ON MONOCHROMATIC ASCENDING WAVES

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Abstract

A sequence of positive integers $w_1, w_2, \ldots, w_n$ is called an ascending wave if $w_{i+1} - w_i \geq w_i - w_{i-1}$ for $2 \leq i \leq n-1$. For integers $k, r \geq 1$, let $AW(k;r)$ be the least positive integer such that under any r-coloring of $[1, AW(k;r)]$ there exists a k-term monochromatic ascending wave. The existence of $AW(k;r)$ is guaranteed by van der Waerden's theorem on arithmetic progressions since an arithmetic progression is, itself, an ascending wave. Originally, Brown, Erdős, and Freedman defined such sequences and proved that $k^2 - k + 1 \leq AW(k;2) \leq \frac{1}{3}(k^3 - 4k + 9)$. Alon and Spencer then showed that $AW(k;2) = \Theta(k^3)$. In this article, we show that $AW(k;3) = \Theta(k^5)$ as well as offer a proof of the existence of $AW(k;r)$ independent of van der Waerden's theorem. Furthermore, we prove that for any $\epsilon > 0$ and any fixed $r \geq 1$,

$$\frac{k^{2r-1-\epsilon}}{2^{r-1}(40r)^{r^2-1}}(1+o(1)) \leq AW(k;r) \leq \frac{k^{2r-1}}{(2r-1)!}(1+o(1)),$$

which, in particular, improves upon the best known upper bound for $AW(k;2)$. Additionally, we show that for fixed $k \geq 3$,

$$AW(k;r) \leq \frac{2^{k-2}}{(k-1)!}\, r^{k-1}(1+o(1)).$$

0. Introduction

A sequence of positive integers $w_1, w_2, \ldots, w_n$ is called an *ascending wave* if $w_{i+1} - w_i \geq w_i - w_{i-1}$ for $2 \leq i \leq n-1$. For $k, r \in \mathbb{Z}^+$, let $AW(k;r)$ be the least positive integer such that under any r-coloring of $[1, AW(k;r)]$ there exists a monochromatic k-term ascending wave. Although guaranteed by van der Waerden's theorem, the existence of $AW(k;r)$ can be proven independently, as we will show.

Bounds on $AW(k;2)$ have appeared in the literature. Brown, Erdős, and Freedman [2] showed that for all $k \geq 1$,

$$k^2 - k + 1 \leq AW(k;2) \leq \frac{k^3}{3} - \frac{4k}{3} + 3.$$

[1]This work was done as part of a high honor thesis in mathematics while the first author was an undergraduate at Colgate University, under the directorship of the second author.

Soon after, Alon and Spencer [1] showed that for sufficiently large k,

$$AW(k;2) > \tfrac{k^3}{10^{21}} - \tfrac{k^2}{10^{20}} - \tfrac{k}{10} + 4.$$

Recently, Landman and Robertson [4] proposed the refinement of the bounds on $AW(k;2)$ and the study of $AW(k;r)$ for $r \geq 3$. (Note: Since [4] concerns descending waves, we remark that in any finite interval, descending waves are ascending waves when we transverse the interval from right to left.) Here, we offer bounds on $AW(k;r)$ for all $r \geq 1$, improving upon the previous upper bound for $AW(k;2)$.

1. An Upper Bound

To show that $AW(k;r) \leq \Theta(k^{2r-1})$ is straightforward. We will first show that $AW(k;r) \leq k^{2r-1}$ by induction on r. The case $r=1$ is trivial; for $r \geq 2$, assume $AW(k;r-1) \leq k^{2r-3}$ and consider any r-coloring of $[1, k^{2r-1}]$. Set $w_1 = 1$ and let the color of 1 be red. In order to avoid a monochromatic k-term ascending wave there must exist an integer $w_2 \in [2, k^{2r-3}+1]$ that is colored red, lest the inductive hypothesis guarantee a k-term monochromatic ascending wave of some color other than red (and we are done). Similarly, there must be an integer $w_3 \in [w_2+(w_2-w_1), w_2+(w_2-w_1)+k^{2r-3}-1]$ that is colored red to avoid a monochromatic k-term ascending wave. Iterating this argument defines a monochromatic (red) k-term ascending wave $w_1, w_2, \ldots, w_k$, provided that $w_k \leq k^{2r-1}$. Since for $i \geq 2$, $w_{i+1} \leq w_i + (w_i - w_{i-1}) + k^{2r-3}$ we see that $w_{i+1} - w_i \leq ik^{2r-3}$ for $i \geq 1$. Hence, $w_k - w_1 = \sum_{i=1}^{k-1}(w_{i+1}-w_i) \leq \sum_{i=1}^{k-1} ik^{2r-3} \leq k^{2r-1} - 1$ and we are done.

In this section we provide a better upper bound. Our main theorem in this section follows.

Theorem 1.1 For fixed $r \geq 1$,

$$AW(k;r) \leq \frac{k^{2r-1}}{(2r-1)!}(1+o(1)).$$

We will prove Theorem 1.1 via a series of lemmas, but first we introduce some pertinent notation.

