On Generalized Van der Waerden Triples

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Abstract

Van der Waerden's classical theorem on arithmetic progressions states that for any positive integers k and r, there exists a least positive integer, $w(k,r)$, such that any r-coloring of $\{1,2,\ldots,w(k,r)\}$ must contain a monochromatic k-term arithmetic progression $\{x, x+d, x+2d, \ldots, x+(k-1)d\}$. We investigate the following generalization of $w(3,r)$. For fixed positive integers a and b with $a \le b$, define $N(a,b;r)$ to be the least positive integer, if it exists, such that any r-coloring of $\{1,2,\ldots,N(a,b;r)\}$ must contain a monochromatic set of the form $\{x, ax+d, bx+2d\}$. We show that $N(a,b;2)$ exists if and only if $b \ne 2a$, and provide upper and lower bounds for it. We then show that for a large class of pairs (a,b), $N(a,b;r)$ does not exist for r sufficiently large. We also give a result on sets of the form $\{x, ax+d, ax+2d, \ldots, ax+(k-1)d\}$.

1. Introduction

B.L. van der Waerden [8] proved that for any positive integers k and r, there exists a least positive integer, $w(k,r)$, such that any r-coloring of $[1,w(k,r)] = \{1,2,\ldots,w(k,r)\}$ must contain a monochromatic k-term arithmetic progression $\{x, x+d, x+2d, \ldots, x+(k-1)d\}$. The only known non-trivial values of $w(k,r)$ are $w(3,2)=9$, $w(4,2)=35$, $w(5,2)=178$, $w(3,3)=27$ and $w(3,4)=76$. The estimation of the function $w(k,r)$ has long been one of the more elusive problems in Ramsey theory. In 1974, Szemerédi [7] proved the long-standing conjecture of Erdős and Turán that any set of positive integers with positive upper density must contain arbitrarily long arithmetic progressions. The proof of the conjecture made use of van der Waerden's theorem, and did not yield any reasonable upper bounds for $w(k,r)$. In a major breakthrough in 1988, Shelah [6] showed that there exists a primitive recursive upper bound on the van der Waerden numbers. Then, in 1998, Timothy Gowers announced that he had proven that $w(k,2) \le 2^{2^{2^{2^{2^{k+10}}}}}$, a remarkable achievement (at the time of this writing, Gowers' result has not yet been published).

The function $w(k,r)$ is sometimes called the *Ramsey function* for the collection of arithmetic progressions. In [1] the authors considered a generalization of van der Waerden's theorem, by considering, for a given function $f : \mathbf{N} \to \mathbf{N}$, the Ramsey function corresponding to the collection of arithmetic progressions $\{a, a+d, a+2d, ..., a+(k-1)d\}$ with the property that $d \geq f(a)$. The Ramsey functions for other "substitutes" for the set of arithmetic progressions were studied in [2], [4], and [5]. In this paper we consider a new generalization of $w(k, r)$. Unlike the classical van der Waerden numbers, which exist regardless of the number of colors used, the existence of these new numbers depends on the number of colors. To help describe this generalization, we begin with three definitions.

Definition 1.1: Fix $1 \leq a \leq b$. A set, S, of three natural numbers is called an (a, b)-*triple* if there exist natural numbers x and d such that $S = \{x, ax + d, bx + 2d\}$.

Definition 1.2: Fix $1 \leq a \leq b$. Define $N(a, b; r)$ to be the least positive integer, if it exists, such that any r-coloring of $[1, N(a, b; r)]$ must contain a monochromatic (a, b)-triple.

Definition 1.3: Fix $1 \leq a \leq b$. Define (a, b) to be *regular* if $N(a, b; r)$ exists for all positive integers r. If (a, b) is not regular, the *degree of regularity of* (a, b) is the largest r such that $N(a, b; r)$ exists. Denote this by $dor(a, b)$.

We note here that $N(1, 1; r)$ is the van der Waerden number $w(3, r)$ so that $N(a, b; r)$ is a generalization of $w(3, r)$, and obviously $(1, 1)$ is regular. We now discuss the sections which follow.

In Section 2 we consider $r = 2$. We show that, except for the case in which $b = 2a$, $N(a, b; 2)$ does exist; we also find upper and lower bounds on $N(a, b; 2)$ (for $b \neq 2a$). For certain pairs (a, b), we obtain stronger bounds; in particular, we use a result of [1] to deal with $N(a, 2a-1; 2)$ (when $a = 1$ this is just $w(3, 2)$). In Section 3 we establish that (a, b) is not regular for a rather large class of pairs (a, b), and give an upper bound (for these pairs) on the degree of regularity. We then give lower bounds on $N(a, b; r)$ for all $1 \leq a \leq b$ and $r > 2$. In Section 4 we make some observations about monochromatic sets of the form $\{x, ax + d, ax + 2d, \ldots, ax + (k-1)d\}$ for $a \geq 1$. We establish that for $a > 1$ and k sufficiently large (dependent upon a), we can 4-color the natural numbers so that no monochromatic such k-set exists (this is in contrast to van der Waerden's theorem which says that there are arbitrarily long monochromatic arithmetic progressions in any r-coloring of the natural numbers).

2. Using Two Colors

Our first theorem categorizes those (a, b) pairs for which $N(a, b; 2)$ exists, i.e., those pairs for which $dor(a, b) \geq 2$. It also provides an upper bound on $N(a, b; 2)$ whenever it exists.

