

# Difference Ramsey Numbers and Issai Numbers

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## Abstract

We present a recursive algorithm for finding good lower bounds for the classical Ramsey numbers. Using notions from this algorithm we then give some results for a generalization of the Generalized Schur numbers, which we call Issai numbers.

## 1 Introduction

We present two main ideas in this paper. The first pertains to the classical Ramsey numbers, where we give a recursive algorithm for finding the (to be defined) difference Ramsey numbers. Using the notions developed in this first part we then define Issai numbers, a generalization of the Generalized Schur numbers (see [BB]). We prove their existence and give some basic results for these Issai numbers.

Let  $G$  be a graph. Then  $E(G)$  is the edge set of  $G$ , and  $V(G)$  is the set of vertices of  $G$ . Let  $r \geq 2$  and  $k_i \geq 2$  for  $i \in \{1, 2, \dots, r\}$ . Recall that the Ramsey number  $N = R(k_1, k_2, \dots, k_r)$  is the minimal integer with the following property.

**Ramsey Property:** *Let  $\chi : E(K_N) \rightarrow \{1, 2, \dots, r\}$  be an  $r$ -coloring of the complete graph on  $N$  vertices. Then for some  $j \in \{1, 2, \dots, r\}$  there exists a  $K_{k_j}$  with  $V(K_{k_j}) \subset V(K_N)$  such that  $\chi(E(K_{k_j})) = \{j\}$ .*

To find a lower bound,  $L$ , for one of these Ramsey numbers, it suffices to find an edgewise coloring of  $K_L$  which avoids the Ramsey property. To this end, we will restrict our search to the subclass of *difference graphs* (defined below). After presenting some results, we will show that the Issai numbers are a natural consequence of the difference Ramsey numbers, and a natural extension of the Generalized Schur numbers.

The Difference Ramsey Numbers section of this article is accompanied by the Maple package `AUTORAMSEY`. It has also been translated into Fortran77 and is available as `DF.f`. The Issai Numbers section (section 4) of this article is accompanied by the Maple package `ISSAI`. All computer packages are available for download at the author's website (<http://math.colgate.edu/~aaron/>).

## 2 Difference Ramsey Numbers

Our goal here is to find good lower bounds for the classical Ramsey numbers. Hence, we wish to find edgewise colorings of complete graphs which avoid the Ramsey Property. Our approach is to construct a recursive algorithm to find the best possible colorings among those colorings we search. Since searching *all* possible colorings of a complete graph on any nontrivial number of vertices is not feasible by today's computing standards, we must restrict the class of colored graphs to be searched. The class of graphs we will search will be the class of *difference graphs*.

**Definition 2.1:** *r-colored Difference Graph.* Let  $r \in \mathbf{N}$ . Consider the complete graph on  $n$  vertices,  $K_n$ . Number the vertices 1 through  $n$ . Partition  $\{1, 2, \dots, n - 1\}$  arbitrarily into  $r$  classes,  $C_1, C_2, \dots, C_r$ . Let  $i < j$  be two vertices of  $K_n$ . Using colors  $1, 2, \dots, r$  color the edges of  $K_n$  as follows: determine  $t$  so that  $j - i \in C_t$  ( $t \in \{1, 2, \dots, r\}$ ). Color the edge connecting  $i$  and  $j$  with color  $t$ . Repeat this procedure for all pairs  $(i, j)$ . The resulting colored graph will be called an *r-colored difference graph*.

We now focus our attention on the 2-colored difference graphs (using blue and red as our colors). Given  $k$  and  $l$ , a 2-colored difference graph with the maximal number of vertices which avoids both a blue  $K_k$  and a red  $K_l$  will be called a *maximal 2-colored difference Ramsey graph*, or simply a maximal difference Ramsey graph when no confusion arises. Let the number of vertices of a maximal difference Ramsey graph be  $V$ . Then we will define the *2-colored difference Ramsey number*, denoted  $D(k, l)$ , to be  $V + 1$ . Further, since the class of difference graphs is a subclass of all 2-colored complete graphs, we have  $D(k, l) \leq R(k, l)$ . Hence, by finding the difference Ramsey numbers, we are finding lower bounds for the classical Ramsey numbers.

