SOME TWO COLOR, FOUR VARIABLE RADO NUMBERS

Aaron Robertson
Department of Mathematics, Colgate University, Hamilton, NY 13346
aaron@math.colgate.edu

and

Kellen Myers[1]

Abstract

There exists a minimum integer N such that any 2-coloring of $\{1, 2, \ldots, N\}$ admits a monochromatic solution to $x+y+kz = \ell w$ for $k, \ell \in \mathbb{Z}^+$, where N depends on k and ℓ. We determine N when $\ell - k \in \{0, 1, 2, 3, 4, 5\}$, for all k, ℓ for which $\frac{1}{2}((\ell-k)^2-2)(\ell-k+1) \leq k \leq \ell - 4$, as well as for arbitrary k when $\ell = 2$.

1. Introduction

For $r \geq 2$, an r-coloring of the positive integers $\mathbb{Z}^+$ is an assignment $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$. Given a diophantine equation $\mathcal{E}$ in the variables $x_1, \ldots, x_n$, we say a solution $\{\bar{x}_i\}_{i=1}^n$ is monochromatic if $\chi(\bar{x}_i) = \chi(\bar{x}_j)$ for every i, j pair. A well-known theorem of Rado states that a linear homogeneous equation $c_1x_1 + \ldots + c_nx_n = 0$ with each $c_i \in \mathbb{Z}$ admits a monochromatic solution in $\mathbb{Z}^+$ under any r-coloring of $\mathbb{Z}^+$, for any $r \geq 2$, if and only if some nonempty subset of $\{c_i\}_{i=1}^n$ sums to zero. Such an equation is said to satisfy Rado's regularity condition. The smallest N such that any r-coloring of $\{1, 2, \ldots, N\} = [1, N]$ satisfies this condition is called the r-color Rado number for the equation $\mathcal{E}$. Rado also proved the following, much lesser known, result.

Theorem 1 (*Rado [R]*) Let $\mathcal{E} = 0$ be a linear homogeneous equation with integer coefficients. Assume that $\mathcal{E}$ has at least 3 variables with both positive and negative coefficients. Then any 2-coloring of $\mathbb{Z}^+$ admits a monochromatic solution to $\mathcal{E} = 0$.

For Rado's original proof (in German) see [R]; for a proof in English see [MR].

In this article we study the equation $x + y + kz = \ell w$ for positive integers k and ℓ. As such, we make the following notation.

Notation For k a positive integer and $j > -k$ an integer, let $\mathcal{E}(k, j)$ represent the equation

$$x + y + kz = (k + j)w.$$

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2. A General Upper Bound

Definition Let $\mathcal{E}$ be any equation that satisfies the conditions in Theorem 1. Denote by $RR(\mathcal{E})$ the minimum integer N such that any 2-coloring of $[1,N]$ admits a monochromatic solution to $\mathcal{E}$.

Part of the following result is essentially a result due to Burr and Loo [BL] who show that, for $j \geq 4$, we have $RR(x+y=jw) = \binom{j+1}{2}$. Their result was never published. Below we present our (independently derived) proof.

Theorem 2 Let $k,j \in \mathbb{Z}^+$ with $j \geq 4$. Then $RR(\mathcal{E}(k,j)) \leq \binom{j+1}{2}$. Furthermore, for all $k \geq \frac{(j^2-2)(j+1)}{2}$, we have $RR(\mathcal{E}(k,j)) = \binom{j+1}{2}$.

Proof. Let $\mathcal{F}$ denote the equation $x+y=jw$. We will show that $RR(\mathcal{F}) \leq \binom{j+1}{2}$. Since any solution to $\mathcal{F}$ is also a solution to $\mathcal{E}(k,j)$ for any $k \in \mathbb{Z}^+$, the first statement will follow.

Assume, for a contradiction, that there exists a 2-coloring of $\left[1,\binom{j+1}{2}\right]$ with no monochromatic solution to $\mathcal{F}$. Using the colors red and blue, we let R be the set of red integers and B be the set of blue integers. We denote solutions of $\mathcal{F}$ by (x,y,w) where $x,y,w \in \mathbb{Z}^+$.

Since $(x,y,w) = (j-1,1,1)$ solves $\mathcal{F}$, we may assume that $1 \in R$ and $j-1 \in B$. We separate the proof into two cases.

`Case 1.` $j+1 \in B$. Assume $i \geq 1$ is red. Considering $(1, ij-1, i)$ gives $ij-1 \in B$. If $i \leq \left\lfloor \frac{j+1}{2} \right\rfloor$, this, in turn, gives us $i+1 \in R$ by considering $(ij-1, j+1, i+1)$. Hence $1,2,\ldots,\left\lfloor \frac{j+1}{2} \right\rfloor + 1$ are all red. But then $\left(\left\lfloor \frac{j}{2} \right\rfloor, \left\lfloor \frac{j+1}{2} \right\rfloor, 1\right)$ is a red solution, a contradiction.

`Case 2.` $j+1 \in R$. This implies that $j\binom{j+1}{2} \in B$. Note also that the solutions $(1,j-1,1)$ and $\left(j\binom{j-1}{2}, j\binom{j-1}{2}, j-1\right)$ give us $j-1 \in B$ and $j\binom{j-1}{2} \in R$.

First consider the case when j is even. By considering $\left(\frac{j}{2},\frac{j}{2},1\right)$ we see that $\frac{j}{2} \in B$. Assume, for $i \geq 1$, that $\frac{(2i-1)j}{2} \in B$. Considering $\left(j\binom{j+1}{2}, \frac{(2i-1)j}{2}, \frac{j}{2}+i\right)$ we have $\frac{j}{2}+i \in R$. This, in turn, implies that $\frac{(2i+1)j}{2} \in B$ by considering $\left(j\binom{j-1}{2}, \frac{(2i+1)j}{2}, \frac{j}{2}+i\right)$. Hence we have $\frac{j}{2}+i$ is red for $1 \leq i \leq \frac{j}{2}$. This gives us that $j-1 \in R$ (when $i = \frac{j}{2}-1$), a contradiction.

Now consider the case when j is odd. We consider two subcases.

`Subcase i.` $j \in B$. For $i \geq 1$, assume that $ij \in B$. We obtain $\frac{j+1}{2}+i \in R$ by considering the solution $\left(j\binom{j+1}{2}, ij, \frac{j+1}{2}+i\right)$. This gives us $(i+1)j \in B$ by considering $\left(j\binom{j-1}{2}, (i+1)j, \frac{j+1}{2}+i\right)$ Hence, we have that $j, 2j, \ldots, \binom{j+1}{2} j$ are all blue, contradicting the deduction that $j\binom{j-1}{2} \in R$.