Notation For $k \geq 2$ and $M \geq AW(k;r)$, let $\Psi^M(r)$ be the collection of all r-colorings of $[1, M]$. For $\psi \in \Psi^M(r)$, let $\chi_k(\psi)$ be the set of all monochromatic k-term ascending waves under ψ. For each monochromatic k-term ascending wave $w = \{w_1, w_2, \ldots, w_k\} \in \chi_k(\psi)$, define the i^{th} *difference*, $d_i(w) = w_{i+1} - w_i$, for $1 \leq i \leq k-1$. For $\psi \in \Psi^M(r)$, define

$$\delta_k(\psi) = \min\{d_{k-1}(w) | w \in \chi_k(\psi)\},$$

i.e., the minimum last difference over all monochromatic k-term ascending waves under ψ. Lastly, define

$$\Delta_{k,r}^M = \max\{\delta_k(\psi) | \psi \in \Psi^M(r)\}.$$

These concepts will provide us with the necessary tools to prove Theorem 1.1.

We begin with an upper bound for $AW(k;r)$, which is the recusively defined function in the following definition.

Definition 1.2 For $k, r \geq 1$, let $M(k;1) = k$, $M(1;r) = 1$, $M(2;r) = r+1$, and define, for $k \geq 3$ and $r \geq 2$,

$$M(k;r) = M(k-1;r) + \Delta_{k-1,r}^{M(k-1;r)} + M(k;r-1) - 1.$$

Using this definition, we have the following result.

Lemma 1.3 For all $k, r \geq 1$, $AW(k;r) \leq M(k;r)$.

Proof. Noting that the cases $k + r = 2, 3,$ and 4 are, by definition, true, we proceed by induction on $k + r$ using $k + r = 5$ as our basis. We have $M(3;2) = 7$. An easy calculation shows that $AW(3;2) = 7$. So, for some $n \geq 5$, we assume Lemma 1.1 holds for all $k, r \geq 1$ such that $k+r = n$. Now, consider $k+r = n+1$. The result is trivial when $k = 1$ or 2, or if $r = 1$, thus we may assume $k \geq 3$ and $r \geq 2$. Let ψ be an r-coloring of $[1, M(k;r)]$. We will show that ψ admits a monochromatic k-term ascending wave, thereby proving Lemma 1.3.

By the inductive hypothesis, under ψ there must be a monochromatic $(k-1)$-term ascending wave $w = \{w_1, w_2, \ldots, w_{k-1}\} \subseteq [1, M(k-1;r)]$ with $d_{k-2}(w) \leq \Delta_{k-1,r}^{M(k-1;r)}$. Let

$$N = [w_{k-1} + \Delta_{k-1,r}^{M(k-1;r)}, w_{k-1} + \Delta_{k-1,r}^{M(k-1;r)} + M(k;r-1) - 1].$$

If there exists $q \in N$ colored identically to w, then $w \cup \{q\}$ is a monochromatic k-term ascending wave, since $q - w_{k-1} \geq \Delta_{k-1,r}^{M(k-1;r)} \geq d_{k-2}(w)$. If there is no such $q \in N$, then N contains integers of at most $r-1$ colors. Since $|N| = M(k;r-1)$, the inductive hypothesis guarantees that we have a monochromatic k-term ascending wave in N. As

$$w_{k-1} + \Delta_{k-1,r}^{M(k-1;r)} + M(k;r-1) - 1 \leq M(k;r),$$

the proof is complete. □

We now proceed to bound $M(k;r)$. We start with the following lemma.

Lemma 1.4 Let $k \geq 3$ and $r \geq 2$. Let $M(k;r)$ be as in Definition 1.2. Then

$$\Delta_{k,r}^{M(k;r)} \leq \Delta_{k-1,r}^{M(k-1;r)} + M(k;r-1) - 1.$$

Proof. Let ψ, w, and N be as defined in the proof of Lemma 1.3. If there exists $q \in N$ colored identically to w, then

$$\delta_k(\psi) \leq d_{k-1}(w \cup \{q\}) \leq \Delta_{k-1,r}^{M(k-1;r)} + M(k;r-1) - 1.$$

If there is no such $q \in N$, then there exists a monochromatic k-term ascending wave, say v, in N. Hence, $\delta_k(\psi) \le d_{k-1}(v) \le M(k; r-1) - (k-1)$. Since ψ was chosen arbitrarily, it follows that

$$\Delta_{k,r}^{M(k;r)} \le \Delta_{k-1,r}^{M(k-1;r)} + M(k; r-1) - 1.$$

□

The following lemma will provide a means for recursively bounding $M(k; r)$.

