INTERMINGLED ASCENDING WAVE M-SETS

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Abstract

Given a coloring of $\mathbb{Z}^+$, we call a monochromatic set $A = \{a_1 < a_2 < \cdots < a_m\}$ an m-set. The diameter of A is $a_m - a_1$. Given two m-sets A and B, we say that they are non-overlapping if $\max(A) < \min(B)$ or $\max(B) < \min(A)$. The original study of non-overlapping m-sets, done by Bialostocki, Erdős, and Lefmann, concerned non-decreasing diameters. We investigate an "intermingling" of certain subset diameters of non-overlapping m-sets. In particular, we show that, for every integer $m \geq 2$, the minimum integer $n(m)$ such that every 2-coloring of $[1, n(m)]$ admits two m-sets $\{a_1 < a_2 < \cdots < a_m\}$ and $\{b_1 < b_2 < \cdots < b_m\}$ with $a_m < b_1$, such that $b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_m - a_m$ is $n(m) = 6m - 5$. The r-coloring case is also investigated.

1. Introduction

This article falls under the general heading of Ramsey theory on the integers. In order to discuss some of the history, we first have need of some definitions and notation. Recall that a 2-coloring of a set S is a map $\chi : S \to T$, where $|T| = 2$. We will use $T = \{G, R\}$ (to stand for green and red).

Definition 1. Let $m \geq 2$ be an integer and consider an arbitrary 2-coloring χ of $\mathbb{Z}^+$. We say that $A \subseteq \mathbb{Z}^+$ is an *m-set* if $|A| = m$ and A is monochromatic under χ (i.e., $|\chi(A)| = 1$).

Definition 2. For $A = \{a_1 < a_2 < \cdots < a_m\}$ we call $a_m - a_1$ the *diameter of A* and write $\text{diam}(A)$; we refer to the differences $a_{i+1} - a_i$, $1 \leq i \leq m - 1$, as *gaps*.

Definition 3. For A and B both m-sets (possibly of different colors, but both monochromatic by defintion), we say that $A = \{a_1 < a_2 < \cdots < a_m\}$ and $B =$

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$\{b_1 < b_2 < \cdots < b_m\}$ are *non-overlapping* if either $a_m < b_1$ or $b_m < a_1$. When we have $a_m < b_1$ we write $A \prec B$.

The research into m-sets with diameter restrictions was started by Bialostocki, Erdős, and Lefmann. In their paper [3] they proved, in particular, that for $m \geq 2$, the integer $s = s(m) = 5m - 3$ is the minimum integer such that every 2-coloring of $[1, s]$ admits two m-sets A and B with $A \prec B$, such that $\text{diam}(A) \leq \text{diam}(B)$. They also provided the associated Ramsey-type number for three colors, while Grynkiewicz [6] proved that $12m - 9$ is the correct interval length for two colors for four m-sets with non-decreasing diameters. Bollobás, Erdős, and Jin [5] investigated the situation for strictly increasing diameters. If we insist that two m-sets have equal diameters, the answer is still unknown; Bialostocki and Wilson [4] conjecture that the correct interval length is $6m - 4$. Schultz [8] investigated the minimal interval length for r-colorings that admit m-sets with $2 \cdot \text{diam}(A) \leq \text{diam}(B)$, finding exact values for $r = 2, 3, 4$.

Perhaps the closest diameter requirement to what we study in this article was done by Grynkiewicz and Sabar [7], in which they study pairs of non-overlapping m-sets that satisfy $b_j - a_j \geq b_1 - a_1$ for a fixed $j \in \{2, 3, \ldots, m\}$. Our requirement on the relationship between non-overlapping m-sets is examined in Section 2. There, not only do we require non-decreasing diameters, but also non-decreasing gaps between corresponding pairs of elements in each set. Specifically, for $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ we require

$$b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_{m-1} - a_{m-1} \leq b_m - a_m. \qquad (\star)$$

Note that by considering only the first and last arguments above we have $b_1 - a_1 \leq b_m - a_m$, which is equivalent to $\text{diam}(A) \leq \text{diam}(B)$. Also note that $(\star)$ is equivalent to the conditions $b_j - b_{j-1} \geq a_j - a_{j-1}$ for $j = 2, 3, \ldots, m$. Hence, this is also a refinement of what was studied by Grynkiewicz and Sabar [7].