`Subcase ii.` $j \in R$. We easily have $2 \in B$. Next, we conclude that $j\binom{j-3}{2} \in R$ by considering $\left(j\binom{j+1}{2}, j\binom{j-3}{2}, j-1\right)$. Then, the solution $\left(j\binom{j-3}{2}, j\binom{j-1}{2}, j-2\right)$ gives us $j-2 \in B$. We use $\left(j\binom{j-1}{2}, j, \frac{j+1}{2}\right)$ to see that $\frac{j+1}{2} \in B$. From $\left(j\binom{j+1}{2}-2, 2, \frac{j+1}{2}\right)$ we have $j\binom{j+1}{2}-2 \in R$. To avoid $\left(j\binom{j-1}{2}+2, j\binom{j+1}{2}-2, j\right)$ being a red solution, we have $j\binom{j-1}{2}+2 \in B$. This gives us a contradiction; the solution $\left(j\binom{j-1}{2}+2, j-2, \frac{j+1}{2}\right)$ is blue.

This completes the proof of the first statement of the theorem.

For the proof of the second statement of the theorem, we need only provide a lower bound of $\binom{j+1}{2}-1$. We first show that any solution to $x+y+kz=(k+j)w$ with $x,y,z,w<\binom{j+1}{2}$ must have $z=w$ when $k \geq \frac{(j^2-2)(j+1)}{2}$. Assume, for a contradiction, that $z \neq w$. If $z<w$, then $(k+j)w \geq (k+j)(z+1) > kz+k$. However, $x+y<j(j+1)$ while $k>j(j+1)$ for $j \geq 3$. Hence, $z \not< w$. If $z>w$, then $(k+j)w \leq k(z-1)+j\left(\binom{j+1}{2}-1\right)$. Since we have $x+y+kz=(k+j)w$ we now have $2+kz \leq x+y+kz \leq k(z-1)+j\left(\binom{j+1}{2}-1\right)$. Hence, $k \leq j\left(\binom{j+1}{2}-1\right)-2$, contradicting the given bound on k. Thus, $z=w$.

Now, any solution to $x+y+kz=(k+j)w$ with $z=w$ is a solution to $x+y=jw$. From Burr and Loo's result, there exists a 2-coloring of $\left[1,\binom{j+1}{2}-1\right]$ with no monochromatic solution to $x+y=jw$. This provides us with a 2-coloring with no monochromatic solution to $x+y+kz=(k+j)w$, thereby finishing the proof of the second statement. □

3. Some Specific Numbers

In this section we determine the exact values for $RR(\mathcal{E}(k,j))$ for $j \in \{0,1,2,3,4,5\}$, most of which are cases not covered by Theorem 2.

Theorem 3 For $k \geq 2$,

$$RR(\mathcal{E}(2k,0)) = 2k \text{ and } RR(\mathcal{E}(2k-1,0)) = 3k-1.$$

Furthermore $RR(\mathcal{E}(2,0)) = 5$ and $RR(\mathcal{E}(1,0)) = 11$.

Proof. The cases $RR(\mathcal{E}(2,0)) = 5$, $RR(\mathcal{E}(4,0)) = 4$, and $RR(\mathcal{E}(3,0) = 5$ are easy calculations, as is $RR(\mathcal{E}(1,0)) = 11$, which first appeared in [BB]. Hence, we may assume $k \geq 3$ in the following arguments.

We start with $RR(\mathcal{E}(2k,0)) = 2k$. To show that $RR(\mathcal{E}(2k,0)) \geq 2k$ consider the 2-coloring of $[1,2k-1]$ defined by coloring the odd integers red and the even integers blue. To see that there is no monochromatic solution to $x+y+2kz=2kw$, note that we must have $2k \mid (x+y)$. This implies that $x+y=2k$ since $x,y \leq 2k-1$. Thus, $w=z+1$. However, no 2 consecutive integers have the same color. Hence, any solution to $\mathcal{E}(2k,0)$ is necessarily bichromatic.

Next, we show that $RR(\mathcal{E}(2k,0)) \leq 2k$. Assume, for a contradiction, that there exists a 2-coloring of $[1,2k]$ with no monochromatic solution to our equation. Using the colors red and blue, we may assume, without loss of generality, that k is red. This gives us that $k-1$ and $k+1$ are blue, by considering $(x,y,z,w)=(k,k,k-1,k)$ and $(k,k,k,k+1)$. Using these in the solution $(2k,2k,k-1,k+1)$ we see that $2k$ must be red, which implies that $k-2$ is blue (using $(2k,2k,k-2,k)$). However, this gives the blue solution $(k-1,k+1,k-2,k-1)$, a contradiction.

We move on to $RR(\mathcal{E}(2k-1,0))$. To show that $RR(\mathcal{E}(2k-1,0)) \leq 3k-1$ consider the following 2-colorings of $[1, 3k-2]$, dependent on k (we use r and b for red and blue, respectively):

$$\begin{array}{ll} brrbbrrbb \dots bbr & \text{if } k \equiv 0 \pmod 4 \\ rrbbrrbb \dots bbr & \text{if } k \equiv 1 \pmod 4 \\ brrbbrrbb \dots rrb & \text{if } k \equiv 2 \pmod 4 \\ rrbbrrbb \dots rrb & \text{if } k \equiv 3 \pmod 4. \end{array}$$

Since we need $(2k-1) \mid (x+y)$ and $x, y \leq 3k-2$, we have $x+y \in \{2k-1, 4k-2\}$. By construction, if $x+y = 2k-1$, then x and y have different colors. Hence, the only possibility is $x+y = 4k-2$. But then $w = z+2$ and we see that w and z must have different colors.

Next, we show that $RR(\mathcal{E}(2k-1,0)) \leq 3k-1$. Assume, for a contradiction, that there exists a 2-coloring of $[1, 3k-1]$ with no monochromatic solution to our equation. Using the colors red and blue, we may assume, without loss of generality, that $2k-1$ is red. To avoid $(2k-1, 2k-1, z, z+2)$ being a red solution, we see that $2k+1$ and $2k-3$ are blue (using $z = 2k-1$ and $2k-3$, respectively).