Lemma 1.5 Let $k \ge 3$ and $r \ge 2$. Let $M(k; r)$ be as in Definition 1.2. Then

$$M(k;r) \quad \le \sum_{i=0}^{k-3}((i+1)M(k-i; r-1)) - \frac{k^2}{2} + \frac{3k}{2} + (k-1)r.$$

Proof. We proceed by induction on k. Consider $M(3; r)$. We have

$$M(3; r) = M(2; r) + \Delta_{2,r}^{M(2;r)} + M(3; r-1) - 1.$$

Since $M(2; r) = r+1$ and $\Delta_{2,r}^{M(2;r)} = r$, we have $M(3; r) = M(3; r-1) + 2r$, thereby finishing the case $k = 3$ and arbitrary r. Now assume that Lemma 1.5 holds for some $k \ge 3$. The inductive hypothesis, along with Lemma 1.4, give us

$$\begin{aligned}
M(k+1; r) \quad &= M(k; r) + \Delta_{k,r}^{M(k;r)} + M(k+1; r-1) - 1 \\
&\le \sum_{i=0}^{k-3}((i+1)M(k-i; r-1)) - \frac{k^2}{2} + \frac{3k}{2} + (k-1)r \\
&\qquad + \Delta_{k,r}^{M(k;r)} + M(k+1; r-1) - 1 \\
&\le \sum_{i=0}^{k-3}((i+1)M(k-i; r-1)) - \frac{k^2}{2} + \frac{3k}{2} + (k-1)r \\
&\qquad + \Delta_{2,r}^{M(2;r)} + \textstyle\sum_{i=0}^{k-3} M(k-i; r-1) \\
&\qquad + M(k+1; r-1) - (k-2) - 1 \\
&\le \sum_{i=0}^{k-2}((i+1)M(k+1-i; r-1)) - \frac{(k+1)^2}{2} + \frac{3(k+1)}{2} + kr
\end{aligned}$$

as desired. □

Now, for $r \ge 2$, an upper bound on $M(k; r)$ can be obtained by using Lemma 1.5. We offer one additional lemma, from which Theorem 1.1 will follow by application of Lemma 1.3.

Lemma 1.6 For $r \ge 1$, there exists a polynomial $p_r(k)$ of degree at most $2r-2$ such that

$$M(k; r) \le \frac{k^{2r-1}}{(2r-1)!} + p_r(k)$$

for all $k \geq 3$.

Proof. We have $M(k;1) = k$, so we can take $p_1(k) = 1$, having degree 0. We proceed by induction on r. Let $r \in \mathbb{Z}^+$ and assume the lemma holds for r. Lemma 1.5 gives

$$\begin{aligned} M(k;r+1) &\leq \sum_{j=3}^{k}((k-j+1)M(j;r)) - \frac{k^2}{2} + \frac{3k}{2} + (k-1)(r+1) \\ &\leq k\sum_{j=3}^{k}\left(\frac{j^{2r-1}}{(2r-1)!} + p_r(j)\right) - \sum_{j=3}^{k}\left((j-1)\left(\frac{j^{2r-1}}{(2r-1)!} + p_r(j)\right)\right) \\ &\qquad -\frac{k^2}{2} + \frac{3k}{2} + (k-1)(r+1). \end{aligned}$$

By Faulhaber's formula [3], for some polynomial $p_{r+1}(k)$ of degree at most $2r$, we now have

$$M(k;r+1) \leq k\frac{\frac{k^{2r}}{2r}}{(2r-1)!} - \frac{\frac{k^{2r+1}}{2r+1}}{(2r-1)!} + p_{r+1}(k) = \frac{k^{2r+1}}{(2r+1)!} + p_{r+1}(k)$$

and the proof is complete. □

Theorem 1.1 now follows from Lemmas 1.3 and 1.6.

Interestingly, Lemma 1.5 can also be used to show the following result.

Theorem 1.7 For fixed $k \geq 3$,

$$AW(k;r) \leq \frac{2^{k-2}}{(k-1)!}\, r^{k-1}(1+o(1)).$$

Proof. In analogy to Lemma 1.6, we show that for $k \geq 3$ and $r \geq 2$, there exists a polynomial $s_k(r)$ of degree at most $k-2$ such that

$$M(k;r) \leq \frac{2^{k-2}}{(k-1)!}\, r^{k-1} + s_k(r). \tag{1}$$

We proceed by induction on k. Let $r \geq 2$ be arbitrary. By definition we have

$$M(3;r) = M(3;r-1) + 2r.$$

Since $M(3;1) = 3$, we get

$$M(3;r) = M(3;1) + \textstyle\sum_{i=2}^{r} 2i = r^2 + r + 1,$$

for $r \geq 2$, which serves as our basis. Now, for given $k \geq 4$, let $\hat{s}_3(r) = (k-1)r - \frac{k^2}{2} + \frac{3k}{2}$ and assume (1) holds for all integers $3 \leq j \leq k-1$ and for all $r \geq 2$. Lemma 1.5 yields

$$\begin{aligned} M(k;r) &\leq \sum_{i=0}^{k-3}((i+1)M(k-i;r-1)) + \hat{s}_3(r) \\ &= M(k;r-1) + \sum_{i=1}^{k-3}((i+1)M(k-i;r-1)) + \hat{s}_3(r). \end{aligned}$$