Based on our refinement, we make the following definition.

Definition 4. Let $r \geq 2$ be an integer. Define $n(m; r)$ to be the minimum integer n such that every r-coloring of $[1, n]$ admits two non-overlapping m-sets $\{a_1, a_2, \ldots, a_m\} \prec \{b_1, b_2, \ldots, b_m\}$ with $b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_m - a_m$.

The main result is in Section 2; namely, $n(m; 2) = 6m - 5$. In Section 3 we investigate $n(m; r)$ and end with some open questions.

Remark. We refer to sets that satisfy $(\star)$ as *intermingled ascending wave m-sets.* A set $S = \{s_1, s_2, \ldots\}$ is called an *ascending wave* if $s_{j+1} - s_j \geq s_j - s_{j-1}$ for $j = 2, 3, \ldots$ (see [1]). In other words, successive gaps between adjacent elements of S are non-decreasing (which has been labeled in the literature as "ascending" instead of the more appropriate "non-descending"). We are intermingling the sets by considering $b_j - a_j$ and requiring that these "intermingled gaps" be non-decreasing.

Notation. Throughout the paper, unless otherwise stated, we will use $A = \{a_1 < a_2 < \cdots < a_m\}$ and $B = \{b_1 < b_2 < \cdots < b_m\}$, sometimes without explicitly stating so. Since we will be dealing strictly with the positive integers, we use the notation $[i, j] = \{i, i+1, \ldots, j\}$ for positive integers $i < j$. For all of our 2-colorings, we use green and red for the colors and denote these colors by G and R, respectively.

2. Intermingled Ascending Wave m-Sets Using Two Colors

We prove that $n(m) = 6m - 5$ in the standard way: by providing matching lower and upper bounds for $n(m)$. We start with the lower bound.

Lemma 5. *For $m \geq 2$, we have $n(m) \geq 6m - 5$.*

Proof. Consider the following 2-coloring $\gamma : [1, 6m-6] \to \{G, R\}$ defined by

$$\underbrace{RGRG \ldots GR}_{2m-1}\underbrace{GGRRGGRR \ldots GGR}_{4m-5}.$$

We will show that γ does not admit two m-sets $A \prec B$ that satisfy $(\star)$.

Consider the case when $A \subseteq [1, 2m]$. Then we have $a_j - a_{j-1} = 2$ for all $j \in [2, m]$. Hence, we must have $B \subseteq [2m+1, 6m-6]$ with $b_j - b_{j-1} \geq 2$. Thus, B contains at most one element from each of the following $2m - 2$ monochromatic sets: $\{2m+1\}, [2m+2, 2m+3], \ldots, [4m-7, 4m-6], \{4m-5\}$. Each color appears in exactly $m - 1$ of these sets. Hence, B cannot have m elements.

Next, consider the case when $A \not\subseteq [1, 2m]$. In this situation, A may contain gaps of size 1. Let there be exactly j elements of A that are in $[2m, 6m-6]$, where $j \geq 2$. This gives us $a_m \geq 2m + 2j - 3$ so that $b_1 \geq 2m + 2j - 2$. Turning our attention to B, we see that the $m - j$ gaps defined by the first $m - j + 1$ elements of B must have average gap size at least 4 since the corresponding gaps defined by the first $m - j + 1$ elements of A are necessarily greater than 1. Hence, we have $b_{m-j+1} \geq b_1 + 4(m-j) \geq (2m + 2j - 2) + 4(m-j) = 6m - 2j - 2$ as we can only use at most one element from any four consecutive elements with color pattern $GGRR$ or $RRGG$. There are at most $2j - 4$ elements larger than b_{m-j+1}, of which at most $j - 2$ are the same color as $b_1, b_2, \ldots, b_{m-j+1}$. However, we require $j - 1$ more elements in order for B to contain m total elements.