Theorem 2.1: Let $a, b \in \mathbf{N}$ with $a \leq b$. Then $dor(a, b) = 1$ if and only if $b = 2a$. Furthermore, if $b \neq 2a$,

$$N(a, b; 2) \leq \begin{cases} 4a(b^3 + b^2 - 3b - 3) + 2b^3 + 6b^2 + 6b & \text{for } b > 2a \\ 4a(b^3 + 2b^2 + 2b) - 4b^2 & \text{for } b < 2a \end{cases}$$

Proof. We first consider the case in which $b = 2a$. To show that $N(a, 2a; 2)$ does not exist, we exhibit a 2-coloring of $\mathbf{N}$ which avoids monochromatic $(a, 2a)$-triples. Namely, color the natural numbers so that the odd numbers are colored arbitrarily, and so that for each even number $2n$, the color of $2n$ is different from the color of n. Such a coloring avoids monochromatic $(a, 2a)$-triples since such a triple has the form $\{x, y, z\}$ where $z = 2y$.

We next consider the case $b > 2a$. Let $M = 4a(b^3 + b^2 - 3b - 3) + 2b^3 + 6b^2 + 6b$ and let $\chi : [1, M] \to \{0, 1\}$ be a 2-coloring. Assume there is no monochromatic (a, b)-triple. Then within the set $\{2, 4, \ldots, 2b+4\}$ there exist x and $x+2$ that are not the same color, since otherwise $\{2, 2a+2, 2b+4\}$ would be a monochromatic (a, b)-triple. Without loss of generality, assume $\chi(x) = 0$ and $\chi(x+2) = 1$. Let z be the least integer greater than $a(x+2)$ such that $b - 2a$ divides z.

Let $S = \{z, az + (b - 2a), bz + 2(b - 2a)\}$. Since $z \le 2a(b+1) + b$, we have

$$bz + 2(b - 2a) \le 2a(b^2 + b - 2) + b^2 + 2b \le M. \tag{1}$$

Hence, since S is an (a, b)-triple, some member, say s, of S has color 1. Let

$$T = \{s + i(b - 2a) : 0 \le i \le \frac{s(b-1)}{b - 2a} + 2\}.$$

Note that $bs + 2(b - 2a)$ is the largest member of T, and that $as + (b - 2a) \in T$ since $b - 2a$ divides s. Also, by (1)

$$bs + 2(b - 2a) \le 2a(b^3 + b^2 - 2b - 2) + b^3 + 2b^2 + 2b \le M,$$

so some member of T must have color 0 (otherwise $\{s, as + (b - 2a), bs + 2(b - 2a)\}$ is monochromatic).

Let t be the least member of T with color 0. Then $\chi(t - (b - 2a)) = 1$. Note that since $x \le 2b + 2$, $z \le a(x + 2) + (b - 2a)$, $s \le bz + 2(b - 2a)$, and $t \le bs + 2(b - 2a)$ we have

$$b(x + 2) + 2(t - ax - b) = 2t + x(b - 2a) \le M.$$

Thus, since $\chi(x+2) = \chi(t-(b-2a)) = 1$, we must have $\chi(b(x+2)+2(t-ax-b)) = 0$ (that $t-(b-2a) > a(x+2)$ follows from the definition of t). This implies that $\{x, t, bx+2(t-ax)\}$ is a monochromatic (a, b)-triple, a contradiction.

The case for $b < 2a$ is very similar. Let $M = 4a(b^3 + 2b^2 + 2b) - 4b^2$ and let χ be a 2-coloring of $[1, M]$. Then the set $\{2, 4, \ldots, 2b+4\}$ contains $x - 2$ and x that are not the same color. Assume $\chi(x) = 0$ and $\chi(x - 2) = 1$, and let z be the least integer greater than $ax - (2a - b)$ such that $2a - b$ divides z. Let $S = \{z, az+(2a-b), bz+2(2a-b)\}$. Let $s \in S$ have color 1 and define $T = \{s, s+(2a-b), s+2(2a-b), \ldots, bs+2(2a-b)\}$. As in the previous case, $as+(2a-b) \in T$ and $T \subseteq [1, M]$. Hence, T must have a least member, t, with color 0. Then $\chi(t - (2a - b)) = 1$, and since $\chi(x - 2) = \chi(t - (2a - b)) = 1$, we must have $\chi(b(x - 2) + 2(t - ax + b)) = 0$. This gives the monochromatic (a, b)-triple $\{x, t, bx + 2(t - ax)\}$, a contradiction. (That $bx + 2(t - ax) \le M$ follows easily from the definitions of z, s, and t, and the fact that $x \ge 4$.) $\square$

For certain pairs (a, b), we are able to improve the upper bounds of Theorem 2.1. The next theorem deals with the case in which $a = b$. Theorem 2.1 gives an upper bound for this case of $O(a^4)$. The following theorem improves this to $O(a^2)$.

Theorem 2.2: $N(a,a;2) \leq \begin{cases} 3a^2+a & \text{for } 4 \leq a \text{ even} \\ 8a^2+a & \text{for } a \text{ odd} \end{cases}$

Proof. We start with the case when a is even. We may assume that $a \geq 6$ since $N(4,4;2) = 40$ was obtained by computer search (for other exact values see Table 1 at the end of this section). For readability, we will use red-blue colorings instead of $0-1$ colorings in this proof. We will show that every red-blue coloring of $S = [1, 3a^2+a]$ yields a monochromatic (a,a)-triple by considering all possible 2-colorings of the set $\{1, a+1, (3/2)a^2+a, 2a^2+a\}$. Assume, by way of contradiction, that there is a 2-coloring χ of S that yields no monochromatic (a,a)-triple.

