon$ as needed.

8.4) Find N such that $\forall n > N$ we have $|s_n| < \frac{\epsilon}{M}$. Then $|s_n t_n - 0| = |s_n t_n| \leq M|s_n| \leq M \cdot \frac{\epsilon}{M} = \epsilon$.

8.5) a) Let $\epsilon > 0$. Let N_1 be such that for all $n > N_1$ we have $|a_n - s| < \epsilon$. Let N_2 be such that for all $n > N_2$ we have $|b_n - s| < \epsilon$. Let $N = \max(N_1, N_2)$. Then, for all $n > N$, we have $a_n, b_n \in (s - \epsilon, s + \epsilon)$. Hence, $s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon$ so that $|s_n - s| < \epsilon$ for all $n > N$.

b) Apply part (a) with $a_n = -t_n$ and $b_n = t_n$.

8.6) a) If $s_n \to 0$, then for $\epsilon > 0$ we have $|s_n - 0| = |s_n| < \epsilon$ for all $n > N$. Consider $||s_n| - 0| = ||s_n|| = |s_n|$, which is less than ϵ for all $n > N$. Now assume, $|s_n| \to 0$. We want to show that $s_n \to 0$. We have $-|s_n| \leq s_n \leq |s_n|$, so by exercise 8.5a, we have $s_n \to 0$.

b) So part (a) only holds if the limit is 0, not just any real number.

8.7) a) For n a multiple of 3, the sequence value at n is either 1 or -1. Thus, for any possible limit s we have $|s_n - s| \geq 1$ for infinitely many values of n.

b) Assume this has a limit, say $s \in \mathbb{R}$. Consider $|(-1)^n n - s| \geq |(-1)^n n| - |s| = n - |s|$. Now let $N = |s| + 1$. Then, for all $n > N$ we have $|(-1)^n n - s| \geq n - |s| > |s| + 1 - |s| = 1$.

c) For n a multiple of 2, the sequence value at n is either $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$. Thus, for any possible limit s we have $|s_n - s| \geq \frac{\sqrt{3}}{2}$ for infinitely many values of n.

8.8) a) Let $\epsilon > 0$. Let $N = \frac{1}{2\epsilon}$. Then $n > N$ (i.e., $\frac{1}{2n} < \epsilon$) implies that $\left|\sqrt{n^2+1} - n - 0\right| = \frac{1}{\sqrt{n^2+1}+n} \leq \frac{1}{2n} < \epsilon$.

b) Let $\epsilon > 0$. We may assume $\epsilon < \frac{1}{2}$ by the following argument. If $\epsilon \geq \frac{1}{2}$, we need only find N such that for all

$n > N$ we have $\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| < \frac{1}{2}$. We have

$$\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| = \left|\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2}\right| = \left|\frac{n-\sqrt{n^2+n}}{2(\sqrt{n^2+n}+n)}\right| = \frac{\sqrt{1+1/n}-1}{2(\sqrt{1+1/n}+1)} < \frac{\sqrt{2}-1}{4} < \frac{1}{2}.$$

So, let $\epsilon < \frac{1}{2}$. Let $N = \left(\left(\frac{1+2\epsilon}{1-2\epsilon}\right)^2 - 1\right)^{-1}$. Then $n > N$ implies that $\frac{1}{n} < \left(\frac{1+2\epsilon}{1-2\epsilon}\right)^2 - 1$ so that $\sqrt{1+1/n} < \left(\frac{1+2\epsilon}{1-2\epsilon}\right)$. With some algebra this gives us $\frac{\sqrt{1+1/n}-1}{2(\sqrt{1+1/n}+1)} < \epsilon$. Hence, we have $\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| = \frac{\sqrt{1+1/n}-1}{2(\sqrt{1+1/n}+1)} < \epsilon$.

c) Let $\epsilon > 0$. We may assume $\epsilon < \frac{1}{4}$ by the following argument. If $\epsilon \geq \frac{1}{4}$, we need only find N such that for all $n > N$ we have $\left|\sqrt{4n^2+n} - 2n - \frac{1}{4}\right| < \frac{1}{4}$. We have

$$\left|\sqrt{4n^2+n} - 2n - \frac{1}{4}\right| = \left|\frac{n}{\sqrt{4n^2+n}+2n} - \frac{1}{4}\right| = \left|\frac{2n-\sqrt{4n^2+n}}{4(\sqrt{4n^2+n}+2n)}\right| = \frac{\sqrt{1+1/4n}-1}{4(\sqrt{1+1/4n}+1)} < \frac{\sqrt{2}-1}{4} < \frac{1}{4}.$$