Before we present the computational aspect of these difference Ramsey numbers, we note that  $D(k, l) \leq D(k - 1, l) + D(k, l - 1)$ , which is analogous to the upper bound derived from Ramsey's proof [GRS p. 3], does *not* follow from Ramsey's proof. (This is easily shown and will be omitted.) Hence it is not clear as to whether or not  $D(k, k)$  has the same asymptotic behavior as  $R(k, k)$ . This may lead to an improved asymptotic bound for  $R(k, k)$ . However, in this paper we are concerned with small Ramsey numbers, for which  $D(k, l)$  appears to behave very much like  $R(k, l)$ .

The set of difference graphs is a superclass of the often searched circular (or cyclic) graphs (see the survey [CG] by Chung and Grinstead), which are similarly defined. Let  $n$  be the number of vertices of a graph. The distinction is that for a colored graph to be circular we require that  $b$  and  $n - b$  must have the same color. By removing this circular condition, we remove from the coloring the dependence on  $n$ , and can thereby construct a recursive algorithm to find the set of maximal difference Ramsey graphs.

The recursive step in the algorithm is described as follows. A *good* 2-colored difference graph on  $n$  vertices consists of a set of blue differences,  $B_n$ , and a set of red differences,

$R_n$ , for which the Ramsey Property is avoided. Note that  $B_n \dot{\cup} R_n = \{1, 2, 3, \dots, n-1\}$ . To obtain a *good* difference graph on  $n+1$  vertices, we consider the difference  $d = n$ . If  $B_n \cup \{d\}$  avoids a blue clique, then we have a *good* difference graph on  $n+1$  vertices where  $B_{n+1} = B_n \cup \{d\}$  and  $R_{n+1} = R_n$ . Likewise, if  $R_n \cup \{d\}$  avoids a red clique, then we have a different *good* difference graph on  $n+1$  vertices with  $B_{n+1} = B_n$  and  $R_{n+1} = R_n \cup \{d\}$ . Hence, we have a simple recursion which is not possible with circular graphs. We can now use our recursive algorithm to find *automatically* (and we must note theoretically due to time and memory constraints, but much less time and memory than would be required to search all colored graphs) all maximal difference Ramsey graphs for any given  $k$  and  $l$ .

## 2.1 About the Maple Package AUTORAMSEY

AUTORAMSEY is a Maple package that *automatically* computes the maximal difference graph(s) for a given  $k$  and  $l$ . Hence, this package automatically finds lower bounds for the Ramsey number  $R(k, l)$ . In the spirit of automation, AUTORAMSEY can create a verification Maple program tailored to the maximal graph(s) calculated in AUTORAMSEY (that can be run at your leisure) and can write a L<sup>A</sup>T<sub>E</sub>X paper giving the lower bound for the Ramsey number  $R(k, l)$  along with a maximal difference graph.

The computer generated program is an independent, straightforward program that can be used to (double) check that the results obtained in AUTORAMSEY do indeed avoid both a blue  $K_k$  and a red  $K_l$ . AUTORAMSEY has also been translated into Fortran77 as DF.f to speed up the algorithm implementation.

## 2.2 The Algorithm

Below we give the pseudocode which produces the maximal difference Ramsey graph(s) for a given  $k$  and  $l$ . Hence, it finds the exact value of the difference Ramsey number  $D(k, l)$ . Let  $n$  be the number of vertices of a graph to be searched. Because the number of difference graphs is of order  $2^n$  as compared to  $2^{n^2/2}$  for all 2-colored graphs, the algorithm can feasibly work on larger Ramsey numbers.

Let  $\mathcal{D}_n$  be the class of 2-colored difference graphs on  $n$  vertices. Let **GoodSet** be the set of difference graphs that avoid both a blue  $K_k$  and a red  $K_l$ .