Now, by the inductive hypothesis, for $1 \le i \le k-3$, we have that

$$\begin{aligned} M(k-i;r-1) &\le \frac{2^{k-i-2}}{(k-i-1)!}\,(r-1)^{k-i-1} + s_{k-i}(r-1) \\ &= \frac{2^{k-i-2}}{(k-i-1)!}\,r^{k-i-1} + \tilde{s}_{k-i}(r)\,, \end{aligned}$$

where $\tilde{s}_{k-i}(r)$ is a polynomial of degree at most $k-i-2 \le k-3$. This gives us that

$$\begin{aligned} \sum_{i=1}^{k-3}(i+1)M(k-i;r-1) + \hat{s}_3(r) &\le \sum_{i=1}^{k-3}\left((i+1)\left(\frac{2^{k-i-2}}{(k-i-1)!}\,r^{k-i-1} + \tilde{s}_{k-i}(r)\right)\right) + \hat{s}_3(r) \\ &= 2\cdot\frac{2^{k-3}}{(k-2)!}\,r^{k-2} + \check{s}_{k-1}(r), \end{aligned}$$

where $\check{s}_{k-1}(r)$ is a polynomial of degree at most $k-3$. Hence, we have

$$\begin{aligned} M(k;r) &\le M(k;r-1) + 2\cdot\frac{2^{k-3}}{(k-2)!}\,r^{k-2} + \check{s}_{k-1}(r) \\ &= M(k;r-1) + \frac{2^{k-2}}{(k-2)!}\,r^{k-2} + \check{s}_{k-1}(r). \end{aligned}$$

As $M(k;1) = k$, we have a recursive bound on $M(k;r)$ for $r \ge 2$. Faulhaber's formula [3] yields

$$M(k;r) \le M(k;1) + \sum_{i=2}^{r}\left(\frac{2^{k-2}}{(k-2)!}i^{k-2} + \check{s}_{k-1}(r)\right) \le \frac{2^{k-2}}{(k-1)!}r^{k-1} + s_k(r),$$

where $s_k(r)$ is a polynomial of degree at most $k-2$. By Lemma 1.3, the result follows. □

2. A Lower Bound for more than Three Colors

We now provide a lower bound on $AW(k;r)$ for arbitrary fixed $r \ge 1$. We generalize an argument of Alon and Spencer [1] to provide our lower bound.

We will use $\log x = \log_2 x$ throughout. Also, by $k = x$ for $x \notin \mathbb{Z}^+$ we mean $k = \lfloor x \rfloor$.

We proceed by defining a certain type of random coloring. To this end, let $r \ge 2$ and consider the $r \times 2r$ matrix $A_0 = (a_{ij})$:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 & \dots & (r-1) & (r-1) \\ 0 & 1 & 1 & 2 & 2 & 3 & \dots & (r-1) & 0 \\ 0 & 2 & 1 & 3 & 2 & 4 & \dots & (r-1) & 1 \\ \vdots & & & & & \vdots & & \vdots & \vdots \\ 0 & (r-1) & 1 & 0 & 2 & 1 & \dots & (r-1) & (r-2) \end{bmatrix},$$

i.e., for $j \in [0, r-1]$, we have $a_{i,2j+1} = j$, for all $1 \leq i \leq r$, and $a_{i,2j+2} \equiv i + j - 1 \pmod{r}$.

Next, we define $A_j = A_0 \oplus \mathbf{j}$ where $\oplus$ means that entry-wise addition is done modulo r and $\mathbf{j}$ is the $r \times 2r$ matrix with all entries equal to j.

Consider the $r^2 \times 2r$ matrix $A = [A_0 \ A_1 \ A_2 \ \ldots \ A_{r-1}]^t$.

In the sequel, we will use the following notation.

Notation For $k, r \geq 1$, let

$$N_r = \frac{1}{2^{r-1}(40r)^{r^2-1}} \quad \text{and} \quad b = AW\left(\frac{k}{10(4r-4)}; r-1\right) - 1.$$

Furthermore, let γ_i be an $(r-1)$-coloring of $[1, b]$ with no monochromatic $\frac{k}{10(4r-4)}$-term ascending wave, where the $r-1$ colors used are $\{0, 1, \ldots, i-1, i+1, i+2, \ldots, r-1\}$ (i.e., color i is not used, and hence the subscript on γ).

Fix $\epsilon > 0$. We next describe how we randomly r-color $[1, M_\epsilon]$, where

$$M_\epsilon = N_r k^{2r-1-\epsilon}.$$

We partition the interval $[1, M_\epsilon]$ into consecutive intervals of length b and denote the i^{th} such interval by B_i and call it a *block* (note that the last block may be a block of length less than b). For $i = 1, 2, \ldots, \lceil \frac{M_\epsilon}{2rb} \rceil$, let

$$C_i = \bigcup_{j=1}^{2r} B_{2r(i-1)+j}.$$

For each C_i, we randomly choose a row in A, say $(s_1, s_2, \ldots, s_{2r})$. We color the j^{th} block of C_i by γ_{s_j}. By $col(B_i)$ we mean the coloring of the i^{th} block, $1 \leq i \leq \lceil \frac{M_\epsilon}{b} \rceil$, which is one of $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$. In the case when $2r \cdot \lceil \frac{M_\epsilon}{2rb} \rceil \neq \lceil \frac{M_\epsilon}{b} \rceil$, the j^{th} block (and block of length less than b, if present) of $C_{\lceil \frac{M_\epsilon}{2rb} \rceil}$ is colored by γ_{s_j} for all possible j (so that the entries in the row of A chosen for $C_{\lceil \frac{M_\epsilon}{2rb} \rceil}$ may not all be used).