In both cases, under γ we do not have m-sets $A \prec B$ that satisfy $(\star)$. Hence, we conclude that $n(m) > 6m - 6$. $\square$

We now move on to our main result.

Theorem 6. *For $m \geq 2$, let $n = n(m)$ be the minimal integer such that every 2-coloring of $[1, n]$ admits m-sets $A \prec B$ that satisfy $b_{j+1} - a_{j+1} \geq b_j - a_j$ for $1 \leq j \leq m-1$. Then $n(m) = 6m - 5$.*

Proof. From Lemma 5, it remains to show that $n(m) \leq 6m-5$. Consider an arbitary coloring $\chi : [1, 6m-5] \to \{G, R\}$. We start by partitioning $[1, 6m-5]$ as $X \cup Y$, where $X = [1, 2m-1]$ and $Y = [2m, 6m-5]$. In X, we are guaranteed a monochromatic m-set by the pigeonhole principle. Without loss of generality, let this m-set be red and call it A. We consider two cases.

`Case 1.` Y contains at least $2m-1$ elements of the same color, say green. Choose the first $2m-1$ greens in Y and call this set C. Let $A = \{a_1 < a_2 < \cdots < a_m\}$ and $C = \{c_1 < c_2 < \cdots < c_{2m-1}\}$, and consider $B = \{c_{a_1}, c_{a_2}, \cdots, c_{a_m}\}$. Since $c_{a_j} - c_{a_{j-1}} \geq a_j - a_{j-1}$ for all $j \in \{1, 2, \ldots, m\}$ we have m-sets A and B with $A \prec B$ that satisfy $(\star)$.

`Case 2.` Y contains $2m-2$ green elements and $2m-2$ red elements. If X has more than m reds, then we can assume that $A \subseteq [1, 2m-2]$. By the same reasoning as in Case 1, by letting C be the set of green (or red, if preferred) elements in Y, we can find an m-set $B \subseteq C$ such that $A \prec B$ that satisfy $(\star)$. Thus, we assume that X has m red elements and $m-1$ green elements. Furthermore, the first and last element in X must be red or else we have $A \subseteq [1, 2m-2]$ or $A \subseteq [2, 2m-1]$ and we are in the same situation as when X has more than m red elements. Thus, X must be colored, under χ, as follows (or we are done):

$$R_1 \underbrace{G \cdots G}_{k_1} R_2 \underbrace{G \cdots G}_{k_2} R_3 \underbrace{G \cdots G}_{k_3} \cdots \underbrace{G \cdots G}_{k_{m-1}} R_m,$$

for some nonnegative k_i's with $\sum_{i=1}^{m-1} k_i = m-1$. We will call the spacing $(k_1, k_2, \ldots, k_{m-1})$ the *gap profile*.

Now, we focus on the coloring in Y. Choose a red m-set in Y via a greedy algorithm, i.e, let r_1 be the minimal red element in Y and, for successive $i \in \{2, 3, \ldots, m\}$, let r_i be the minimal red element such that $r_i - r_{i-1} \geq k_{i-1} + 1$.

This fails to produce a red m-set B with $A \prec B$ that satisfy $(\star)$ only if we "run out" of red elements in Y. Note that we can have at most k_i red elements between r_i and r_{i+1} that are not chosen via our greedy algorithm. This accounts for at most $m-1$ elements. Thus, we "run out" only if we have k_i red elements between r_i and r_{i+1} for each $i \in \{1, 2, \ldots, m-1\}$.