If $2k$ is red, then $2k-2$ is blue (using $(2k-2, 2k, 2k-2, 2k)$). From $(3k-1, 3k-1, 2k-2, 2k+1)$ we see that $3k-1$ is red. This, in turn, implies that $2k-4$ is blue (using $(3k-1, 3k-1, 2k-4, 2k-1)$). So that $(2k-4, 2k+2, 2k-4, 2k-2)$ is not a blue solution, we require $2k+2$ to be red. But then $(2k-1, 2k-1, 2k, 2k+2)$ is a red solution, a contradiction.

If $2k$ is blue, then $2k-2$ must be red. So that $(3k-1, 3k-1, 2k-3, 2k)$ is not a blue solution, we have that $3k-1$ is red. Also, $2k+2$ must be red by considering $(2k-3, 2k+1, 2k, 2k+2)$. But this implies that $(3k-1, 3k-1, 2k-1, 2k+2)$ is a red solution, a contradiction. □

We proceed with a series of results for the cases $j = 1, 3, 4, 5$. When $j = 2$, the corresponding number is trivially 1 for all $k \in \mathbb{Z}^+$.

Below, we will call a coloring of $[1, n]$ *valid* if it does not contain a monochromatic solution to $\mathcal{E}(k, j)$.

Theorem 4 For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k,1)) = \begin{cases} 4 & \text{for } k \leq 3 \\ 5 & \text{for } k \geq 4. \end{cases}$$

Proof. Assume, for a contradiction, that there exists a 2-coloring of $[1, 5]$ with no monochromatic solution to $x + y + kz = (k+1)w$. We may assume that 1 is red. Considering the solutions $(1,1,2,2), (2,2,4,4), (1,3,4,4)$, and $(2,3,5,5)$, in order, we find that 2 is blue, 4 is red, 3 is blue, and 5 is red. But then $(1,4,5,5)$ is a red solution, a contradiction. Hence, $RR(\mathcal{E}(k,1)) \leq 5$ for all $k \in \mathbb{Z}^+$.

We see from the above argument that the only valid colorings of $[1, 3]$ (assuming, without loss of generality, that 1 is red) are rbr and rbb (where we use r for red and b for blue). Furthermore, the only valid coloring of $[1, 4]$ is $rbbr$. We use these colorings to finish the proof.

First consider the valid coloring rbr. The possible values of $x + y + kz$ when x, y, z are all red form the set $\{k+2, k+4, k+6, 3k+2, 3k+4, 3k+6\}$. The possible values when x, y, z are all blue is $2k+4$. The possible values of $(k+1)w$ when w is red form the set $\{k+1, 3k+3\}$; when w is blue, $2k+2$ is the only possible value. We denote these results by:

$$\begin{aligned}
R_{x,y,z} &= \{k+2, k+4, k+6, 3k+2, 3k+4, 3k+6\} \\
B_{x,y,z} &= \{2k+4\} \\
R_w &= \{k+1, 3k+3\} \\
B_w &= \{2k+2\}.
\end{aligned}$$

Next, we determine those values of k, if any, for which $R_{x,y,z} \cap R_w \neq \emptyset$ or $B_{x,y,z} \cap B_w \neq \emptyset$. Clearly, there is no such k for these sets. Hence, we conclude that $RR(\mathcal{E}(k,1)) \geq 4$ for all k. (We need not consider the valid coloring rbb since we now know that $RR(\mathcal{E}(k,1)) \geq 4$ for all k.)

We move on to the valid coloring of $[1,4]$, which is $rbbr$. We find that

$$\begin{aligned}
R_{x,y,z} &= \{k+2, k+5, k+8, 4k+2, 4k+5, 4k+8\} \\
B_{x,y,z} &= \{2k+4, 2k+5, 2k+6, 3k+4, 3k+5, 3k+6\} \\
R_w &= \{k+1, 4k+4\} \\
B_w &= \{2k+2, 3k+3\}.
\end{aligned}$$

We see that $B_{x,y,z} \cap B_w \neq \emptyset$ when $k=1$ $(2k+4 = 3k+3)$, $k=2$ $(2k+5 = 3k+3)$, and $k=3$ $(2k+6 = 3k+3)$. For all other values of k, $B_{x,y,z} \cap B_w = \emptyset$ and $R_{x,y,z} \cap R_w = \emptyset$. Hence, we conclude that $RR(\mathcal{E}(k,1)) \geq 5$ for $k \geq 3$, while, since $rbbr$ is the only valid coloring of $[1,4]$, $RR(\mathcal{E}(k,1)) \leq 4$ for $k=1,2,3$. This completes the proof of the theorem. □

The proofs below refer to the small Maple package `FVR`. The description of `FVR` is given in Section 3.1, which follows the next 3 theorems.

Theorem 5 For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k,3)) = \begin{cases} 4 & \text{for } k \leq 5 \text{ and } k=7 \\ 6 & \text{for } k = 8, 11 \\ 9 & \text{for } k = 6, 9, 10 \text{ and } k \geq 12 \end{cases}$$

Proof. The method of proof is the same as that for Theorem 4, but we will work it out in some detail commenting on the use of the Maple package `FVR`.

It is easy to check that the only valid 2-colorings (using r for red, b for blue, and assuming that 1 is red) of $[1,n]$ for $n = 4, 5, \ldots, 8$ are as in the following table. The determinations of

$R_{x,y,z}, R_w, B_{x,y,z}$, and B_w are equally easy.

n	coloring	sets
4	*rbrr*	$R_{x,y,z} = \{ik+j : i = 1,3,4; j = 2,4,5,6,7,8\}; R_w = \{i(k+3) : i = 1,3,4\}$
		$B_{x,y,z} = \{2k+2\}; B_w = \{2k+6\}$
5	*rbrrb*	$R_{x,y,z} = \{ik+j : i = 1,3,4; j = 2,4,5,6,7,8\}; R_w = \{i(k+3) : i = 1,3,4\}$
		$B_{x,y,z} = \{ik+j : i = 2,5; j = 2,4,7,10\}; B_w = \{2k+6, 5k+15\}$
6	*rbrrbb*	$R_{x,y,z} = \{ik+j : i = 1,3,4; j = 4,5,6,7,8\}; R_w = \{i(k+3) : i = 1,3,4\}$
		$B_{x,y,z} = \{ik+j : i = 2,5,6; j = 4,7,8,10,11,12\}; B_w = \{2k+6, 5k+15, 6k+18\}$
7	*rbrrbbr*	$R_{x,y,z} = \{ik+j : i = 1,3,4,7; j = 4,\ldots,8,10,11,14\}; R_w = \{i(k+3) : i = 1,3,4,7\}$
		$B_{x,y,z} = \{ik+j : i = 2,5,6; j = 4,7,8,10,11,12\}; B_w = \{2k+6, 5k+15, 6k+18\}$
8	*rbrrbbrb*	$R_{x,y,z} = \{ik+j : i = 1,3,4,7; j = 4,\ldots,8,10,11,14\}; R_w = \{i(k+3) : i = 1,3,4,7\}$
		$B_{x,y,z} = \{ik+j : i = 2,5,6; j = 4,7,8,10,\ldots,14,16\}; B_w = \{i(k+3) : i = 2,5,6,8\}$