So, let $\epsilon < \frac{1}{4}$. Let $N = \frac{1}{4}\left(\left(\frac{1+4\epsilon}{1-4\epsilon}\right)^2 - 1\right)^{-1}$. Then $n > N$ gives $\frac{1}{4n} < \left(\frac{1+4\epsilon}{1-4\epsilon}\right)^2 - 1$ so that $\sqrt{1+1/4n} < \left(\frac{1+4\epsilon}{1-4\epsilon}\right)$. With some algebra this gives us $\frac{\sqrt{1+1/4n}-1}{4(\sqrt{1+1/4n}+1)} < \epsilon$. Hence, we have $\left|\sqrt{4n^2+n} - 2n - \frac{1}{4}\right| = \frac{\sqrt{1+1/4n}-1}{4(\sqrt{1+1/4n}+1)} < \epsilon$.

8.9) a) Assume, for a contradiction, that $\lim s_n = s < a$. Let $\epsilon = \frac{a-s}{2}$. Then, there exists N so that $n > N \Rightarrow |s_n - s| < \epsilon = \frac{a-s}{2}$. Hence, for all $n > N$ we have $s_n < a$, i.e., for all but finitely many n, $s_n < a$, a contradiction.

b) Assume, for a contradiction, that $\lim s_n = s > b$. Let $\epsilon = \frac{s-b}{2}$. Then, there exists N so that $n > N$ implies $|s_n - s| < \epsilon = \frac{s-b}{2}$. Hence, for all $n > N$ we have $s_n > b$, i.e., for all but finitely many n, $s_n > b$, a contradiction.

c) This is immediate from (a) and (b).

8.10) Let $s_n \to s > a$. Let $\epsilon = \frac{s-a}{2}$. Then, $\exists N$ so that $n > N \Rightarrow |s_n - s| < \epsilon$. By choice of ϵ, this gives us $s_n > a$.

9.1) a) We have $\frac{n+1}{n} = 1 + \frac{1}{n}$. Since the limit of a sum is the sum of limits, and $\frac{1}{n} \to 0$, we are done.

b) We have $\frac{3+7/n}{6-5/n}$. Use $3 + 7/n \to 3$, $6 - 5/n \to 6$, and the limit of a quotient is the quotient of the limits.

c) Divide through by n^5: $\frac{17+73/n-18/n^3+3/n^5}{23+13/n^2}$. Use numerator $\to 17$, denominator $\to 23$, and Theorem 9.6.

9.2) a) 10 b) $\frac{18}{49}$

9.3) Since $a_n \to a$ we have $(a_n)^3 \to a^3$ and $4a_n \to 4a$. Hence, since limit of a sum is the sum of limits (when the sequences are convergent) we have $(a_n)^3 + 4a_n \to a^3 + 4a$. Similarly, $(b_n)^2 + 1 \to b^2 + 1$. Since the numerator and denominator both converge, the limit is the quotient of the limits and we are done.

9.4) a) $1, \sqrt{2}, \sqrt{\sqrt{2}+1}, \sqrt{\sqrt{\sqrt{2}+1}+1}$

b) Let $L = \lim s_n$. Let we have $\lim s_{n+1} = \lim \sqrt{s_n + 1}$. Since we are assuming s_n converges, $\lim \sqrt{s_n+1} = \sqrt{\lim(s_n) + 1} = \sqrt{L+1}$. Hence, $L = \sqrt{L+1}$, i.e., $L^2 - L - 1$. Using the quadratic formula gives us $\frac{1\pm\sqrt{5}}{3}$. We disregard $\frac{1-\sqrt{5}}{2}$ since it is negative and we clearly have a positive limit (assuming we have a limit).

9.5) The justification for the following is very similar to that in Exercise 9.4b. Let t be the limit. Then $t = \frac{t^2+2}{2t}$ so that $2t^2 = t^2 + 2$, i.e., $t^2 = 2$. Hence, $t = \sqrt{2}$.

9.6) Under this assumption, we would have $a = 3a^2$ so that $a = 0$ or $a = \frac{1}{3}$. Since x_n is clearly increasing at $x_1 = 1$, neither limit makes sense.

b) No, x_n diverges to ∞.

c) The method used in these last 3 problems assumes the sequence converges. This problem shows that you can use the method and get and answer, but that answer is only justified if you have first shown that the series converges.

