OFF-DIAGONAL GENERALIZED SCHUR NUMBERS

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Abstract: We determine all values of the 2-colored off-diagonal generalized Schur numbers (also called Issai numbers), an extension of the generalized Schur numbers. These numbers, denoted $S(k,l)$, are the minimal integers such that any red and blue coloring of the integers from 1 to $S(k,l)$ must admit either a solution to $\sum_{i=1}^{k-1} x_i = x_k$ consisting of only red integers, or a solution to $\sum_{i=1}^{l-1} x_i = x_l$ consisting of only blue integers. We show that $S(3,l) = 3l-4$ for odd $l \geq 3$, $S(3,l) = 3l-5$ for even $l \geq 4$, and $S(k,l) = kl-l-1$ for $4 \leq k \leq l$.

1 Introduction

In 1916, Issai Schur proved that any r-coloring of $[N] = \{1,2,\ldots,N\}$ must admit a monochromatic solution to $x+y=z$, provided N is sufficiently large. We denote the minimal N to satisfy this criterion by the Schur number $S(3,3,\ldots,3)$, where the number of 3's equals r, and the 3's are due to the three variables in the equation $x+y=z$. In 1982, Beutelspacher and Brestovansky [1] defined the generalized Schur number, denoted $S = S(k,k,\ldots,k)$ (where the number of k's equals r), to be the least integer such that any r-coloring of $[S]$ must admit a monochromatic solution to the equation $\sum_{i=1}^{k-1} x_i = x_k$. Such numbers exist by Rado's Theorem (see, for example, [2]). In [1], it is shown that $S(k,k) = k^2-k-1$ for all $k \geq 3$.

Let us define the following extension of the generalized Schur numbers. Let $r \geq 2$ and $k_i \geq 3$ for $i = 1,\ldots,r$. Let $M = S(k_1,k_2,\ldots,k_r)$ be the minimal integer such that any r-coloring of $[M]$ must admit a j-colored solution to $\sum_{i=1}^{k_j-1} x_i = x_{k_j}$ for some $j \in \{1,2,\ldots,r\}$. In this paper we focus on the 2-colored off-diagonal generalized Schur numbers, $S(k,l)$. These numbers are given their name because of their similarity to the classical off-diagonal Ramsey numbers. The off-diagonal generalized Schur numbers are also referred to as Issai numbers (see [3]).

The existence of off-diagonal generalized Schur numbers can be directly deduced from Ramsey's Theorem (see [3]) or from the following generalization of the single equation version of Rado's Theorem [see, for example, [2]].

Theorem 1.1 *Consider the set of r linear homogeneous equations*

$$\left\{ \sum_{i=1}^{n_j} c_i^{(j)} x_i = 0 : 1 \leq j \leq r \right\}.$$

If, for each $j \in \{1, 2, \ldots, r\}$, there exists a nonempty subset of the $c_i^{(j)}$'s which sums to 0, then for any r-coloring of the natural numbers there must exist $J \in \{1, 2, \ldots, r\}$ such that $\sum_{i=1}^{n_J} c_i^{(J)} x_i = 0$ has a J-colored solution in the variables x_i, $i = 1, 2, \ldots, n_J$.

The proof of Theorem 1 is an easy extension of the proof of Rado's Theorem (Abridged) found in [2] and will be omitted.

In this article we completely determine the values of all 2-colored off-diagonal generalized Schur numbers.

Let $\mathcal{L}(t)$ represent the equation $\sum_{m=1}^{t-1} x_m = x_t$. Let $S(k, l)$ be the minimal integer such that any 2-coloring of the integers from 1 to $S(k, l)$ must admit either a red solution to $\mathcal{L}(k)$ or a blue solution to $\mathcal{L}(l)$. We now present our main theorem.

Theorem 1.2 $S(k, l) = \begin{cases} 3l - 4 & \text{if } k = 3 \text{ and } l \geq 3 \text{ is odd;} \\ 3l - 5 & \text{if } k = 3 \text{ and } l \geq 4 \text{ is even;} \\ kl - l - 1 & \text{for } 4 \leq k \leq l. \end{cases}$

We prove this theorem in an elementary way by matching the lower bounds to the upper bounds for each case. For all colorings below, we let R be the set of red integers and B be the set of blue integers.

2 The Lower Bounds

We start with the lower bounds. For each case we exhibit a coloring which avoids both a red solution to $\mathcal{L}(k)$ and a blue solution to $\mathcal{L}(l)$ to obtain the lower bounds. We call such a coloring a *good* coloring.

`Case I:` $l \geq 3$ and odd.

We exhibit a good coloring of $[1, 3l - 5]$:

$$R = \{n : 1 \leq n \leq l - 2,\ n \text{ odd}\} \cup \{n : 2l - 2 \leq n \leq 3l - 5,\ n \text{ even}\};$$
$$B = [1, 3l - 5] \setminus R.$$

Let $x_1 \leq x_2 < x_3$ be red integers. We first show that (x_1, x_2, x_3) is not a solution to $\mathcal{L}(3)$.