The sets $R_{x,y,z}, R_w, B_{x,y,z}$, and B_w are automatically found by `FVR`, which then gives us the values of k that induce a nonempty intersection of either $R_{x,y,z} \cap R_w$ or $B_{x,y,z} \cap B_w$. For completeness, we give the details.

For the coloring *rbrr*, we have $R_{x,y,z} \cap R_w \neq \emptyset$ when $k = 1$ $(k+3 = 3k+5)$, $k = 2$ $(3k+9 = 4k+7)$, $k = 3$ $(3k+9 = 4k+6)$, $k = 4$ $(3k+9 = 4k+5)$, $k = 5$ $(3k+9 = 4k+4)$, and $k = 7$ $(3k+9 = 4k+2)$. Since *rbrr* is the only valid coloring of $[1,4]$, we have $RR(\mathcal{E}(k,3)) = 4$ for $k = 1,2,3,4,5,7$.

For the coloring *rbrrb*, we have no new additional elements in $R_{x,y,z} \cap R_w$. Hence, any possible additional intersection point comes from $B_{x,y,z} \cap B_w$. However, $B_{x,y,z} \cap B_w = \emptyset$ for all $k \in \mathbb{Z}^+$. Hence, $RR(\mathcal{E}(k,3)) \geq 6$ for $k \in \mathbb{Z}^+ \setminus \{1,2,3,4,5,7\}$.

For *rbrrbb*, we again have no new additional elements in the red intersection. We do, however, have additional elements in $B_{x,y,z} \cap B_w$. When $k = 8$ $(5k+15 = 6k+7)$ and $k = 11$ $(5k+15 = 6k+4)$ we have a blue intersection. We conclude that $RR(\mathcal{E}(k,3)) = 6$ for $k = 8,11$ and $RR(\mathcal{E}(k,3)) \geq 7$ for $k \in \mathbb{Z}^+ \setminus \{1,2,3,4,5,7,8,11\}$.

Considering *rbrrbbr*, we have no new additional elements in $B_{x,y,z} \cap B_w$. Furthermore, we have no new additional intersection points in $R_{x,y,z} \cap R_w$. Hence, $RR(\mathcal{E}(k,3)) \geq 8$ for $k \in \mathbb{Z}^+ \setminus \{1,2,3,4,5,7,8,11\}$.

Lastly, we consider *rbrrbbrb*, which gives no new additional elements in $R_{x,y,z} \cap R_w$. Furthermore, we have no new additional intersection points in $B_{x,y,z} \cap B_w$. Thus, $RR(\mathcal{E}(k,3)) \geq 9$ for $k \in \mathbb{Z}^+ \setminus \{1,2,3,4,5,7,8,11\}$.

Analyzing the valid coloring of $[1,8]$ we see that we cannot extend it to a valid coloring of $[1,9]$. Hence, $RR(\mathcal{E}(k,3)) \leq 9$ for all k so that $RR(\mathcal{E}(k,3)) = 9$ for $k \in \mathbb{Z}^+ \setminus \{1,2,3,4,5,7,8,11\}$. □

Theorem 6 For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k,4)) = \begin{cases} 3 & \text{for } k = 2,3,4 \\ 5 & \text{for } k = 6,7,8,10,11,14 \\ 6 & \text{for } k = 5,9,12,13,15,18 \\ 8 & \text{for } k = 17,19,22 \\ 9 & \text{for } k = 1,23,24 \\ 10 & \text{for } k = 16,20,21 \text{ and } k \geq 25. \end{cases}$$

Proof. Use the Maple package `FVR` with the following valid colorings (which are easily obtained):

n	valid colorings
3	rbb
4	$rbbr$
5	$rbbrr$
6	$rbbrrr$
7	$rbbrrrr, rbbrrrb$
8	$rbbrrrrb, rbbrrrbb$
9	$rbbrrrrbr, rbbrrrbbr$
10	none

Note that if $[1,n]$ has more than one valid coloring, we can conclude that $RR(\mathcal{E}(k,4)) \leq n$ for $k = \hat{k}$ only if $\hat{k}$ is an intersection point for *all* valid colorings. Otherwise, there exists a coloring of $[1,n]$ that avoids monochromatic solutions to $\mathcal{E}(k,4)$ when $k = \hat{k}$. □

Theorem 7 For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k,5)) = \begin{cases} 4 & \text{for } k = 1,2,3 \\ 6 & \text{for } k = 4,13,14 \\ 7 & \text{for } k = 16,17,18,23 \\ 8 & \text{for } 5 \leq k \leq 12 \text{ and } k = 21 \\ 10 & \text{for } k = 19,24,26,27,28,29,33 \\ 11 & \text{for } k = 22,30,31,32,34,36,37,38,39,41,42,43,48 \\ 12 & \text{for } k = 15,35,44,46,47,53 \\ 13 & \text{for } k = 51,52 \\ 15 & \text{for } k = 20,25,40,45,49,50 \text{ and } k \geq 54. \end{cases}$$

Proof. Use the Maple package `FVR` with the following valid colorings (which are easily obtained):

n	valid colorings
4	$rrrb, rrbb, rbrb, rbbb$
5	$rrbbb, rbrbr, rbbbr$
6	$rrbbbr, rbrbrr, rbbbrr$
7	$rrbbbrr, rrbbbrb, rbrbrrr, rbrbrrb, rbbbrrr$
8	$rrbbbrrb, rbrbrrbr, rbbbrrrr$
9	$rrbbbrrbb, rbrbrrbrb, rbbbrrrrr.rbbbrrrrb$
10	$rrbbbrrbbr, rbbbrrrrbr, rbbbrrrrrr$
11	$rrbbbrrbbrr, rbbbrrrrrrr, rbbbrrrrbrr$
12	$rrbbbrrbbrrr, rbbbrrrrrrrr, rbbbrrrrbrrr$
13	$rrbbbrrbbrrrr, rrbbbrrbbrrrb$
14	$rrbbbrrbbrrrrr, rrbbbrrbbrrrrb, rrbbbrrbbrrrbr, rrbbbrrbbrrrbb$
15	none.