9.7) Let $\epsilon > 0$. Let $N = 1 + \frac{2}{\epsilon^2}$. Then $n > N$ implies that $n - 1 \geq \frac{2}{\epsilon^2}$, i.e., $\sqrt{\frac{2}{n-1}} < \epsilon$. Hence, since s_n is nonnegative, $|s_n - 0| = s_n < \sqrt{\dfrac{2}{n-1}} < \epsilon$, which shows that $s_n \to 0$, as desired.

9.8) a) ∞ b) $-\infty$ c) NOT EXIST d) ∞ e) ∞

9.9 a) Let $M > 0$. There exists N_1 so that $n > N_1 \Rightarrow s_n > M$. Let $N = \max(N_0, N_1)$. Then $t_n > M$, $\forall n > N$.

b) Let $m < 0$. There exists N_1 so that $n > N_1 \Rightarrow t_n < m$. Let $N = \max(N_0, N_1)$. Then $s_n < m$ for all $n > N$.

c) Consider $x_n = t_n - s_n$. For all $n > N_0$ we have $x_n \geq 0$. Since $\lim t_n$ and $\lim s_n$ exist, so does there difference. Clearly $\lim x_n \geq 0$. Hence, $\lim(t_n - s_n) = \lim t_n - \lim s_n \geq 0$ and we are done.

9.10 a) We have that, given $\frac{M}{k}$, there exists N so that for all $n > N$, $s_n > \frac{M}{k}$. Thus, $ks_n > M$ as needed.

b) Assume $\lim s_n = \infty$ so that for any $M > 0$, there exists N so that $n > N$ gives us $s_n > M$. Hence, $-s_n < -M$. Thus, for any $m < 0$, there exists N so that $n > N$ gives us $-s_n < m$.

c) Sketch: We have $s_n > \left|\frac{M}{k}\right|$ so that $ks_n < M$.

9.11 a) We can't have $\inf t_n = \infty$, so let $\inf t_n = x \in \mathbb{R}$. Then $t_n \geq x$ for all $n \in \mathbb{Z}^+$. Hence, there exists N so that $n > N \Rightarrow s_n > M - x$ (since $\lim s_n = \infty$). Thus, $s_n + t_n > M$ for all $n > N$, showing that $\lim(s_n + t_n) = \infty$.

b) Let $\lim t_n = t \in \mathbb{R}$. The case $t = \infty$ is essentially the arguement in part (a). Since t_n converges, there exists N so that $n > N \Rightarrow t_n \geq t - 1$ (using $\epsilon = 1$ in the definition). Repeat the argument in part (a) with $x = t - 1$.

c) If t_n is bounded then there exists $x \in \mathbb{R}$ such that $t_n \geq x$. Repeat the argument in part (a).

9.12) Use the hint. Since $\left|\frac{s_{n+1}}{s_n}\right|$ converges, we take $\epsilon = \frac{a-L}{2}$ in the definition to see that $\left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < \frac{a-L}{2}$ for all $n \geq N$ (for some N). This implies that for all $n \geq N$, $\left|\frac{s_{n+1}}{s_n}\right| \in (L - (a-L)/2, L + (a-L)/2)$. By construction, $L + (a-L)/2 < a$ (since $L < a$). Hence, for all $n \geq N$, we have $\left|\frac{s_{n+1}}{s_n}\right| < a$ which gives $|s_{n+1}| < a|s_n|$. At N we have $|s_{N+1}| < a|s_N|$. In turn, we have $|s_{N+2}| < a|s_{N+1}|$. Continuing, $|s_n| < a^{n-N}|s_N| = \frac{|s_N|}{a^N}$ for any $n > N$. Since N is fixed, we have $|s_n| < ka^n$ for some constant $k > 0$. Since $a^n \to 0$ we see that $|s_n| \to 0$ so that $s_n \to 0$.

b) The argument is just like part (a), except we choose $\epsilon = \frac{L-a}{2}$ and reverse all inequalities to get $|s_n| > ka^n$. Since a_n diverges to ∞, by Exercise 9.10a, so must $|s_n|$.

9.13) a) Let $|a| < 1$. By Exercise 9.12a, since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a^{n+1}}{a^n}\right| = |a| < 1$, we have $a^n \to 0$.

b) If $|a| = 1$, clearly $a^n \to 1$ since it is the constant sequence.

c) If $|a| > 1$, by Exercise 9.12b, we have $\lim |a^n| = \infty$. In the case when $a > 1$, $|a^n| = a^n$ so $\lim a^n = \infty$. In the case when $a < 1$, we clearly have $\lim a^{2n} = \infty$ while $\lim a^{2n+1} = -\infty$ so that the limit doesn't exist.