The following is immediate by construction.

Lemma 2.1

(i) For all $1 \leq i \leq 2rb$, $P(col(B_i) = \gamma_c) = \frac{1}{r}$ for each $c = 0, 1, \ldots, r-1$.

(ii) For any i, $P(col(B_i) = \gamma_c \text{ and } col(B_{i+1}) = \gamma_d) = \frac{1}{r^2}$ for any c and d.

(iii) The colorings of blocks with at least $2r$ other blocks between them are mutually independent.

The approach we take, following Alon and Spencer [1], is to show that there exists a coloring such that for any monochromatic $\frac{k}{2}$-term ascending wave $w_1, w_2, \ldots, w_{k/2}$ we have

$w_{k/2} - w_{k/2-1} \geq \delta k^{2r-2-\epsilon/2}$ for some $\delta > 0$. The following definition and lemma, which are generalizations of those found in [1], will give us the desired result.

Definition 2.2 An arithmetic progression $x_1 < x_2 < \cdots < x_t$ is called a *good progression* if for each $c \in \{0, 1, \ldots, r-1\}$, there exist i and j such that $x_i \in B_j$ and $col(B_j) = col(B_{j+1}) = \gamma_c$. An arithmetic progression that is not good is called a *bad progression.*

Lemma 2.3 For $k, r \geq 2$, let $t = \frac{(4r-2)(2r+1)}{\log(r^2/(r^2-1))} \log k + \frac{(2r+1)(\log r+1)}{\log(r^2/(r^2-1))}$. For k sufficiently large, the probability that there is a bad progression in a random coloring of $[1, M_\epsilon]$ with difference greater than b, of t terms, is at most $\frac{1}{2}$.

Proof. Let $x_1 < x_2 < \cdots < x_t$ be a progression with $x_2 - x_1 > b$. Then no 2 elements belong to the same block. For each i, $1 \leq i \leq \frac{t}{2r+1}$, let D_i be the block in which $x_{(2r+1)i}$ resides, and let E_i be the block immediately following D_i. Then, the probability that the progression is bad is at most

$$p = \sum_{j=1}^{r} P\left(\nexists\, i \in \left[1, \frac{t}{2r+1}\right] : col(D_i) = col(E_i) = \gamma_j\right).$$

We have

$$\begin{aligned} p &\leq rP\left(\nexists\, i \in \left[1, \tfrac{t}{2r+1}\right] : col(D_i) = col(E_i) = \gamma_0\right) \\ &= r\left(\frac{r^2-1}{r^2}\right)^{\frac{t}{2r+1}} \\ &\leq r\left(\frac{r^2-1}{r^2}\right)^{\frac{(4r-2)}{\log(r^2/(r^2-1))}\log k + \frac{\log r+1}{\log(r^2/(r^2-1))}} \\ &\leq \frac{2^{-1}}{k^{4r-2}} \end{aligned}$$

for k sufficiently large.

Since the number of t-term arithmetic progressions in $[1, M_\epsilon]$ is less than $M_\epsilon^2 < k^{4r-2}$, the probability that there is a bad progression is less than

$$k^{4r-2} \cdot \frac{2^{-1}}{k^{4r-2}} = \frac{1}{2},$$

thereby completing the proof. □

Lemma 2.4 Consider any r-coloring of $[1, M_\epsilon]$ having no bad progression with difference greater than b of t terms (t from Lemma 2.3). Then, for any $\epsilon > 0$, for k sufficiently large, any monochromatic $\frac{k}{2}$-term ascending wave $w_1, w_2, \ldots, w_{k/2}$ has $w_{k/2} - w_{k/2-1} \geq bk^{1-\epsilon/2} = \Theta(k^{2r-2-\epsilon/2})$.

Proof. At most $4r-4$ consecutive blocks can have a specific color in all of them. (To achieve this, say the color is 0. The random coloring must have chosen row 1 followed by row $r+1$, to have $\gamma_0\gamma_0\gamma_1\gamma_1 \cdots \gamma_{r-1}\gamma_{r-1}\gamma_1\gamma_1\gamma_2\gamma_2 \cdots \gamma_0\gamma_0$.) Since each block has a monochromatic ascending wave of length at most $\frac{k}{10(4r-4)} - 1$, any $4r-4$ consecutive blocks contribute less than $\frac{k}{10}$ terms to a monochromatic ascending wave. After that, the next difference must be more than b.