Let R be the set of red elements of S under χ, and B the set of blue elements of S under χ. Without loss of generality we assume $1 \in R$. We will come to a contradiction in each case by an obvious "forcing" argument. By this we mean that if we consider an (a,a)-triple having two elements of the same color, then the third element of the triple, by assumption, must be of the opposite color. We will be explicit in the first case below and then present only the relevant (a,a)-triples for the remaining cases.

`Case I:` $a+1, 2a^2+a \in R$. For a contradiction, we will show that the (a,a)-triple $\{a+2, (3/2)a^2+2a, 2a^2+2a\}$ is monochromatic. First consider the (a,a)-triple (with $d=1$) $\{1, a+1, a+2\}$. Since $1, a+1 \in R$ the color of $a+2$ is forced to be blue. Likewise, since $\{1, a^2+a, 2a^2+a\}$ is an (a,a)-triple (with $d=a^2$) and $1, 2a^2+a \in R$ we have $a^2+a \in B$. The (a,a)-triple $\{a+1, (3/2)a^2+a, 2a^2+a\}$ (with $d = a^2/2$) then gives $(3/2)a^2+a \in B$. Next, using $\{a+2, (5/4)a^2+(3/2)a, (3/2)a^2+a\}$ (an (a,a)-triple with $d = (1/4)a^2 - a/2$) we see that $(5/4)a^2+(3/2)a \in R$. The (a,a)-triple $\{a/2+1, a^2+a, (3/2)a^2+a\}$ shows that $a/2+1 \in R$. The (a,a)-triple $\{a+1, (5/4)a^2+(3/2)a, (3/2)a^2+2a\}$ gives us $(3/2)a^2+2a \in B$. Lastly, the (a,a)-triple $\{a/2+1, (5/4)a^2+(3/2)a, 2a^2+2a\}$ forces $2a^2+2a \in B$. This gives our desired contradiction since $\{a+2, (3/2)a^2+2a, 2a^2+2a\}$ is a blue (a,a)-triple.

`Case II:` $a+1, (3/2)a^2+a \in R$ and $2a^2+a \in B$. As in Case I, we must have $a+2 \in B$. Using the following sequence of (a,a)-triples, it is routine to show that the (a,a)-triple $\{a+3, 2a^2+a, 3a^2-a\}$ is blue, giving a contradiction: $\{a+1, (5/4)a^2+a, (3/2)a^2+a\}$, $\{a+2, 2a^2+a, 3a^2\}$, $\{1, (3/2)a^2+a, 3a^2+a\}$, $\{a+2, 2a^2+(3/2)a, 3a^2+a\}$, $\{a+3, 2a^2+(3/2)a, 3a^2\}$, $\{a+2, (5/4)a^2+a, (3/2)a^2\}$, $\{1, (3/2)a^2, 3a^2-a\}$.

`Case III:` $(3/2)a^2+a, 2a^2+a \in B$. This implies $a+1 \in R$, so that again we have $a+2 \in B$. Using the following sequence of (a,a)-triples, it follows that $\{a, (3/2)a^2+a, 2a^2+2a\}$ is a blue (a,a)-triple: $\{a+2, (3/2)a^2+(3/2)a, 2a^2+a\}$, $\{a+2, 2a^2+a, 3a^2\}$, $\{a+1, (3/2)a^2+(3/2)a, 2a^2+2a\}$, $\{a+2, (3/2)a^2+a, 2a^2\}$, $\{a, 2a^2, 3a^2\}$.

`Case IV:` $(3/2)a^2+a, 2a^2+a \in R$. Using this case's assumptions and the fact that $1 \in R$, we have $(3/4)a^2+a, a^2+a, 3a^2+a \in B$. We consider two subcases.

Subcase (i): $2 \in R$. Using the following sequence of (a,a)-triples, we have that the triple $\{2a+2, (5/2)a^2+a, 3a^2\}$ is blue: $\{(a/2)+1, (3/4)a^2+a, a^2+a\}$, $\{(a/2)+1, (3/2)a^2+a, (5/2)a^2+a\}$, $\{2a+1, (5/2)a^2+a, 3a^2+a\}$, $\{2, (3/2)a^2+a, 3a^2\}$, $\{2, 2a+1, 2a+2\}$.

Subcase (ii): $2 \in B$. As in subcase (i), $a/2+1 \in R$. Then the following sequence of (a,a)-triples yields the monochromatic triple $\{a, a^2+a/2, a^2+a\}$: $\{2, (3/4)a^2+a, (3/2)a^2\}$, $\{2, a^2+a, 2a^2\}$, $\{a/2+1, a^2+a/2, (3/2)a^2\}$, $\{a, (3/2)a^2, 2a^2\}$.

`Case V:` $a+1$, $2a^2+a \in B$. In this case $(3/2)a^2+a, 3a^2+a \in R$, so that the (a,a)-triple $\{1, (3/2)a^2+a, 3a^2+a\}$ is monochromatic.

`Case VI:` $a+1$, $(3/2)a^2+a \in B$. This assumption implies that $(5/4)a^2+a, 2a^2+a \in R$. By considering the (a,a)-triples $\{1, a^2+a, 2a^2+a\}$ and $\{a/2+1, a^2+a, (3/2)a^2+a\}$, we have that the (a,a)-triple $\{a/2+1, (5/4)a^2+a, 2a^2+a\}$ must be red.

We now consider a odd. We may assume that $a \geq 5$ since $N(1,1;2)$ is the van der Waerden number $w(3;2)$, which equals nine, and $N(3,3;2) = 39$ (see Table 1). Our method is very similar to that of the even case. Here we 2-color $T = [1, 8a^2+a]$ and consider the various ways in which the set $\{4a+1, 5a^2+a, 8a^2+a\}$ may be colored. The following six cases cover all possibilities.