Assuming that we have k_i red elements between r_i and r_{i+1}, if there exists a green element between r_i and r_{i+1} then we cannot have a red element between such a green element and r_{i+1}; otherwise we would not have chosen r_{i+1} via the greedy algorithm (we would have chosen a smaller red element).

The above analysis of the last three paragraphs also holds if we consider the green elements instead of the red elements (since we have exactly $2m-2$ of each color in Y). Hence, we are either done or are now in the situation where our coloring of Y

under χ is either

$$\underbrace{RR\ldots R}_{k_1+1}\underbrace{GG\ldots G}_{k_1+1}\underbrace{RR\ldots R}_{k_2+1}\underbrace{GG\ldots G}_{k_2+1}\cdots\underbrace{RR\ldots R}_{k_{m-1}+1}\underbrace{GG\ldots G}_{k_{m-1}+1} \qquad \text{(Coloring I)}$$

or

$$\underbrace{GG\ldots G}_{k_1+1}\underbrace{RR\ldots R}_{k_1+1}\underbrace{GG\ldots G}_{k_2+1}\underbrace{RR\ldots R}_{k_2+1}\cdots\underbrace{GG\ldots G}_{k_{m-1}+1}\underbrace{RR\ldots R}_{k_{m-1}+1}. \qquad \text{(Coloring II)}$$

Hence, the gap profile of A implies that the above two colorings are the only possible colorings that may provide a counterexample to our theorem. Thus, if we can prove the existence of an m-set A' with a gap profile that is different from A's, while retaining $2m-2$ elements of the same color that are all larger than the maximal element of A', then we will be done. To this end, we consider $[1, 2m+k_1]$ and note than we still have $2m-2$ elements of the same color in $[2m+k_1+1, 6m-6]$.

Recall that X has exactly m red elements and $m-1$ green elements. If Y has Coloring I, then $[1, 2m+k_1]$ contains at least $m+1$ red elements. In this situation we clearly have another red m-set with a gap profile that is different from A's and we are done. Similarly, if Y has Coloring II and $k_1 \neq 0$, we have a green m-set with a gap profile that is different from A's and we are done. Hence, we may now assume that Y has Coloring II with $k_1 = 0$.

Within $[1, 2m] = [1, 2m+k_1]$ we have m red elements and m green elements, and they each must produce the same gap profile (else we are done). Let $r_1, r_2, \ldots, r_m$ and $g_1, g_2, \ldots, g_m$ be these red and green elements, respectively. We know that $r_1 = 1$. Let $[1, w]$ be entirely red for some $w \in \mathbb{Z}^+$. For the green elements to have the same gap profile as $1, 2, \ldots, w$, we must have $[w+1, 2w]$ entirely green. This, in turn, implies that $[2w+1, 3w]$ must be entirely red. To see this, note that if $x \in [2w+1, 3w]$ is green then $(2w+1)-w \neq x-2w$ so that $1, 2, \ldots, w, 2w+1$ does not have the same gap profile as $w+1, w+2, \ldots, 2w, x$. Continuing this line of reasoning, we deduce that, within $[1, 2m]$, the intervals $[1, w], [2w+1, 3w], [4w+1, 5w], \ldots$ are each red, while $[w+1, 2w], [3w+1, 4w], [5w+1, 6w], \ldots$ are each green. Now, we know that $r_m = 2m-1$ and $g_m = 2m$. This implies that $w = 1$. Hence, we have deduced that the only possible way for χ (assuming $\chi(1) = R$) to provide a counterexample is if χ is

$$\underbrace{RGRG\ldots GR}_{2m-1}\underbrace{GGRRGGRR\ldots GGRR}_{4m-6} \qquad (\dagger)$$

(note the similarity with the coloring from Lemma 5).

We finish the proof by showing that the coloring given by ($\dagger$) admits m-sets $A \prec B$ that satisfy ($\star$). To this end, let $A = \{4, 6, \ldots, 2m, 2m+1\}$ and $B = \{2m+2, 2m+6, 2m+10, \ldots, 6m-6, 6m-5\}$. It is easy to check that these m-sets satisfy the given requirements. $\square$

Having proved the above theorem, we can now gives bounds on a function related to work done by Grynkiewicz and Sabar in [7].