Let $Z = a_1, a_2, \ldots, a_{k/2}$ be monochromatic ascending wave under our random coloring. Then, there exists $i < \frac{k}{10}$ such that $a_{i+1} - a_i \geq b+1$. Now let $X = x_1, x_2, \ldots, x_t$ be a t-term good progression with $x_1 = a_i$ and $d = x_2 - x_1 = a_{i+1} - a_i \geq b+1$.

Assume, without loss of generality, that the color of Z is 0. Since X is a good progression, there exists $x_j \in B_\ell$ with $col(B_\ell) = col(B_{\ell+1}) = \gamma_0$ for some block B_ℓ. Since $a_{i+j} \geq x_j$ as Z is an ascending wave, we see that $a_{i+j} - a_i \geq jd + b + 1$. We conclude that $a_{i+t} - a_i \geq td + b + 1$ so that $a_{i+t+1} - a_{i+t} \geq d + \frac{b+1}{t}$. Now, redefine $X = x_1, x_2, \ldots, x_t$ to be the t-term good progression with $x_1 = a_{i+t}$ and $d' = x_2 - x_1 = a_{i+t+1} - a_{i+t} \geq d + \frac{b+1}{t} \geq (b+1)\left(1 + \frac{1}{t}\right)$. Repeating the above argument, we see that $a_{i+2t} - a_{i+t} \geq td' + b + 1$ so that $a_{i+2t} - a_{i+2t-1} \geq d' + \frac{b+1}{t} \geq (b+1)\left(1 + \frac{2}{t}\right)$. In general,

$$a_{i+st} - a_{i+st-1} \geq (b+1)\left(1 + \frac{s}{t}\right)$$

for $s = 1, 2, \ldots \frac{2k-5t}{5t}$. Thus, we have (with $s = (k^{1-\epsilon/2} - 1)t \leq \frac{2k-5t}{5t}$ for k sufficiently large)

$$a_{k/2} - a_{k/2-1} \geq (b+1)\left(1 + \frac{(k^{1-\epsilon/2} - 1)t}{t}\right) = (b+1)k^{1-\epsilon/2}.$$

□

We are now in a position to state and prove this section's main result.

Theorem 2.5 For fixed $r \geq 1$ and any $\epsilon > 0$, for k sufficiently large,

$$AW(k;r) \geq \frac{k^{2r-1-\epsilon}}{2^{r-1}(40r)^{r^2-1}}.$$

Proof. Fix $\epsilon > 0$ and let $M_\epsilon = N_r k^{2r-1-\epsilon}$ for $r \geq 1$. We use induction on r, with $r = 1$ being trivial (since $AW(k;1) = k$) and $r = 2$ following from Alon and Spencer's result [1]. Hence, assume $r \geq 3$ and assume the theorem holds for $r-1$. Using Lemma 2.4, there exists an r-coloring χ of $[1, M_\epsilon]$ such that any monochromatic $\frac{k}{2}$-term ascending wave has last difference at least $(b+1)k^{1-\epsilon/2}$. This implies that the last term of any monochromatic k-term ascending wave under χ must be at least $\frac{k}{2} + (b+1)k^{1-\epsilon/2} \cdot \frac{k}{2} > \frac{1}{2}(b+1)k^{2-\epsilon/2}$.

We have, by the inductive hypothesis and the definition of b,

$$b+1 \geq N_{r-1}\frac{k^{2r-3-\epsilon/2}}{40^{2r-3-\epsilon/2}(r-1)^{2r-3-\epsilon/2}} \geq N_{r-1}\frac{k^{2r-3-\epsilon/2}}{40^{2r-1}r^{2r-1}}.$$

Hence, for k sufficiently large, the last term of any monochromatic k-term ascending wave under χ must be greater than

$$N_{r-1} \cdot \frac{1}{40^{2r-1}r^{2r-1}}k^{2r-3-\epsilon/2} \cdot \frac{k^{2-\epsilon/2}}{2} = N_r k^{2r-1-\epsilon} = M_\epsilon.$$

Hence, we have an r-coloring of $[1, M_\epsilon]$ with no k-term monochromatic ascending wave, for k sufficiently large. $\square$

3. A Lower Bound for Three Colors

We believe that $AW(k;r) = \Theta(k^{2r-1})$, however, we have thus far been unable to prove this. The approach of Alon and Spencer [1], which is to show that there exists an r-coloring (under a random coloring scheme) such that every monochromatic $\frac{3k}{4}$-term ascending wave has $d_{3k/4-1} > ck^{2r-2}$ does not work for an arbitrary number of colors with our generalization. However, for 3 colors, we can refine their argument to prove that $AW(k;3) = \Theta(k^5)$.

Theorem 3.1

$$\frac{k^5}{2^{13} \cdot 10^{39}} \leq AW(k;3) \leq \frac{k^5}{120}(1 + o(1))$$

The upper bound comes from Theorem 1.1, hence we need only prove the lower bound. We use the same coloring scheme as in Section 2 and proceed with a series of lemmas.