`Case I:` $4a+1, 5a^2+a \in R$. In this case $6a^2+a \in B$ and, since $1 \in R$, $(5a+1)/2 \in B$. We consider two subcases.

Subcase (i): $2 \in B$. The following sequence of (a,a)-triples yields the red triple $\{4a+2, 5a^2+a, 6a^2\}$: $\{2, (5a+1)/2, 3a+1\}$, $\{1, 3a+1, 5a+2\}$, $\{5a+2, 6a^2+a, 7a^2\}$, $\{1, (7a^2+a)/2, 7a^2\}$, $\{(5a+1)/2, (7a^2+a)/2, (9a^2+a)/2\}$, $\{4a, (9a^2+a)/2, 5a^2+a\}$, $\{2, 4a, 6a\}$, $\{6a, 7a^2, 8a^2\}$, $\{4a+2, 6a^2+a, 8a^2\}$, $\{4a, 6a^2, 8a^2\}$.

Subcase (ii): $2 \in R$. The triple $\{2, 5a+2, 8a+4\}$ is forced to be red by considering the following sequence of triples: $\{2, (5a^2+3a)/2, 5a^2+a\}$, $\{(5a+1)/2, (5a^2+3a)/2, (5a^2+5a)/2\}$, $\{4, (5a^2+5a)/2, 5a^2+a\}$, $\{2, 4a+1, 6a+2\}$, $\{(5a+1)/2, (5a^2+3a)/2, (5a^2+5a)/2\}$, $\{1, (5a^2+5a)/2, 5a^2+4a\}$, $\{2, (5a^2+5a)/2, 5a^2+3a\}$, $\{5a+2, 5a^2+3a, 5a^2+4a\}$, $\{4, 6a+2, 8a+4\}$.

`Case II:` $4a+1, 8a^2+a \in R$ and $5a^2+a \in B$. The triple $\{1, 3a, 5a\}$ is forced to be red by considering the following triples: $\{1, (5a+1)/2, 4a+1\}$, $\{1, 4a^2+a, 8a^2+a\}$, $\{4a+1, 6a^2+a, 8a^2+a\}$, $\{(5a+1)/2, 4a^2+a, (11a^2+3a)/2\}$, $\{4a+1, (11a^2+3a)/2, 7a^2+2a\}$, $\{3a, 5a^2+a, 7a^2+2a\}$, $\{5a, 6a^2+a, 7a^2+2a\}$.

`Case III:` $4a+1 \in R$ and $5a^2+a, 8a^2+a \in B$. The triple $\{1, 2a, 3a\}$ is shown to be red by the following triples: $\{2a+1, 5a^2+a, 8a^2+a\}$, $\{1, 2a+1, 3a+2\}$, $\{3a+2, 5a^2+a, 7a^2\}$, $\{3a+2, (11a^2+3a)/2, 8a^2+a\}$, $\{1, (7a^2+a)/2, 7a^2\}$, $\{4a+1, (11a^2+3a)/2, 7a^2+2a\}$, $\{2a, (7a^2+a)/2, 5a^2+a\}$, $\{3a, 5a^2+a, 7a^2+2a\}$.

`Case IV:` $5a^2+a, 8a^2+a \in R$. The sequence of triples $\{1, 4a^2+a, 8a^2+a\}$, $\{2a+1, 5a^2+a, 8a^2+a\}$, $\{2a+1, 3a^2+a, 4a^2+a\}$, $\{2a+1, 4a^2+a, 6a^2+a\}$ shows that $\{1, 3a^2+a, 6a^2+a\}$ is red.

`Case V:` $5a^2+a \in R$ and $4a+1, 8a^2+a \in B$. In this case we have $6a^2+a \in R$ and, since $1 \in R$, we have $3a^2+a \in B$. We consider two subcases.

Subcase (i): $2 \in B$. The sequence of (a,a)-triples $\{2, 3a^2+a, 6a^2\}$, $\{4a+2, 5a^2+a, 6a^2\}$,

$\{6a-1, 6a^2, 6a^2+a\}$, $\{6a-1, 7a^2, 8a^2+a\}$, $\{2, 3a+1, 4a+2\}$, $\{3a+1, 4a^2+a, 5a^2+a\}$, $\{2, 4a^2+a, 8a^2\}$, $\{4a, 6a^2, 8a^2\}$, $\{6a, 7a^2, 8a^2\}$ leads to the conclusion that $\{2, 4a, 6a\}$ is blue.

Subcase (ii): $2 \in R$. We have that the triple $\{4a+1, (9a^2+5a)/2, 5a^2+4a\}$ is blue by using the following sequence of triples: $\{2, (5a^2+3a)/2, 5a^2+a\}$, $\{2a+2, (5a^2+3a)/2, 3a^2+a\}$, $\{2, 2a+2, 2a+4\}$, $\{2, 2a+1, 2a+2\}$, $\{2a+2, (7a^2+3a)/2, 5a^2+a\}$, $\{2a+4, (5a^2+5a)/2, 3a^2+a\}$, $\{(3a+3)/2, (5a^2+3a)/2, (7a^2+3a)/2\}$, $\{2a+1, (5a^2+3a)/2, 3a^2+2a\}$, $\{1, (5a^2+5a)/2, 5a^2+4a\}$, $\{(3a+3)/2, 3a^2+2a, (9a^2+5a)/2\}$.