Corollary 7. *For $m \geq 2$, define $t = t(m)$ to be the minimal integer such that every 2-coloring of $[1, t]$ admits m-sets $A \prec B$ such that $b_j - a_j \geq b_1 - a_1$ for all $j \in \{2, 3, \ldots, m\}$. Then $5m - 3 \leq t(m) \leq 6m - 4$.*

Proof. The upper bound follows immediately from Theorem 6. For the lower bound, an exhaustive computer search shows that $t(2) = 7, t(3) = 13$, and $t(4) = 18$ so that the result holds for $m \leq 4$. For $m \geq 5$, we provide a 2-coloring of $[1, 5m - 4]$ that contains no m-sets satisfying the hypothesis. We leave it to the reader to verify this. The coloring is:

$$RG\underbrace{R\cdots R}_{m-2}\underbrace{G\cdots G}_{m-1}\underbrace{R\cdots R}_{m-1}\underbrace{G\cdots G}_{m}RR\underbrace{G\cdots G}_{m-4}.$$ □

3. Other Results and Open Questions

Using part of the proof of Theorem 6, we have the following upper bound for an arbitrary number of colors.

Corollary 8. *Let $m, r \geq 2$ and let $a(m, r) = r(r + 1)(m - 1) + 2$. Then every r-coloring of $[1, a(m, r)]$ admits two non-overlapping intermingled ascending wave m-sets.*

Proof. By the pigeonhole principle, some color occurs at least m times in $[1, r(m - 1) + 1]$. Let $1 \leq a_1 < a_2 < \cdots < a_m \leq r(m-1)+1$ be represent those integers that have that color. Appealing to the pigeonhole principle again, some color occurs at least $r(m - 1) + 1$ times in $[r(m - 1) + 2, r(r + 1)(m - 1) + 2]$. Let $c_1 < c_2 < \cdots < c_{r(m-1)+1}$ represent the first $r(m - 1) + 1$ integers of this guaranteed color. Define $B = \{c_{a_1}, c_{a_2}, \ldots, c_{a_m}\}$ and note that $A \prec B$ satisfy the corollary. □

For $r = 3$, Corollary 8 gives an upper bound of $n(m; 3) \leq 12m - 10$. We have been unable find a lower bound that matches this. The best lower bound we have found is $n(m; 3) \geq 10m - 9$ given by the coloring (using R, G, and Y to represent the colors red, green, and yellow, respectively)

$$G^{m-1}Y^{m-1}R^{m-1}G^{m-1}Y^{m-1}R^{m-1}Y^{m-1}R^{m-1}G^{2m-2}.$$

There are still many questions left unanswered; we list some below.

Question 1. What is the value of $n(m; 3)$, or, more generally, $n(m; r)$?

Question 2. What is the value of $t(m)$ in Corollary 7?

Question 3. Using strict inequalities in $(\star)$, what can be said about the minimal integer corresponding to $n(m;2)$?

Question 4. What can be said if only the first $k < m$ inequalities of $(\star)$ need be met?

Question 5. What can be said for three sets? Specifically, for an integer $r \geq 2$, define $n_3(m;r)$ to be the minimum integer n_3 such that every r-coloring of $[1, n_3]$ admits three non-overlapping m-sets $\{a_1, a_2, \ldots, a_m\} \prec \{b_1, b_2, \ldots, b_m\} \prec \{c_1, c_2 \ldots, c_m\}$ with $b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_m - a_m$ and $c_1 - b_1 \leq c_2 - b_2 \leq \cdots \leq c_m - b_m$.

Question 6. What can be said about intermingled ascending wave m-sets "in the sense of Erdős-Ginzberg-Ziv?" See, for example, [3] for information about this.

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