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Let  $m = \min(k, 1)$ 
Find  $D_{m-1}$ , our starting point.
Set GoodSet =  $D_{m-1}$ .
Set  $j = m - 1$ 
WHILE flag  $\neq 0$  do

    FOR  $i$  from 1 to  $| \text{GoodSet} |$  do

        Take  $T \in \text{GoodSet}$ , where  $T$  is of the form  $T = [B_j, R_j]$ ,
            where  $B_j$  and  $R_j$  are the blue and red difference sets on  $j$  vertices
        Consider  $S_B = [B_j \cup \{j\}, R_j]$  and  $S_R = [B_j, R_j \cup \{j\}]$ 
        If  $S_B$  avoids a blue  $K_k$  then
            NewGoodSet := NewGoodSet  $\cup S_B$ 
        If  $S_R$  avoids a red  $K_l$  then
            NewGoodSet := NewGoodSet  $\cup S_R$ 

        Repeat FOR loop with a new  $T$ 

    If  $| \text{NewGoodSet} | = 0$  then RETURN GoodSet and set flag = 0
    Otherwise, set GoodSet = NewGoodSet, NewGoodSet =  $\{\}$ , and  $j = j + 1$ 

Repeat WHILE loop

```

For this algorithm to be efficient we must have the subroutine which checks whether or not a monochromatic  $K_k$  (or  $K_l$ ) is avoided be very quick. We use the following lemma to achieve quick results in the Fortran77 code. (The Maple code is mainly for independently checking (with a different, much slower but more straightforward, algorithm) the Fortran77 code for small cases.)

**Lemma 2.1:** Define the binary operation  $*$  to be  $x * y = |x - y|$ . Let  $D$  be a set of differences. Consider  $D$  to be one of the classes associated with an  $r$ -colored difference graph. If  $D$  induces a monochromatic  $K_k$ , then there exists  $K \subset D$ , with  $|K| = k - 1$ , such that for all  $x, y \in K$ ,  $x * y \in D$ .

*Proof.* Let  $\{v_0, v_1, \dots, v_{k-1}\}$  be the vertices of the monochromatic  $K_k$ . Let  $d_i = v_i - v_0$ . Let  $K = \{d_1, d_2, \dots, d_{k-1}\}$ . We must show that for all  $d_i, d_j \in K$  with  $i \neq j$ ,  $d_i * d_j \in D$ . Assume not; i.e. there exists  $I < J$  such that  $d_I * d_J \notin D$ . Since  $v_J - v_I = (d_J - d_0) - (d_I - d_0) = d_I * d_J$ , the edge connecting vertices  $I$  and  $J$  cannot have the color defined by  $D$ , a contradiction.  $\square$

By using this lemma we need only check *pairs* of elements in a  $k$ -set, rather than constructing all possible colorings using the  $k$ -set. Further, we need not worry about the ordering of the pairs; the operation  $*$  is commutative.

## 2.3 Some Results

It is easy to find lower bounds for  $R(k, l)$ , so we must show that the algorithm gives “good” lower bounds. Below are two tables of the 2-colored difference Ramsey number results obtained so far. The first table is of the 2-colored difference Ramsey number values. The second table is of the number of different maximal difference Ramsey graphs produced by `DF.f`. If we are considering the diagonal difference Ramsey number  $D(k, k)$ , then the number of maximal difference graphs takes into account the symmetry of colors; i.e. we do not count a reversal of colors as a different difference graph. Where lower bounds are listed we have made constraints on the size of the set `GoodSet` in the algorithm due to memory and/or (self-imposed) time restrictions.

**Difference Ramsey Numbers**

$l$	3	4	5	6	7	8	9	10	11
$k$									
3	6	9	14	17	22	27	36	39	46
4		18	25	34	47	$\geq 53$	$\geq 62$		
5			42	$\geq 57$					

**Number of Maximal Difference Ramsey Graphs**

$l$	3	4	5	6	7	8	9	10	11
$k$									
3	1	2	3	7	13	13	4	21	6
4		1	6	24	21	n/a	n/a		
5			11	n/a					

When we compare our test results to the well known maximal Ramsey graphs for  $R(3, 3)$ ,  $R(3, 4)$ ,  $R(3, 5)$ ,  $R(4, 4)$  [GG], and  $R(4, 5)$  [MR], we find that the program has found maximal colorings for all of these numbers. The classical coloring in [GRS] for  $R(3, 4)$  is not a difference graph, and hence is not found by the program. More importantly, however, is that the program does find a difference graph on 8 vertices that avoids both a blue  $K_3$  and a red  $K_4$ . Hence, for the Ramsey numbers found by Gleason and