`Case VI:` $4a+1, 5a^2+a \in B$. Using the following sequence of (a,a)-triples, we find that $\{4a-1, 6a^2-2a, 8a^2-3a\}$ is blue: $\{4a+1, 5a^2+a, 6a^2+a\}$, $\{1, 3a^2+a, 6a^2+a\}$, $\{a+1, 3a^2+a, 5a^2+a\}$, $\{1, a+1, a+2\}$, $\{a+2, 3a^2+a, 5a^2\}$, $\{4a-1, 5a^2, 6a^2+a\}$, $\{1, (5a^2+a)/2, 5a^2\}$, $\{2a, (5a^2+a)/2, 3a^2+a\}$, $\{a+2, (5a^2+a)/2, 4a^2-a\}$, $\{2a, 4a^2-a, 6a^2-2a\}$, $\{1, 4a^2-a, 8a^2-3a\}$. □

Another circumstance for which we can improve the upper bounds of Theorem 2.1 is the case in which $b = 2a-1$ (for $a = 1$ this is the van der Waerden number $w(3,2)$). By Theorem 2.1, $N(a, 2a-1; 2)$ is bounded above by a function having order of magnitude $32a^4$. We can improve this to $16a^3$ by making use of the following theorem which is taken from [1].

First, we introduce some notation. Let $f : \mathbf{N} \to \mathbf{R}^+$ be a non-decreasing function. Denote by $w(f,k)$ the least positive integer (if it exists) such that whenever $[1, w(f,k)]$ is 2-colored, there must exist a monochromatic k-term arithmetic progression $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ with $d \geq f(a)$. In [1] it is shown that $w(f,3)$ always exists, and bounds for this function are given as follows.

Theorem 2.3: (Brown and Landman [1]) Let $f : \mathbf{N} \to \mathbf{R}^+$ be a non-decreasing function. Let $b = 1 + 4\left\lceil \frac{f(1)}{2} \right\rceil$. Then

$$w(f,3) \leq \left\lceil 4f\left(b + 4\left\lceil \frac{f(b)}{2} \right\rceil\right) + 14\left\lceil \frac{f(b)}{2} \right\rceil + 7b/2 - 13/2 \right\rceil .$$

Further, if f maps into $\mathbf{N}$ with $f(n) \geq n$ for all $n \in \mathbf{N}$, then $w(f,3,2) \geq 8f(h)+2h+2-c$, where $h = 2f(1)+1$ and c is the largest integer such that $f(c)+c \leq 4f(h)+h+1$.

Relating Theorem 2.3 to (a,b)-triples, we have the following corollary.

Corollary 2.1: For all $a \geq 2$,

$$16a^2 - 12a + 6 \leq N(a, 2a-1; 2) \leq \begin{cases} 16a^3 - 2a^2 + 4a - 3 & \text{for } a \text{ even} \\ 16a^3 + 14a^2 + 2a - 3 & \text{for } a \text{ odd} \end{cases}$$

Proof. Note that $\{x,y,z\}$ is an $(a, 2a-1)$-triple if and only if it is an arithmetic progression with $y - x \geq (a-1)x + 1$. By applying Theorem 2.3 with $f(x) = (a-1)x+1$ we obtain the desired bounds. □

We now present some lower bounds for all (a, b)-triples. This is done by providing 2-colorings which avoid monochromatic (a, b)-triples.

Theorem 2.4: If $b \geq 2a$ then $N(a, b; 2) \geq 2b^2 + 5b - (2a - 4)$. If $b < 2a$ then $N(a, b; 2) \geq 3b^2 - (2a - 5)b - (2a - 4)$.

Proof. For the case $b \geq 2a$, we will exhibit a 2-coloring of $[1, 2b^2 + 5b - 2a + 3]$ with no monochromatic (a, b)-triple. Color $[b + 2, b^2 + 2b + 1]$ red and its complement blue. It is an easy exercise to show that monochromatic (a, b)-triples are avoided.

For the case where $b < 2a$ the 2-coloring of $[1, 3b^2 - (2a - 5)b - (2a - 4)]$ with $[b + 2, b^2 + 2b + 1]$ colored red and its complement colored blue is easily seen to avoid monochromatic (a, b)-triples. □

We are able to improve slightly the lower bound given in Theorem 2.4 for the case when $a = 1$. In fact, from computer calculations (see Table 1 below), it appears that this inequality may in fact be an equality.

Theorem 2.5: $N(1, b; 2) \geq 2b^2 + 5b + 6$ for all $b \geq 3$.

Proof. Consider the following red-blue coloring of $[1, 2b^2 + 5b + 5]$: color $[1, b + 1]$, $\{b + 3\}$, and $[b^2 + 2b + 4, 2b^2 + 5b + 5]$ red and the other integers blue. We now show that this coloring avoids monochromatic $(1, b)$-triples.

Assume $\{x, y, z\} = \{x, x + d, bx + 2d\}$ is a blue $(1, b)$-triple. Since the largest blue element is $b^2 + 2b + 3$, we must have $x = b + 2$. Thus, since we must have $d \geq 2$, we see that $z > b^2 + 2b + 3$, which is not possible.

Now assume $\{x, y, z\}$ is red. First, if $x, x + d \in \{1, 2, \ldots, b, b + 1, b + 3\}$ then we have $b + 2 \leq bx + 2d \leq b^2 + b + 4$. Hence, the only possibility here is $z = b + 3$, but $bx + 2d = b + 3$ has no solution in x for $b \geq 3$. Second, if $y \in [b^2 + 2b + 4, 2b^2 + 5b + 5]$ then $bx + 2d \geq bx + 2b^2 + 2b + 2$. Hence, we must have $x \in \{1, 2, 3, 4\}$ (4 is possible if $b = 3$). However, this gives $bx + 2d \geq bx + 2b^2 + 4b$, which implies that $x \in \{1, 2\}$ (2 is possible if $b = 3$). This in turn implies that $z \geq bx + 2b^2 + 2b + 4$ which gives $x = 1$ as the only possibility. However with $x = 1$ we must have $z > 2b^2 + 5b + 5$, which is out of bounds. □

In Table 1 we present computer-generated values for $N(a, b; 2)$ for small a and b. We also include computer-generated lower bounds for those cases where the computer time became excessive (the program is available for download as the Fortran77 program $VDW.f$ at http://math.colgate.edu/ aaron/).