Definition 3.2 We call a sequence $x_1, x_2, \ldots, x_n$ with $x_2 - x_1 \geq 1$ an *almost ascending wave* if, for $2 \leq i \leq n-1$, we have $d_i = x_{i+1} - x_i$ with $d_i \geq d_{i-1} - 1$, with equality for at least one such i and with the property that if $d_i = d_{i-1} - 1$ and $d_j = d_{j-1} - 1$ with $j > i$ there must exist s, $i < s < j$, such that $d_s \geq d_{s-1} + 1$.

The upper bound of the following proposition is a slight refinement of a result of Alon and Spencer [1, Lemma 1.7].

Proposition 3.3 Denote by $aw(n)$ the number of ascending waves of length n with first term given and $d_{n-1} < \frac{n}{10^{14}}$. Analogously, let $aaw(n)$ be the number of almost ascending waves of length n with first term given and $d_{n-1} \leq \frac{n}{10^{14}}$. Then, for all n sufficiently large,

$$2^{\frac{n}{2}-1} < aw(n) + aaw(n) \leq 2^{\frac{13n}{25}} \cdot \left(\frac{3}{2}\right)^{n/100}.$$

Proof. We start with the lower bound by constructing a sequence of differences that contribute to either $aw(n)$ or $aaw(n)$. We start by constructing a sequence where all of $\frac{n}{2} - 1$ slots contain 2 terms of a sequence. From a list of $\frac{n}{2} - 1$ empty slots, choose j, $0 \leq j \leq \frac{n}{2} - 1$, of them. In these slots place the pair $-1, 1$. In the remaining slots put the pair $0, 0$. We now have a sequence of length $n-2$ or $n-3$. If the length is $n-2$, put a 2 at the end; if the length is $n-3$, put $2, 2$ at the end. We now have, for each j and each choice of j slots, a distinct sequence of length $n-1$. Denote one such sequence by $s_1, s_2, \ldots, s_{n-1}$. Using this sequence, we define a sequence of difference $\{d_i\}$ that will correspond to either an ascending wave or an almost ascending wave. To this end, let $d_1 = 1$ and $d_i = d_{i-1} + s_{i-1}$ for $i = 2, 3, \ldots, n$. Since we have the first term of an almost ascending, or ascending, wave $w_1, \ldots, w_n$ given, such a wave is determined by its sequence of differences $w_{i+1} - w_i$. Above, we have constructed a sequence $\{d_i\}$ of differences that adhere to the rules of an almost ascending, or ascending, wave. Hence, $aw(n;r) + aaw(n;r) > \sum_{j=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{j} = 2^{\frac{n}{2}-1}$.

For the upper bound, we follow the proof of Alon and Spencer [1, Lemma 1.7], improving the bound enough to serve our purpose. Their lemma includes the term $\binom{n+\lceil 10^{-6}n\rceil-1}{n-1}$ which we will work on to refine their upper bound on $aw(n) + aaw(n)$.

First, we have

$$\binom{n+\lceil 10^{-6}n\rceil - 1}{n-1} \leq \binom{(1+10^{-5})n}{n}$$

for n sufficiently large.

Let $q = (1+10^{-5})^{-1}$, $m = \frac{n}{q}$, and let $H(x) = -x\log x - (1-x)\log(1-x)$ for $0 \leq x \leq 1$ be the binary entropy function. Then we have[2]

$$\binom{m}{qm} \leq 2^{mH(q)}.$$

Applying this, we have

$$H(q) \quad = \tfrac{1}{1+10^{-5}} \log(1+10^{-5}) - \tfrac{10^{-5}}{1+10^{-5}} \log \tfrac{10^{-5}}{1+10^{-5}}$$

so that

$$\begin{aligned} mH(q) &= \left[\log(1+10^{-5}) - \tfrac{1}{10^5} \log \tfrac{10^{-5}}{1+10^{-5}}\right] n \\ &= \left[\tfrac{1}{10^5} \log 10^5 (1+10^{-5})^{10^5+1}\right] n \\ &\leq \left[\tfrac{1}{10^5} \log e(10^5+1)\right] n. \end{aligned}$$

We proceed by noting that

$$\left[\frac{\log e(10^5+1)}{10^5}\right] n \leq \left[\frac{1}{100} \log \frac{3}{2}\right] n.$$

Hence, $2^{mH(q)} \leq 2^{\frac{n}{100}\log\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{n}{100}}$. Now, using Alon and Spencer's result [1, Lemma 1.7], the result follows. □

We are now in a position to prove the fundamental lemma of this section. In the proof we refer to the following definition.

[2] Here's a quick derivation: For all $n \geq 1$, we have $\sqrt{2\pi n}e^{1/(12n+1)}(n/e)^n \leq n! \leq \sqrt{2\pi n}e^{1/(12n)}(n/e)^n$ (see [5]). Hence, $\binom{m}{qm} \leq \frac{c}{\sqrt{m(1-q)}} \left(q^{-q}(1-q)^{-(1-q)}\right)^m$ for some positive $c < e^{-2}$ (so that $\frac{c}{\sqrt{m(1-q)}} < 1$ for m sufficiently large). Using the base 2 log, this gives $\binom{m}{qm} \leq 2^{mH(q)}$.