Greenwood [GG], and for  $R(4, 5)$  found by McKay and Radziszowski [MR] we have found maximal difference Ramsey graphs which are also maximal Ramsey graphs.

The algorithm presented above can be trivially extended to search  $r$ -colored difference graphs for  $r > 2$ . The progress made so far in this direction follows.

### 3 Multicolored Difference Ramsey Numbers

The algorithm presented here can be applied to an arbitrary number of colors. The recursive step in the algorithm simply becomes the addition of the next difference to each of the three color set  $B_n$ ,  $R_n$ , and  $G_n$  ( $G$  for green). Everything else remains the same. Hence, the alteration of the program to any number of colors is a simple one. The main hurdle encountered while searching difference graphs of more than two colors is that the size of the set `GoodSet` in the algorithm grows *very* quickly. In fact, the system's memory (1 GB) while fully searching all difference graphs was consumed within seconds for most multicolored difference Ramsey numbers. The results obtained so far are given below.

$$\begin{aligned} D(3, 3, 3) &= 15 \\ D(3, 3, 4) &= 30 \\ D(3, 3, 5) &= 42 \\ D(3, 3, 6) &\geq 60 \end{aligned}$$

We note here that  $D(3, 3, 6) \geq 60$  implies that  $R(3, 3, 6) \geq 60$ , which is a new result. The previous best lower bound was 54 [SLZL]. The coloring on 59 vertices is circular, hence we need only list the differences up to 29:

Color 1: 5,12,13,14,16,20,22

Color 2: 10,15,19,24,26,27

Color 3: 1,2,3,4,6,7,8,9,11,17,18,21,23,25,28,29

#### 3.1 Future Directions

Currently, the algorithm which searches for the maximal difference Ramsey graphs is a straightforward search. In other words, if the memory requirement exceeds the space in the computer, the algorithm will only return a lower bound. In the future this algorithm should be adapted to backtrack searches and/or network searching.

## 4 Issai Numbers

We now turn our attention to an easy consequence of the  $r$ -colored Difference Ramsey Numbers. Issai Schur proved (in 1916) the following theorem which is considered the first Ramsey-type theorem to spark activity in Ramsey Theory. (The first Ramsey-type theorem is generally acknowledged to be due to Hilbert.)

**Schur's Theorem:** *Given  $r \in \mathbf{N}$ , there exists an integer  $N = N(r)$  such that any  $r$ -coloring of the integers 1 through  $N$  must admit a monochromatic solution to  $x + y = z$ .*

We may extend this to the following theorem.

**Theorem 4.1:** Given  $r \in \mathbf{N}$  and  $k \geq 2$ , there exists an integer  $N = N(r, k)$  such that any  $r$ -coloring of the integers 1 through  $N$  must admit a monochromatic solution to  $\sum_{i=1}^{k-1} x_i = x_k$ .

This is not a new theorem. In fact it is a special case of Rado's Theorem [see, for example, GRS p. 56]. The numbers  $N(r, k)$  are called the Generalized Schur Numbers and have been studied in [BB]. In this article, we present a simple proof which relies only on the notions already presented in this paper.

*Proof.* Consider the  $r$ -colored difference Ramsey number  $N = D(k, k, \dots, k)$ . Then any  $r$ -coloring of  $K_N$  must admit a monochromatic  $K_k$  subgraph. Let the vertices of this subgraph be  $\{v_0, v_1, \dots, v_{k-1}\}$ , with the differences  $d_i = v_i - v_0$ . By ordering and renaming the vertices we may assume that  $d_1 < d_2 < \dots < d_{k-1}$ . Since  $K_k$  is monochromatic, we have that the edges  $\overline{v_{i-1}v_i}$ ,  $i = 1, 2, \dots, k-1$ , and  $\overline{v_{k-1}v_0}$  must all be the same color. Since the  $r$ -colored  $K_N$  is a difference graph we have that  $(d_{i+1} - d_i)$ ,  $i = 1, 2, \dots, k-1$ ,  $d_1$ , and  $d_{k-1}$  must all be assigned the same color. Hence we have the monochromatic solution  $d_1 + \sum_{i=1}^{k-2} (d_{i+1} - d_i) = d_{k-1}$ , in any  $r$ -coloring of  $\{1, 2, \dots, N-1\}$ .  $\square$