$\mathbf{N(a, b; 2)}$ Values

$a \setminus b$	1	2	3	4	5	6	7
1	9	dne	39	58	81	≥ 108	≥ 139
2		16	46	dne	139	≥ 106	≥ 133
3			39	60	114	dne	≥ 135
4				40	87	≥ 124	≥ 214
5					70	100	≥ 150
6						78	≥ 105
7							95

Table 1

3. The Degree of Regularity of $(\mathbf{a}, \mathbf{b})$

In this section we consider $N(a, b; r)$ for general r. We begin by showing, in Theorems 3.1 and 3.2, that for many choices of a and b, the pair (a, b) is not regular. For such pairs we find an upper bound on $dor(a, b)$.

Theorem 3.1: Let $1 \leq a < b$, and assume that $b \geq (2^{3/2} - 1)a + 2 - 2^{3/2}$. Let $c = \lceil b/a \rceil$. Then $dor(a, b) \leq \lceil 2 \log_2 c \rceil$.

Proof. Let $r = \lceil 2 \log_2 c \rceil + 1$. We will give an r-coloring of the natural numbers which contains no monochromatic (a, b)-triple. For readability, let $p = \sqrt{2}$. Using the colors $0, 1, \ldots, r-1$, define the coloring χ by letting $\chi(x) \equiv i \pmod{r}$, where $p^i \leq x < p^{i+1}$.

Assume that there exists an (a, b)-triple, say $x < y < z$, that is monochromatic under χ. Let j be the integer such that $p^j \leq y < p^{j+1}$. Since $\{x, y, z\}$ is an (a, b)-triple, $y = ax + d$ and $z = bx + 2d$ for some d. Thus $z \leq cy < p^{r-1}p^{j+1} = p^{j+r}$. Hence, by the way χ is defined and the fact that $\chi(y) = \chi(z)$, we must have $p^j \leq y < z < p^{j+1}$. We consider two cases.

`Case I:` $b \geq 2a - 1$. In this case, $y - x = (a-1)x + d \leq (b-a)x + d = z - y < p^j(p-1) \leq p^j(1 - 1/p^{r-1})$. Hence, since $y > p^j$, we have $x \geq p^j - p^j(1 - 1/p^{r-1}) = p^{j-r+1}$. Since $\chi(x) = \chi(y)$, and by the definition of χ, we must have $p^j \leq x < y < p^{j+1}$. Thus, all three numbers x, y, z belong to the interval $[p^j, p^{j+1})$. Hence, $z - x = (b-1)x + 2d < p^j(p-1) \leq x(p-1)$, a contradiction (since $b - 1 > p - 1$).

`Case II:` $c = 2$. In this case, $2(a-1) \leq (b-a)/(p-1)$, so that $(a-1)x/(b-a)x \leq 1/(2p-2)$. Therefore, $((a-1)x + d)/((b-a)x + d) \leq 1/(2p-2)$. Hence, $y - x \leq (z-y)/(2p-2) < (p-1)p^j/(2p-2) = p^j/2 = p^{j-2}$. So, $x \geq p^j - p^{j-2} = p^{j-2}$. Since $r = 3$ in this case, and $\chi(x) = \chi(y)$, we must have $p^j \leq x < p^{j+1}$. Thus, as in Case I, x and z both belong to the interval $[p^j, p^{j+1})$, and we again have a contradiction. □

In the following theorem we give an upper bound on $dor(a, b)$ for several pairs (a, b) that are either not covered by Theorem 3.1 or for which we are able to improve the bound of Theorem 3.1.

Theorem 3.2: $dor(1,3) \leq 3$, $dor(2,2) \leq 5$, $dor(2,5) \leq 3$, $dor(2,6) \leq 3$, $dor(3,3) \leq 5$, $dor(3,4) \leq 5$, $dor(3,8) \leq 3$, and $dor(3,9) \leq 3$.

Proof. We give the proof for the pair $(2,2)$, and outline the proofs for the other cases, which are quite similar.

To show that $dor(2,2) \leq 5$, we provide a 6-coloring of the positive integers that avoids monochromatic $(2,2)$-triples. Let $\chi : \mathbf{N} \to \{0,1,2,3,4,5\}$ be defined by

$$\chi(i) \equiv \begin{cases} 2k \pmod 6 & \text{if } i \in [2^k, \lfloor p2^k \rfloor) \\ 2k+1 \pmod 6 & \text{if } i \in [\lfloor p2^k \rfloor, 2^{k+1}) \end{cases}$$

where $p = \sqrt{2}$. Assume $\{x, 2x+d, 2x+2d\}$ is a (2,2)-triple such that $\chi(x) = \chi(2x+d)$. We will show that $\chi(2x+2d) \neq \chi(x)$. We consider two cases.