9.14) Let $|a| < 1$ and let s_n denote the given sequence. By Exercise 9.12a, since $\left|\frac{s_{n+1}}{s_n}\right| = \left|\frac{a^{n+1}n^p}{a^n(n+1)^p}\right| = |a| \cdot \left(\frac{n}{n+1}\right)^p \leq |a| < 1$, we have $s^n \to 0$. If $|a| = 1$, we have $s_n = \frac{1}{n^p}$ which is easily shown (via the standard ϵ-proof) to have limit 0. Hence, $s_n \to 0$ for $|a| \leq 1$.

b) If $a > 1$, by Exercise 9.12b we have $\left|\frac{s_{n+1}}{s_n}\right| = \left|\frac{a^{n+1}n^p}{a^n(n+1)^p}\right| = |a| \cdot \left(\frac{n}{n+1}\right)^p > |a| \cdot \frac{1}{a} = 1$ (since $\left(\frac{n}{n+1}\right)^p \to 1$ and $\frac{1}{a} < 1$). Since $\lim \left|\frac{s_{n+1}}{s_n}\right| > 1$ we have s_n diverges to ∞.

c) If $a < -1$, again $\lim |s_n| = \infty$, but we have $\lim s_{2n} = \infty$ while $\lim s_{2n+1} = -\infty$, so that $\lim s_n$ does not exist.

9.15) Use 9.12a. Let s_n be the sequence. Then $\left|\frac{s_{n+1}}{s_n}\right| = \left|\frac{a^{n+1}(n)!}{a^n(n+1)!}\right| = \left|\frac{a}{n+1}\right| \to 0$ (for any a). Thus, $s_n \to 0$.

9.16) a) Let s_n be the given sequence. Note that $s_n > 0$ for all n. Hence, consider $\frac{1}{s_n}$. We have $\frac{1}{s_n} = \frac{n^2+9}{n^4+8n} > \frac{n^2}{n^4+8n} > \frac{n^2}{n^4+n^4} = \frac{1}{2n^2} \to 0$ (by Exercise 9.14 with $a = 1$ and $p = 2$). By Theorem 9.10, $\lim s_n = \infty$.

b) We have $\inf\{(-1)^n : n \in \mathbb{Z}^+\} = 1$ and by Exercise 9.14, $\frac{2^n}{n^2} \to \infty$. Hence, by Exercise 9.11, $\lim\left((-1)^n + \frac{2^n}{n^2}\right) = \infty$.

c) Using Exercises 9.14 and 9.15 we have $\frac{3^n}{n^3}$ diverges to ∞ while $-\frac{3^n}{n!} \to 0$. Let $t_n = -\frac{3^n}{n!}$, which is always negative. Since $t_n \to 0$, there exists an N such that for all $n > N$, $|t_n| < 1$ (letting $\epsilon = 1$ in the definition). Let $m = \min\{t_1, t_2, \dots, t_N, -1\}$. Then $\inf t_n \geq m > -\infty$. By Exercise 9.11, $\lim \left(\frac{3^n}{n^3} + t_n\right) = \infty$.

9.17) Let $M > 0$. Let $N = \sqrt{M}$. Then for all $n > N$, since $n > \sqrt{M}$ we have $n^2 > M$.

9.18) a) Let S be the given sum. Note that $S - aS = S(1-a) = 1 - a^{n+1}$. If $a \neq 1$, we divide both sides by $(1-a)$ to obtain the resulut.

b) If $|a| < 1$, we have $a^{n+1} \to 0$ so the limit is $\frac{1}{1-a}$. c) Use your answer to (b) with $a = 1/3$ to get $3/2$.

d) If $a \geq 1$, the sum is at least n. Since $\lim n = \infty$, the given limit is ∞.

10.1) Nondecreasing: c Nonincreasing: a, f Bounded: a, b, d, f

10.4) The Completeness Axiom doesn't hold over $\mathbb{Q}$, so sup (inf) may not exist. Both proofs rely on this existence.