□

3.1 About FVR

In the above theorems, we find our lower bounds by considering all valid colorings of $[1,n]$ for some $n \in \mathbb{Z}^+$ and deducing the possible elements that $x+y+kz$ can be when x, y, and z are monochromatic and the possible elements that $(k+j)w$ can be, i.e., determining $R_{x,y,z}$, $B_{x,y,z}$, R_w, and B_w. We then looked for intersections that would make (x,y,z,w) a monochromatic solution. The intersections are specific values of k which show that the given coloring has monochromatic solutions for these values of k.

This process has been automated in the Maple package `FVR`, which is available from the first author's website[2]. The input is a list of all valid colorings of $[1,n]$. The output is a list of values of k for which we have monochromatic solutions. By increasing n we are able to determine the exact Rado numbers for all $k \in \mathbb{Z}^+$. An example of this is explained in detail in the proof of the next theorem.

4. A Formula for $x+y+kz=2w$

In [HS] and [GS] a formula for, in particular, $x+y+kz=w$ is given: $RR(x+y+kz=w)=(k+1)(k+4)+1$. In this section we provide a formula for the next important equation of this form, namely the one in this section's title. To the best of our knowledge this is the first formula given for a linear homogeneous equation $\mathcal{E}=0$ of more than three variables with a negative coefficient not equal to -1 (assuming, without loss of generality, at least as many positive coefficients as negative ones) that does not satisfy Rado's regularity condition.

[2] `http://math.colgate.edu/~aaron/programs.html`

Theorem 8 For $k \in \mathbb{Z}^+$,

$$RR(x+y+kz=2w) = \begin{cases} \frac{k(k+4)}{4}+1 & \text{if } k \equiv 0 \pmod 4 \\ \frac{(k+2)(k+3)}{4}+1 & \text{if } k \equiv 1 \pmod 4 \\ \frac{(k+2)^2}{4}+1 & \text{if } k \equiv 2 \pmod 4 \\ \frac{(k+1)(k+4)}{4}+1 & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Proof. We begin with the lower bounds. Let N_i be one less than the stated formula for $k \equiv i \pmod 4$, with $i \in \{0,1,2,3\}$. We will provide 2-colorings of $[1, N_i]$, for $i = 0,1,2,3$, that admit no monochromatic solution to $x+y+kz = 2w$.

For $i = 0$, color all elements in $\left[1, \frac{k}{2}\right]$ red and all remaining elements blue. If we assume x, y, and z are all red, then $x+y+kz \geq k+2$ so that for any solution we have $w \geq \frac{k}{2}+1$. Thus, there is no red solution. If we assume x, y, and z are all blue, then $x+y+kz \geq \frac{k^2}{2}+2k+2 = 2\left(\frac{k(k+4)}{4}+1\right) > 2N_0$, showing that there is no blue solution.

For $i = 1$, color N_1 and all elements in $\left[1, \frac{k+1}{2}\right]$ red. Color the remaining elements blue. Similarly to the last case, we have no blue solution since $x+y+kz > 2(N_1-1)$. If we assume x, y, z and w are all red, then we cannot have all of x, y, z in $\left[1, \frac{k+1}{2}\right]$. If we do, since k is odd (so that we must have $x+y$ odd), then $x+y+kz \geq 1+2+k(1) = k+3 > 2\left(\frac{k+1}{2}\right)$. Thus, w must be blue. Now we assume, without loss of generality, that $x = N_1$. In this situation, we must have $w = N_1$. Hence, since $N_1+y+kz = 2N_1$ we see that $y+kz = N_1$. Hence, $y, z \leq \frac{k+1}{2}$. But then $y+kz \leq \frac{k^2+2k+1}{2} < N_1$, a contradiction. Hence, there is no red solution under this coloring.

The cases $i = 2$ and $i = 3$ are similar to the above cases. As such, we provide the colorings and leave the details to the reader. For $i = 2$, we color N_2 and all elements in $\left[1, \frac{k}{2}\right]$ red, while the remaining elements are colored blue. For $i = 3$, color all elements in $\left[1, \frac{k+1}{2}\right]$ red and all remaining elements blue.

We now turn to the upper bounds. We let M_i be equal to the stated formula for $k \equiv i \pmod 4$, with $i \in \{0,1,2,3\}$. We employ a "forcing" argument to determine the color of certain elements. We let R denote the set of red elements and B the set of blue elements. We denote by a 4-tuple (x,y,z,w) a solution to $x+y+kz = 2w$. In each of the following cases assume, for a contradiction, that there exists a 2-coloring of $[1, M_i]$ with no monochromatic solution to the equation. In each case we assume $1 \in R$.

`Case 1.` $k \equiv 0 \pmod 4$. We will first show that $2 \in R$. Assume, for a contradiction, that $2 \in B$. Then $2k+2 \in R$ by considering $(2, 2k+2, 2, 2k+2)$. Also, $k+1 \in B$ comes from the similar solution $(1, k+1, 1, k+1)$. Now, from $(3k+3, 1, 1, 2k+2)$ we have $3k+3 \in B$. As a consequence, we see that $3 \in R$ by considering $(3, 3k+3, 3, 3k+3)$. From here we use $(3k+1, 3, 1, 2k+2)$ to see that $3k+1 \in B$. But then $(3k+1, k+1, 2, 3k+1)$ is a blue solution, a contradiction. Hence, $2 \in R$.

Now, since $1, 2 \in R$, in order for $\left(1, 1, 1, \frac{k}{2}+1\right)$ not to be monochromatic, we have $\frac{k}{2}+1 \in B$. Similarly, $\left(2, 2, 1, \frac{k}{2}+2\right)$ gives $\frac{k}{2}+2 \in B$. Consequently, so that $\left(\frac{k}{2}+1, \frac{k}{2}+1, \frac{k}{2}+1, \frac{k^2}{4}+k+1\right)$ is not monochromatic, we have $\frac{k^2}{4}+k+1 \in R$.