`Case I:` $2^k \leq x < \lfloor p2^k \rfloor$ for some $k \in \{0,1,2,\ldots\}$. Since $\chi(x) = \chi(2x+d)$ and $2x+d > p2^k$, there exists an $m \in \mathbf{N}$ such that $2x+d \in [2^{k+3m}, \lfloor p2^{k+3m} \rfloor)$. Hence $d \in [2^{k+3m} - p2^{k+1}, p2^{k+3m} - 2^{k+1}]$. This yields

$$p2^{k+3m} \leq 2^{k+3m} + 2^{k+3m} - p2^{k+1} \leq 2x+2d < p2^{k+3m+1} - 2^{k+1} < 2^{k+3(m+1)}. \quad (2)$$

By (2) it follows that $\chi(2x+2d) \neq \chi(x)$.

`Case II:` $\lfloor p2^k \rfloor \leq x < 2^{k+1}$ for some $k \in \{0,1,2,\ldots\}$. As in Case I, there must exist an $m \in \{1,2,\ldots\}$ such that $2x+d \in [\lfloor p2^{k+3m} \rfloor, 2^{k+3m+1})$. Thus, $d \in [p2^{k+3m} - 2^{k+2} - 1, 2^{k+3m+1} - p2^{k+1}]$. Therefore,

$$2^{k+3m+1} \leq 2^{k+3m+1}(p - 2^{1-3m} - 2^{-k-3m}) \leq 2x+2d < 2^{k+3m+2} - p2^{k+1} < 2^{k+3(m+1)}. \quad (3)$$

It follows from (3) that $\chi(2x+2d) \neq \chi(x)$.

The proofs that $dor(3,3) \leq 5$ and $dor(3,4) \leq 5$ may be done in the same way as that for $dor(2,2)$ except that we use $p = \sqrt{3}$ instead of $p = \sqrt{2}$ and we use powers of 3 instead of 2 in the defined intervals. The cases of $(2,5)$, $(2,6)$, $(3,8)$, and $(3,9)$ are done similarly, where we use a 4-coloring rather than a 6-coloring, which is defined the same as χ except that "mod 6" is replaced by "mod 4;" where we take p to be 1.6, 1.5, 1.9, and 1.9, respectively; and where the powers in the defined intervals are powers of the given value of a. The case $(1,3)$ is done using a "mod 4" coloring with $p = \sqrt{3}$, where the powers in the given intervals of the coloring are powers of 3. $\square$

The following two results extend Theorems 2.4 and 2.5 to arbitrary r. It will be convenient to use the following notation. For b and r positive integers with $r \geq 2$, let

$$b_r = \begin{cases} 2\sum_{i=0}^{r-3} b^i & \text{if } r \geq 3 \\ 0 & \text{if } r = 2\ . \end{cases}$$

Proposition 3.1. Let $r \geq 2$. If $b \geq 2a$ then $N(a,b;r) \geq 2b^r + 5b^{r-1} - (2a-4)b^{r-2} + b_r$. If $b < 2a$ then $N(a,b;r) \geq 3b^r - (2a-5)b^{r-1} - (2a-4)b^{r-2} + b_r$.

Proof. Consider the case $b \geq 2a$. We use induction on r. The case in which $r = 2$ is proved in Theorem 2.4. Assuming $r \geq 3$ and that the result holds for $r-1$, there exists an $(r-1)$-coloring of $[1, 2b^{r-1}+5b^{r-2}-(2a-4)b^{r-3}+b_{r-1}-1]$ with no monochromatic (a,b)-triple. Color the interval $[2b^{r-1}+5b^{r-2}-(2a-4)b^{r-3}+b_{r-1}, 2b^r+5b^{r-1}-(2a-4)b^{r-2}+b_r-1]$ with the remaining color. By construction this r-coloring avoids monochromatic triples.

The case $b < 2a$ is quite similar and will be omitted. □

Proposition 3.2: $N(1,b;r) \geq 2b^r + 5b^{r-1} + 6b^{r-2} + b_r$ for all $b, r \geq 2$.

Proof. We use induction on r. The case $r = 2$ is proved in Theorem 2.5. Assuming $r \geq 3$ and that the inequality is true for $r-1$, we have the existence of an $(r-1)$-coloring of $[1, 2b^{r-1} + 5b^{r-2} + 6b^{r-3} + b_{r-1} - 1]$ which does not contain a monochromatic $(1,b)$-triple. Color the interval $[2b^{r-1} + 5b^{r-2} + 6b^{r-3} + b_{r-1}, 2b^r + 5b^{r-1} + 6b^{r-2} + b_r - 1]$ with the remaining color. It is an easy exercise to show that there is no monochromatic $(1,b)$-triple in this r-coloring. □

We conclude this section with a table which describes what is known about $dor(a,b)$ for some small values of a and b. By Theorem 2.1, we know that if $b \neq 2a$, then $dor(a,b) \geq 2$. In the third column of the table below we give the reason for the given upper bound on $dor(a,b)$.

Values of $\mathbf{dor(a, b)}$

(a,b)	$dor(a,b)$	reason
(1,1)	∞	van der Waerden's Theorem
(1,2)	1	Theorem 2.1
(1,3)	$2-3$	Theorem 3.2
(1,4)	$2-4$	Theorem 3.1
(1,5)	$2-5$	Theorem 3.1
(1,6)	$2-6$	Theorem 3.1
(1,7)	$2-6$	Theorem 3.1
(1,8)	$2-6$	Theorem 3.1
(1,9)	$2-7$	Theorem 3.1
(2,2)	$2-5$	Theorem 3.2
(2,3)	2	Theorem 3.1
(2,4)	1	Theorem 2.1
(2,5)	$2-3$	Theorem 3.2
(2,6)	$2-3$	Theorem 3.2
(2,7)	$2-4$	Theorem 3.1
(2,8)	$2-4$	Theorem 3.1
(2,9)	$2-5$	Theorem 3.1
(3,3)	$2-5$	Theorem 3.2
(3,4)	$2-5$	Theorem 3.2
(3,5)	2	Theorem 3.1
(3,6)	1	Theorem 2.1
(3,7)	$2-4$	Theorem 3.1
(3,8)	$2-3$	Theorem 3.2
(3,9)	$2-3$	Theorem 3.2