10.6) a) First note that, for $m > n$,

$$\begin{aligned} |s_m - s_n| &= |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\ &< 2^{-m} + 2^{-m+1} + \cdots + 2^{-n-1} \\ &= 2^{-m}(1 + 2 + 4 + \cdots + 2^{m-n+1}) \\ &= 2^{-m}\frac{1-2^{m-n+2}}{1-2} = \frac{1}{2^m} \cdot (2^{m-n+2} - 1) \quad \text{(by Ex. 9.18)} \\ &< \frac{4}{2^n}. \end{aligned}$$

Now, let $\epsilon > 0$. Let $N = \log_2(4/\epsilon) = 2 - \log_2(\epsilon)$. Then $n > N$ implies that $2^n > \frac{4}{\epsilon}$ so that $\frac{4}{2^n} < \epsilon$. Let $m > n > N$. Then $|s_m - s_n| < \frac{4}{2^n} < \epsilon$ and we are done (it is Cauchy and hence convergent).

b) No. Consider $s_{n+1} = s_n + \frac{1}{2n}$ with $s_1 = 0$. We have $s_m = s_{m-1} + \frac{1}{2(m-1)} = s_{m-2} + \frac{1}{2(m-1)} + \frac{1}{2(m-2)}$. Continuing this we get $s_m = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1}\right)$. You may recognize this from calculus: for large m, $s_m \approx \frac{1}{2}\log(1+m)$, which diverges to ∞. Since $\lim s_m = \infty$, it cannot be Cauchy (else it would be convergent).

10.7) Let $s = \sup S$. First, note that, for *any* given $\epsilon > 0$, there must be infinitely many $t \in S$ with $t \in (s - \epsilon, s)$. Otherwise, if $(s-\epsilon, s)$ contained only finitely many such t's, say $t_1, \dots, t_n$, we could let $m = \max(t_1, \dots, t_n)$ so that $t \leq m$ for all $s \in S$. Since $m < s$, this would be a contradiction since s is the sup. We define our sequence as follows. Choose any $s_1 \in (s-1, s)$ (taking $\epsilon = 1$). Next, choose $s_2 \in (s - s_1, s)$. Then $s_2 > s_1$. Continue by choosing $s_n \in (s - s_{n-1}, s)$. We have constructed an increasing sequence that converges to s.

10.8) We need to show that $\sigma_{n+1} \geq \sigma_n$. Substituting in, we need to show that $s_1 + s_2 + \cdots + s_{n+1} \geq \frac{n+1}{n}(s_1 + s_2 + \cdots + s_n)$, i.e., we want $s_{n+1} \geq \frac{1}{n}(s_1 + \cdots + s_n)$. Note that $s_1 \leq s_2 \leq \cdots \leq s_n$ so that $\frac{1}{n}(s_1 + \cdots + s_n) \leq \frac{1}{n}(s_n + s_n + \cdots + s_n) = s_n$. Since $s_{n+1} \geq s_n \geq \frac{1}{n}(s_1 + \cdots + s_n)$ we are done.

10.10) a) $s_2 = \frac{2}{3}$; $s_3 = \frac{5}{9}$; $s_4 = \frac{14}{27}$

b) $s_1 = 1 > 1/2$ so assume the result for n. We will show it for $n+1$: $s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}\left(\frac{1}{2} + 1\right) = \frac{1}{2}$.

c) We'll show $s_{n+1} \leq s_n$, i.e., $\frac{1}{3}(s_n + 1) \leq s_n \Leftrightarrow 1 \leq 2s_n \Leftrightarrow s_n \geq \frac{1}{2}$, which was shown to be true in part (b).

d) This is a nonincreasing sequence bounded from below so it converges. To find the limit, let $L = \lim s_n$. Hence, $L = \frac{1}{3}(L+1)$ so that $L = \frac{1}{2}$.

10.11) a) Clearly $t_{n+1} \leq t_n$ so the sequence is nonincreasing. It is bounded below by 0. This is a nonincreasing sequence bounded from below so it converges.

b) Find the first few terms $(1, 3/4, 45/1575/2304, 99225/147456)$. An (educated) guess: $\frac{2}{3}$.

10.12) a) Clearly $t_{n+1} \leq t_n$ so the sequence is nonincreasing. It is bounded below by 0. This is a nonincreasing sequence bounded from below so it converges.

b) First few terms: $1, 3/4, 24/36, 360/576, 8640/14400$. This one is less than $\frac{2}{3}$, so let's guess $\frac{1}{2}$.

c&d) This is true for $n = 1$ so let's assume it is true for n and show for $n+1$. We have $t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) t_n = \left(\frac{(n+1)^2 - 1}{(n+1)^2}\right) \frac{n+1}{2n} = \frac{1}{2(n+1)} \left(\frac{(n+1)^2 - 1}{n(n+1)}\right) = \frac{1}{2(n+1)} \cdot \frac{n^2 + 2n}{n} = \frac{n+2}{2(n+1)} \to \frac{1}{2}$ and we are done.