Our next goal is to show that $\frac{k}{2} \in R$. So that $\left(\frac{k^2+2k}{4}+1, \frac{k^2+2k}{4}+1, 1, \frac{k^2}{4}+k+1\right)$ is not red, we have $\frac{k^2+2k}{4}+1 \in B$. In turn, to avoid $\left(\frac{k}{2}+1, \frac{k}{2}+1, \frac{k}{2}, \frac{k^2+2k}{4}+1\right)$ being blue, we have $\frac{k}{2} \in R$, as desired.

So that $\left(\frac{k}{2}, \frac{k}{2}, 1, k\right)$ and $\left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k^2+2k}{4}\right)$ are not red, we have $k, \frac{k^2+2k}{4} \in B$. Using these in $\left(k, k, \frac{k}{2}-1, \frac{k^2+2k}{4}\right)$ gives $\frac{k}{2}-1 \in R$. Since $\frac{k}{2}$ and $\frac{k}{2}-1$ are both red, $\left(\frac{k}{2}-1, \frac{k}{2}-1, \frac{k}{2}-1, \frac{k^2}{4}-1\right)$ gives us $\frac{k^2}{4}-1 \in B$ while $\left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}-1, \frac{k^2}{4}\right)$ gives us $\frac{k^2}{4} \in B$. This, in turn, gives us $\frac{k}{4} \in R$ by considering $\left(\frac{k^2}{4}, k, \frac{k}{4}, \frac{k^2+2k}{4}\right)$.

Now, from $\left(\frac{k^2+4k}{4}+1, 1, \frac{k}{4}+1, \frac{k^2+4k}{4}+1\right)$ we have $\frac{k}{4}+1 \in B$. We use this in the two solutions $\left(\frac{k^2}{4}-1, \frac{k}{2}+1, \frac{k}{4}+1, \frac{k^2+3k}{4}\right)$ and $\left(\frac{k^2}{4}, k, \frac{k}{4}+1, \frac{k^2+4k}{4}\right)$ to find that $\frac{k^2+3k}{4}, \frac{k^2+4k}{4} \in R$. But this gives us the red solution $\left(\frac{k^2+4k}{4}, \frac{k}{2}, \frac{k}{4}, \frac{k^2+3k}{4}\right)$, a contradiction.

`Case 2.` $k \equiv 1 \pmod 4$. The argument at the beginning of Case 1 holds for this case, so we have $2 \in R$. We consider two subcases.

`Subcase i.` $\frac{k+3}{4} \in R$. From $\left(\frac{k^2+4k+3}{4}, \frac{k+3}{4}, \frac{k+3}{4}, \frac{k^2+4k+3}{4}\right)$ we have $\frac{k^2+4k+3}{4} \in B$. This gives us $\frac{k+1}{2} \in R$ by considering $\left(k+1, \frac{k+1}{2}, \frac{k+1}{2}, \frac{k^2+4k+3}{4}\right)$ (where $k+1 \in B$ comes from $(1, 1, 2, k+1)$). We also have, from $(1, 2, 1, \frac{k+3}{2})$, that $\frac{k+3}{2} \in B$. Consequently, $\frac{k^2+5k+6}{4} \in R$ so that $\left(\frac{k+3}{2}, \frac{k+3}{2}, \frac{k+3}{2}, \frac{k^2+5k+6}{4}\right)$ is not monochromatic.

We next have that $\frac{k+5}{2} \in B$ so that $\left(3, 2, 1, \frac{k+5}{2}\right)$ is not monochromatic (we may assume that $k \geq 9$). Hence, $\frac{k^2+5k+10}{4} \in R$ by considering $\left(\frac{k+5}{2}, \frac{k+5}{2}, \frac{k+3}{2}, \frac{k^2+5k+10}{4}\right)$. But this gives us the monochromatic solution $\left(\frac{k^2+5k+10}{4}, \frac{k+1}{2}, \frac{k+3}{4}, \frac{k^2+5k+6}{4}\right)$, a contradiction.

`Subcase i.` $\frac{k+3}{4} \in B$. Via arguments similar to those in Subcase i, we have $\frac{k^2+4k+3}{4}, \frac{k^2+5k+6}{4} \in R$. From $\left(\frac{k^2+4k+3}{4}, \frac{k^2+2k+9}{4}, 1, \frac{k^2+5k+6}{4}\right)$ we have $\frac{k^2+2k+9}{4} \in B$. This gives us $\frac{k^2+15}{4} \in R$ by considering $\left(\frac{k^2+15}{4}, \frac{k+3}{4}, \frac{k+3}{4}, \frac{k^2+2k+9}{4}\right)$.

We now show that $\frac{k-1}{4} \in R$ by showing that for any $i \leq \frac{k-1}{4}$ we must have $i \in R$. To this end, assume, for a contradiction, that $i-1 \in R$ but $i \in B$ (where $i \geq 3$). From $(i, ik+i, i, ik+i)$ we have $ik+i \in R$. In turn we have $(i+1)k+i \in B$ by considering $(ik+i, ik+i, 2, (i+1)k+i)$. We next see from $((i+1)k+i, i, i+1, (i+1)k+i)$ that $i+1 \in R$. Using our assumption that $i-1 \in R$ in $(i-1, i+1, 2, k+i)$ we have $k+i \in B$. But then $((i+1)k+i, k+i, i, (i+1)k+i)$ is a blue solution, provided $(i+1)k+i \le M_1$, which by the bound given on i is valid. By applying this argument to $i = 3, 4, \ldots, \frac{k-1}{4}$, in order, we see that all positive integers less than or equal to $\frac{k-1}{4}$ must be red. In particular, $\frac{k-1}{4} \in R$.

Using $\frac{k-1}{4} \in R$ in $\left(\frac{k^2+15}{4}, \frac{k+15}{4}, \frac{k-1}{4}, \frac{k^2+15}{4}\right)$ we have $\frac{k+15}{4} \in B$. This, in turn, gives us $\frac{k^2+4k+15}{4} \in R$ by considering $\left(\frac{k^2+4k+15}{4}, \frac{k+15}{4}, \frac{k+3}{4}, \frac{k^2+4k+15}{4}\right)$. For our contradiction, we see now that $\left(\frac{k^2+4k+15}{4}, \frac{k^2+15}{4}, 1, \frac{k^2+4k+15}{4}\right)$ is a red solution.