Table 2

4. A More General Question

In this section we move from (a, b)-triples to sets of the form $\{x, ax+d, ax+2d, \ldots, ax+(k-1)d\}$ for $a \geq 1$ and $k \geq 3$. Let us call such a set a *k-term a-progression.* For $a = 1$ these are simply the k-term arithmetic progressions. Van der Waerden's theorem states that given $r \geq 1$, any r-coloring of the natural numbers must contain arbitrarily long monochromatic arithmetic progressions. Theorem 4.1 shows that a similar result does *not* hold for $a > 1$ and $r > 3$. Let $dor_k(a)$ be the largest number r such that in any r-coloring of $\mathbf{N}$ there is a monochromatic k-term a-progression. Theorem 4.1 shows that for k large enough, $dor_k(a) \leq 3$.

Theorem 4.1: For all $a \geq 2$ and all integers $k \geq \frac{a^2}{a+1} + 2$, $dor_k(a) \leq 3$.

Proof. It suffices to exhibit a 4-coloring of $\mathbf{N}$ which avoids monochromatic k-term a-progressions. Clearly, we may assume $k = \left\lceil \frac{a^2}{a+1} \right\rceil + 2$.

Define a 4-coloring of $\mathbf{N}$ by coloring each interval $[a^j, a^{j+1})$ with the color $j \pmod 4$. We will show that there is no monochromatic k-term a-progression by showing that if x and $ax + d$ are the same color, then $ax + (k-1)d$ is a different color.

Let $x \in [a^i, a^{i+1})$, and assume $ax + d$ has the same color as x. Then clearly $ax + d \notin [a^i, a^{i+1})$. Hence, there exists an $m \in \mathbf{N}$ such that $ax + d \in [a^{i+4m}, a^{i+4m+1})$. From this we conclude that $a^i(a^{4m} - a^2) \leq d \leq a^{i+1}(a^{4m} - 1)$. To complete the proof we will show that the following inequalities hold:

$$ax + (k-1)d < a^{i+4(m+1)}, \tag{4}$$

$$a^{i+4m+1} \leq ax + (k-1)d. \tag{5}$$

From (4) and (5) we conclude that $ax + (k-1)d$ is colored differently than x and $ax + d$.

To prove (4), note that $k < a^3 + 1$ for all $a \geq 2$. Thus, $1 + (k-2)(1 - a^{-4m}) < a^3$, and hence $a^{i+4m+1} + (k-2)a^{i+1}(a^{4m} - 1) < a^{i+4(m+1)}$. This last inequality, together with the fact that $ax + (k-1)d = ax + d + (k-2)d \leq a^{i+4m+1} + (k-2)a^{i+1}(a^{4m} - 1)$, implies (4).

To prove (5), first note that since $k \geq a^2/(a+1) + 2$, we have $(k-2)(a^2 - 1) \geq a^3 - a^2$, and hence $(k-2)(a^2 - a^{-4(m-1)}) \geq a^3 - a^2$. Thus, $a^{i+4m} + (k-2)a^i(a^{4m} - a^2) \geq a^{i+4m+1}$. This last inequality, along with the fact that $ax + (k-1)d \geq a^{i+4m} + (k-2)a^i(a^{4m} - a^2)$, shows that (5) holds. □

According to Theorem 4.1, it is not true that every 4-coloring of $\mathbf{N}$ yields arbitrarily long monochromatic a-progressions. We are not sure if this holds for two or three colors. However, if for $r = 2$ or $r = 3$, every r-coloring of $\mathbf{N}$ does yield arbitrarily long monochromatic a-progressions, then a somewhat stronger result holds, as stated in Proposition 4.1 below. We omit the proof, a trivial generalization of the proof of [3, Theorem 2, p. 70].

Proposition 4.1 Let $a \in \mathbf{N}$ and let $r \in \{2, 3\}$. Assume that for every r-coloring of $\mathbf{N}$ there are arbitrarily long monochromatic a-progressions. Then for all $s \geq 1$ there exists $n = n(a, r, s)$ such that whenever $[1, n]$ is r-colored, then for all $k \in \mathbf{N}$ there exist $\hat{x}, \hat{d}$ so that $\{\hat{x}, a\hat{x} + \hat{d}, a\hat{x} + 2\hat{d}, \ldots, a\hat{x} + k\hat{d}\} \cup \{s\hat{d}\}$ is monochromatic.

5. Some Concluding Remarks

Although we have not proved that $dor(a, b) < \infty$ for general a, b, the results of this paper lead us to believe that this is the case for all $(a, b) \neq (1, 1)$. In particular, we make the following conjecture:

Conjecture: Let $a > 1$ and $r > 3$. Define $K(a, r)$ to be the least positive integer such that $dor_K(a) \leq r$. Then there exists $s > r$ such that $K(a, s) < K(a, r)$.

By Theorem 4.1, we know that $K(a, r)$ exists. Clearly, $K(a, s) \leq K(a, r)$ for $s \geq r$, but if we are able to show that the inequality is strict for some s, then we can conclude that $dor(a, a) < \infty$ for all $a > 1$. In fact it may be true that $dor(a, b) = 2$ for all $b \neq 2a$, although we have presented scant evidence for this.

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