Using the proof of Theorem 4.1 we will define the *Issai numbers*. But first, another definition is in order.

**Definition 4.1:** *Schur  $k$ -tuple.* We will call a  $k$ -tuple,  $(x_1, x_2, \dots, x_k)$ , a *Schur  $k$ -tuple* if  $\sum_{i=1}^{k-1} x_i = x_k$ .

In the case where  $k = 3$ , the 3-tuple  $(x, y, x + y)$  is called a Schur triple. In Schur's theorem the only parameter is  $r$ , the number of colors. Hence, a Schur number is defined to be the minimal integer  $S = S(r)$  such that any  $r$ -coloring of the integers 1 through  $S$  must contain a monochromatic Schur triple. The Schur numbers have been generalized in [BB] and [S] in directions different from what will be presented below. We will extend the Schur numbers in the same fashion as the Ramsey numbers were extended from  $R(k, k)$  to  $R(k, l)$  (in the proof of Ramsey's theorem).

**Definition 4.2:** *Issai Number.* Let  $S = S(k_1, k_2, \dots, k_r)$  be the minimal integer such that any  $r$ -coloring of the integers from 1 to  $S$  must have a monochromatic (of color  $i$ ) Schur  $k_i$ -tuple, for some  $i \in \{1, 2, \dots, r\}$ .  $S$  will be called an *Issai number*. Unlike Theorem 4.1, the existence of Issai numbers is not directly implied by Rado's Theorem. However, existence of these Issai numbers is implied by the existence of the  $r$ -colored difference Ramsey numbers  $D(k_1, k_2, \dots, k_r)$ . In fact, we have the following result.

**Lemma 4.1:**  $S(k_1, k_2, \dots, k_r) \leq D(k_1, k_2, \dots, k_r) - 1$

*Proof.* By definition, there exists a minimal integer  $N = D(k_1, k_2, \dots, k_r)$  such that any  $r$ -coloring of  $K_N$  must contain a monochromatic (of color  $i$ )  $K_{k_i}$ , for some  $i \in \{1, 2, \dots, r\}$ . Using the reasoning in the proof of Theorem 4.1 we have the stated inequality.  $\square$

Using this new notation, it is already known that  $S(3, 3) = 5$ ,  $S(3, 3, 3) = 14$ , and  $S(3, 3, 3, 3) = 45$ . We note here that since  $D(3, 3, 3) = 15$  we immediately have  $S(3, 3, 3) \leq 14$ , whereas before, since  $R(3, 3, 3) = 17$ , we had only that  $S(3, 3, 3) \leq 16$ . Attempts to find general bounds for  $S(k, l)$  have so far been unsuccessful.

## 4.1 Some Issai Values and Colorings

We used the Maple package **ISSAI** to calculate the exact values as well as an exceptional coloring given below. **ISSAI** is written for two colors, but can easily be extended to any number of colors. The value  $S(3, 3) = 5$  has been known since before Schur proved his theorem. The value  $S(4, 4) = 11$  follows from Beutelspacher and Brestovansky in [BB], who more generally show that  $S(k, k) = k^2 - k - 1$ . The remaining values are new.

**Issai Numbers**

$l$	3	4	5	6	7
$k$					
3	5	7	11	13	17
4		11	14		

The maximal colorings which avoid monochromatic sums found by **ISSAI** are as follows. Let  $S(k, l)$  denote the minimal number such that any 2-coloring of the integers from 1 to  $S(k, l)$  must contain either a red Schur  $k$ -tuple or a blue Schur  $l$ -tuple. It is enough to list only those integers colored red.



$S(3, 4) > 6$ : Red: 1,6  
 $S(3, 5) > 10$ : Red: 1,3,8,10  
 $S(4, 4) > 10$ : Red: 1,2,9,10  
 $S(3, 6) > 12$ : Red: 1,3,10,12  
 $S(4, 5) > 13$ : Red: 1,2,12,13  
 $S(3, 7) > 16$ : Red: 1,3,5,12,14,16

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