Definition 3.3 Let $a_1, \ldots, a_n$ be an ascending wave and let $x \in \mathbb{Z}^+$. We call $\lfloor \frac{a_1}{x} \rfloor, \lfloor \frac{a_2}{x} \rfloor, \cdots, \lfloor \frac{a_n}{x} \rfloor$ the associated *x-floor wave.*

Lemma 3.4 Let

$$Q = \frac{k^5}{2^{13} \cdot 10^{39}}$$

and let $b = AW(k/80; 2) - 1$. The probability that in a random 3-coloring of $[1, Q]$ there is a monochromatic $\frac{k}{4}$-term ascending wave whose first difference is greater than $6b$ $(= 2rb)$ and whose last difference is smaller than $\frac{kb}{4 \cdot 10^{14}} = \Theta(k^4)$ is less than $\frac{1}{2}$ for k sufficiently large.

Proof. Let $Y = a_1 < a_2 < \cdots < a_{k/4}$ be an ascending wave and let $\lfloor \frac{a_1}{b} \rfloor < \lfloor \frac{a_2}{b} \rfloor < \cdots < \lfloor \frac{a_{k/4}}{b} \rfloor$ be the associated b-floor wave. Note that this b-floor wave is either an ascending wave or an almost ascending wave with last difference at most $\frac{k/4}{10^{14}}$. Hence, by Proposition 3.2, the number of such b-floor waves is at most, for k sufficiently large,

$$k^2 \cdot 2^{\frac{13k}{100}} \cdot \left(\frac{3}{2}\right)^{k/400} \leq 2^{\frac{14k}{100}} \cdot \left(\frac{3}{2}\right)^{k/400}$$

(we have less than k^2 choices for $\lfloor \frac{a_1}{b} \rfloor$).

Note that Y is monochromatic of color, say c, only if none of the blocks $B_{\lfloor \frac{a_i}{b} \rfloor}$, $1 \leq i \leq \frac{k}{4}$, is colored by γ_c. Note that all of these blocks are at least $6 (= 2r)$ blocks from each other. We use Lemma 2.1 to give us that the probability that Y is monochromatic is no more than

$$3\left(\frac{2}{3}\right)^{k/4}.$$

Thus, the probability that in a random 3-coloring of $[1, Q]$ we have a monochromatic $\frac{k}{4}$-term ascending wave with last difference less than $\frac{kb}{4 \cdot 10^{14}}$ is at most

$$3 \cdot 2^{\frac{14k}{100}} \cdot \left(\frac{2}{3}\right)^{99k/400}.$$

We have $3 < \left(\frac{3}{2}\right)^{3k/400}$ for k sufficiently large, so that the above probability is less than

$$2^{\frac{14k}{100}} \cdot \left(\frac{2}{3}\right)^{24k/100}.$$

The above quantity is, in particular, less than 1/2 for k sufficiently large. □

To finish proving Theorem 3.1, we apply Lemmas 2.3 and 2.4, as well as Lemma 3.5, to show that, for k sufficiently large, there exists a 3-coloring of $[1, Q]$ such that both of the following hold:

1) Any $\frac{k}{2}$-term monochromatic ascending wave has last difference greater than $6b (= 2rb)$.

2) Any $\frac{k}{4}$-term monochromatic ascending wave with first difference greater than $6b(= 2rb)$ has last difference greater than $\frac{kb}{4\cdot 10^{14}}$.

Hence, we conclude that there is a 3-coloring of $[1, Q]$ such that any monochromatic $\frac{3k}{4}$-term ascending wave has last difference greater than $\frac{kb}{4\cdot 10^{14}}$, for k sufficiently large. This implies that the last term of a monochromatic k-term ascending wave must be at least $\frac{3k}{4} + \frac{kb}{4\cdot 10^{14}} \cdot \frac{k}{4}$.

We have $b = AW\left(\frac{k}{10(4r-4)}; r-1\right) - 1$ with $r = 3$. By Alon and Spencer's result [1], this gives us

$$b \geq \frac{k^3}{10^{25} \cdot 8^3}$$

for k sufficiently large.

Hence, for k sufficiently large, the last term of a monochromatic k-term ascending wave must be at least

$$\frac{3k}{4} + \frac{k^2}{4^2 \cdot 10^{14}} \cdot \frac{k^3}{10^{25} \cdot 8^3} > \frac{k^5}{2^{13} \cdot 10^{39}} = Q.$$

Since we have the existence of a 3-coloring of $[1, Q]$ with no monochromatic k-term ascending wave, this completes the proof of Theorem 3.1.

Remark From the lower bound given in Proposition 3.3, it is not possible to show that there exists $c > 0$ such that $AW(k; r) \geq ck^{2r-1}$ for $r \geq 4$, by using the argument presented in Sections 2 and 3. However, we still make the following conjecture.

Conjecture For all $r \geq 1$, $AW(k; r) = \Theta(k^{2r-1})$.

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