`Case 3.` $k \equiv 2 \pmod 4$. From Case 1 we have $\frac{k}{2}, \frac{k^2}{4}+k+1 \in R$ and $\frac{k}{2}+1, \frac{k^2}{4}-1, \frac{k^2+2k}{4} \in B$. From $\left(1, 1, \frac{k}{2}+2, \frac{k^2}{4}+k+1\right)$ we see that $\frac{k}{2}+2 \in B$. This gives us $\frac{k^2}{4}+k+2 \in R$ by considering $\left(\frac{k}{2}+2, \frac{k}{2}+2, \frac{k}{2}+1, \frac{k^2}{4}+k+2\right)$. Using this fact in $\left(\frac{k^2}{4}+k+2, \frac{k}{2}, \frac{k+2}{4}, \frac{k^2}{4}+k+1\right)$ we have $\frac{k+2}{4} \in B$. But then $\left(\frac{k^2}{4}-1, \frac{k}{2}+1, \frac{k+2}{4}, \frac{k^2+2k}{4}\right)$ is a blue solution, a contradiction.

`Case 4.` $k \equiv 3 \pmod 4$. Let $i \in R$. From Case 2 we may assume $i \ge 2$ so that $1, 2, \ldots, i$ are all red, By considering $(i, ik+i, i, ik+i)$ we have $ik+i \in B$ so that we may assume $k+1, 2k+2, \ldots, (i-1)k+(i-1), ik+i$ are all blue. Since $(i+1, (i-1)k+i-1, i+1, ik+i)$ is a solution, we have $i+1 \in R$. Hence, $i \in R$ for $1 \le i \le \frac{k+5}{4}$. In particular, $\frac{k+5}{4} \in R$. From Case 2 we also have $\frac{k+3}{2}, \frac{k+5}{2} \in B$. By considering $\left(\frac{k+3}{2}, \frac{k+5}{2}, \frac{k+3}{2}, \frac{k^2+5k+8}{4}\right)$ we have $\frac{k^2+5k+8}{4} \in R$. But then $\left(\frac{k^2+5k+8}{4}, 2, \frac{k+5}{4}, \frac{k^2+5k+8}{4}\right)$ is a red solution, a contradiction.

4. Conclusion

The next important numbers to determine are in the first row of Tables 1. As such, it would be nice to have a formula for $RR(x+y+z=\ell w)$. We have been unable to discover one.

By analyzing Table 1, given below, we have noticed, to some extent, certain patterns that emerge. In particular, we make the following conjecture.

Conjecture For $\ell \ge 2$ fixed and $k \ge \ell+2$, we have

$$RR(x+y+kz=\ell w) = \left(\left\lfloor \frac{k+\ell+1}{\ell} \right\rfloor\right)^2 + O\left(\frac{k}{\ell^2}\right),$$

where the "$O\left(\frac{k}{\ell^2}\right)$ part" depends on the residue class of k modulo ℓ^2.

For a concrete conjecture, we believe the following holds.

Conjecture Let $k \geq 5$. Then

$$RR(x+y+kz=3w) = \left(\left\lfloor \frac{k+4}{3} \right\rfloor\right)^2 + \begin{cases} -\frac{k}{9} & \text{if } k \equiv 0 \pmod{9} \\ 0 & \text{if } k \equiv 1,6 \pmod{9} \\ -\frac{k+7}{9} - 1 & \text{if } k \equiv 2 \pmod{9} \\ -1 & \text{if } k \equiv 3,8 \pmod{9} \\ 1 & \text{if } k \equiv 4 \pmod{9} \\ -\frac{k+4}{9} & \text{if } k \equiv 5 \pmod{9} \\ \frac{k+2}{9} & \text{if } k \equiv 7 \pmod{9} \end{cases}.$$

Acknowledgment

We thank Dan Saracino for a very careful reading that caught some errors in a previous draft.

We end with a table of calculated values of $RR(\mathcal{E}(k,j))$ for small values of k and j. These were calculated by a standard backtrack algorithm. We thank Joey Parrish for helping with implementation efficiency of the algorithm. The program can be downloaded as `RADONUMBERS` at the second author's homepage (`http://math.colgate.edu/∼aaron/programs.html`).

	$\ell = 1$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
$k=$ **1**	11	4	1	4	9	4	10	12	14	16	18	20	22	24	26	38	40	43	48	50	53	59	62
2	19	5	4	1	4	3	4	5	7	8	9	9	17	18	20	21	23	24	26	27	32	33	35
3	29	8	5	4	1	4	3	4	9	6	7	10	9	10	9	13	14	15	16	16	26	27	28
4	41	9	4	4	5	1	4	3	6	5	6	4	10	8	11	10	11	9	13	14	14	15	16
5	55	15	8	6	8	5	1	4	6	8	5	6	4	10	11	11	11	13	12	14	10	14	15
6	71	17	9	5	6	6	5	1	9	5	8	5	11	4	9	10	11	15	13	12	16	10	12
7	89	23	10	7	6	7	11	5	1	4	5	8	6	11	7	10	12	12	13	13	14	15	10
8	109	25	15	8	6	9	8	8	5	1	6	5	8	5	10	7	10	9	11	12	16	17	15
9	131	34	15	9	9	9	9	7	14	5	1	9	6	8	10	10	12	12	9	11	15	12	14
10	155	37	16	10	10	9	8	7	8	10	5	1	9	5	8	10	10	9	8	9	12	11	12
11	181	46	22	15	10	11	8	11	9	10	17	5	1	6	5	8	6	11	9	8	11	16	11
12	209	49	24	16	11	9	10	10	12	8	10	12	5	1	9	6	8	9	11	9	12	10	12
13	239	61	26	17	10	11	13	13	10	10	13	10	20	5	1	9	6	6	9	11	7	10	13
14	271	65	34	18	14	11	14	11	12	13	8	9	11	14	5	1	9	5	6	9	14	7	9
15	305	77	36	24	15	13	13	15	14	15	12	14	12	12	23	5	1	9	6	12	9	10	7
16	341	81	38	24	16	11	14	12	14	13	12	12	11	10	12	16	5	1	9	10	8	9	12
17	379	96	45	24	17	14	15	15	17	12	19	15	14	13	17	14	26	5	1	9	8	8	9
18	419	101	47	25	18	15	14	15	18	15	12	12	16	10	12	11	15	18	5	1	9	6	8
19	461	116	49	32	24	17	14	17	18	18	16	20	15	14	15	13	15	15	29	5	1	9	8
20	505	121	60	34	25	15	16	16	18	15	14	20	17	19	15	14	14	12	15	20	5	1	9
21	551	139	63	36	26	18	19	16	15	20	20	17	19	18	18	14	14	17	21	16	32	5	1
22	599	145	65	37	27	18	17	15	17	19	18	19	15	17	15	17	16	14	14	13	18	22	5
23	649	163	78	45	28	23	18	18	19	19	23	22	17	22	17	21	14	18	16	16	17	18	35
24	701	169	≥ 81	47	32	24	19	16	17	20	21	18	20	16	20	20	18	16	14	18	15	14	19
25	755	190	≥ 84	49	35	25	19	20	16	20	23	21	25	16	28	23	19	18	21	18	16	17	25
26	811	197	≥ 94	51	35	26	17	19	20	20	23	21	22	21	17	18	20	18	19	15	17	17	17
27	869	218	≥ 97	62	35	27	23	22	27	22	21	20	23	26	20	21	22	27	19	17	21	19	19
28	929	225	≥ 100	64	36	27	25	21	21	18	23	22	22	21	16	18	20	21	19	21	21	16	16
29	991	249	≥ 115	66	44	35	26	22	21	23	22	24	27	25	26	19	30	21	23	25	17	21	19
30	1055	257	≥ 118	68	46	36	24	19	23	22	22	21	25	24	30	17	20	24	22	21	22	18	16
31	1121	281	≥ 122	75	47	37	26	22	23	24	22	25	25	26	22	28	20	30	22	20	19	24	21
32	1189	289	≥ 138	77	49	38	27	24	24	23	25	24	26	24	27	24	18	20	21	22	24	25	19
33	1259	316	≥ 141	≥ 79	50	39	26	25	25	23	27	26	25	25	24	24	29	20	29	29	27	28	18
34	1331	325	≥ 147	≥ 81	60	40	34	24	23	24	26	23	27	28	29	27	28	20	21	26	23	26	24
35	1405	352	≥ 161	≥ 93	62	45	35	26	25	26	26	27	27	28	28	26	25	34	21	33	30	25	26
36	1481	361	≥ 165	≥ 96	63	45	36	24	27	26	27	30	27	24	30	27	29	27	20	22	27	22	27
37	1559	391	≥ 169	≥ 98	65	49	37	26	26	26	30	28	25	27	28	31	27	27	36	22	31	26	25
38	1639	401	≥ 188	≥ 100	65	48	38	28	28	25	26	27	30	30	29	30	31	32	38	22	23	28	24
39	1721	431	≥ 193	≥ 113	77	48	39	35	29	28	28	31	30	29	32	32	31	29	28	32	23	31	31
40	1805	441	≥ 196	≥ 116	79	49	39	35	29	25	28	26	31	27	31	32	31	32	32	30	22	24	29

	24	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
k= **1**	64	84	87	≥ 91	≥ 98	≥ 102	≥ 105	≥ 114	≥ 118	≥ 121	≥ 148	≥ 152	≥ 157	≥ 167	≥ 171	≥ 176	≥ 187
2	36	44	46	48	49	66	68	≥ 70	≥ 73	≥ 77	≥ 80	≥ 82	≥ 84	≥ 93	≥ 95	≥ 98	≥ 100
3	30	31	33	34	35	37	38	44	45	47	48	49	58	60	61	63	64
4	17	21	22	23	24	25	25	37	38	39	40	42	43	45	46	47	48
5	14	17	15	16	16	21	22	23	23	24	25	25	31	32	33	34	35
6	9	16	13	18	14	15	15	16	17	21	22	22	23	24	24	25	25
7	18	9	16	11	18	19	14	21	22	16	17	18	19	21	22	22	23
8	16	18	9	19	12	17	18	19	20	21	15	23	24	18	19	24	16
9	16	16	18	27	19	16	21	18	19	20	21	22	26	24	16	26	27
10	13	18	17	16	17	19	20	21	17	23	19	20	21	22	23	24	25
11	12	15	12	15	16	17	18	20	20	21	21	19	25	21	22	23	24
12	11	14	13	12	12	20	15	16	16	20	21	20	21	22	20	26	22
13	12	10	20	13	12	12	13	15	16	16	17	18	20	20	21	22	22
14	10	10	10	12	13	15	12	13	17	21	16	20	18	19	19	20	21
15	10	15	13	10	12	14	23	15	14	15	15	28	20	17	19	21	22
16	7	11	9	12	11	14	12	16	15	18	14	16	14	23	18	20	20
17	12	8	13	9	11	12	14	12	18	14	26	17	16	16	17	23	20
18	9	10	7	27	9	16	11	18	13	15	14	16	17	29	12	18	17
19	10	10	10	9	13	13	16	12	17	13	15	14	16	16	29	19	18
20	10	15	9	10	8	13	14	16	12	13	15	20	16	16	16	20	19
21	9	10	9	9	16	12	15	11	17	13	17	18	18	15	16	16	20
22	1	9	8	10	7	10	12	13	11	20	13	12	13	19	15	25	16
23	5	1	9	9	9	7	10	12	13	11	16	17	13	13	17	15	17
24	24	5	1	9	9	10	7	15	14	13	11	14	18	14	15	20	17
25	19	38	5	1	9	10	15	10	12	14	13	15	14	18	20	13	20
26	15	20	26	5	1	9	10	10	10	12	11	14	11	15	17	26	13
27	18	21	22	41	5	1	9	10	10	15	11	13	27	14	16	17	18
28	16	19	16	22	28	5	1	9	10	10	10	21	12	18	13	16	17
29	18	19	20	29	22	44	5	1	9	10	10	12	12	11	16	13	17
30	21	20	20	20	17	23	30	5	1	9	10	11	15	12	11	18	14
31	25	22	19	21	21	23	24	47	5	1	9	10	11	10	11	11	16
32	20	26	18	17	20	20	18	24	32	5	1	9	10	11	13	12	16
33	20	23	21	20	25	22	21	33	26	50	5	1	9	10	10	14	8
34	19	23	18	22	18	21	18	21	19	27	34	5	1	9	10	11	11
35	25	22	22	27	25	23	23	23	25	25	27	53	5	1	9	10	12
36	24	23	21	27	24	23	21	21	22	23	20	27	36	5	1	9	10
37	28	23	22	24	29	25	23	23	22	24	25	37	28	56	5	1	9
38	27	24	26	22	26	24	23	24	21	23	22	25	21	30	38	5	1
39	28	29	33	25	23	22	28	27	25	26	25	25	27	29	30	59	5
40	25	28	24	22	23	28	25	26	24	27	22	28	22	27	22	31	40

Table 1: Some Values of $RR(x+y+kz=\ell w)$

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