If $\{x_1, x_2\} \subset [1, l - 2]$ then $x_1 + x_2 \in [2, 2l - 4]$ and is even. Thus $x_1 + x_2$ is colored blue, and hence (x_1, x_2, x_3) is not a solution to $\mathcal{L}(3)$.

If $x_1 \in [1, l-2]$ and $x_2 \in [2l-2, 3l-5]$, then $x_1 + x_2 \in [2l-1, 4l-7]$ and is odd. This shows us that either $x_1 + x_2$ is colored blue or out of bounds, and hence (x_1, x_2, x_3) is not a solution to $\mathcal{L}(3)$.

If $\{x_1, x_2\} \subset [2l-2, 3l-5]$ then $x_1 + x_2 \geq 4l - 4$. In this situation, since the sum is out of bounds, (x_1, x_2, x_3) cannot be a solution to $\mathcal{L}(3)$.

Next, let $x_1 \leq x_2 \leq \ldots \leq x_{l-1} < x_l$ be l integers all colored blue. We will show that $(x_1, x_2, \ldots, x_l)$ is not a solution to $\mathcal{L}(l)$.

If $\{x_1, x_2, \ldots, x_{l-1}\} \subset [2, l-3]$ then the sum $\sum_{i=1}^{l-1} x_i \geq 2l - 2$ and is even. This shows us that either $\sum_{i=1}^{l-1} x_i$ is colored red or out of bounds. Hence $(x_1, x_2, \ldots, x_l)$ is not a solution to $\mathcal{L}(l)$.

If there exists $j \in \{1, 2, \ldots, l-1\}$ such that $x_j \notin [2, l-3]$, then we may assume that $\sum_{i=1}^{l-1} x_i = 3l - 5$, which is colored red. This shows that $(x_1, x_2, \ldots, x_l)$ cannot be a solution to $\mathcal{L}(l)$.

`Case II:` $l \geq 4$ and even.

We exhibit a good coloring of $[1, 3l-6]$:

$$R = \{n : 1 \leq n \leq l-3,\ n \,\text{odd}\} \cup \{n : 2l-2 \leq n \leq 3l-6,\ n \,\text{even}\};$$
$$B = [1, 3l-6] \setminus R.$$

The proof that this coloring avoids both a red solution to $\mathcal{L}(3)$ and a blue solution to $\mathcal{L}(l)$ is very similar to the proof given in `Case I` above and will be omitted.

`Case III:` $4 \leq k \leq l$.

We exhibit a good coloring of $[1, kl-l-2]$:

$$R = \{n : 1 \leq n \leq k-2\} \cup \{n : (k-1)(l-1) \leq n \leq kl-l-2\};$$
$$B = [1, kl-l-2] \setminus R.$$

Let $x_1 \leq x_2 \leq \ldots \leq x_{k-1} < x_k$ be k integers all colored red. We first show that $(x_1, x_2, \ldots, x_k)$ is not a solution to $\mathcal{L}(k)$.

If $\{x_1, x_2, \ldots x_{k-1}\} \subset [1, k-2]$, then $\sum_{i=1}^{k-1} x_i \in [k-1, (k-2)(k-1)]$. This shows that $\sum_{i=1}^{k-1} x_i$ must be blue, and hence $(x_1, x_2, \ldots, x_k)$ is not a solution to $\mathcal{L}(k)$.

If there exists $j \in \{1, 2, \ldots, k-1\}$ such that $x_j \notin [1, k-2]$, then $x_j \geq (k-1)(l-1)$ and $\sum_{i=1}^{k-1} x_i \geq kl - l - 1$. Since the sum is out of bounds, $(x_1, x_2, \ldots, x_k)$ cannot be a solution to $\mathcal{L}(k)$.

Next, let $x_1 \leq x_2 \leq \ldots \leq x_{l-1} < x_l$ be l integers all colored blue. We show that $(x_1, x_2, \ldots, x_l)$ is not a solution to $\mathcal{L}(l)$.

Since $\{x_1, x_2, \ldots, x_{l-1}\} \subset [k-1, (k-1)(l-1)-1]$, it follows that $\sum_{i=1}^{l-1} x_i \geq (k-1)(l-1)$. This implies that either $\sum_{i=1}^{l-1} x_i$ is colored red or is out of bounds. Hence, $(x_1, x_2, \ldots, x_l)$ cannot be a solution to $\mathcal{L}(l)$.

3 The Upper Bounds

We now move on to the upper bounds. For each case we assume, for a contradiction, that there exists a 2-coloring of the integers from 1 to the upper bound

which avoids both a red solution to $\mathcal{L}(k)$ and a blue solution to $\mathcal{L}(l)$.
`Case I:` $l \geq 3$ and odd.

We prove the equivalent statement: $S(3, l+2) \leq 3l+2$ for $l \geq 1$ odd. Assume, for a contradiction, that there exists a 2-coloring of $[3l+2]$ which avoids both a red and a blue solution.

Subcase A: $1 \in R$

Since $1 \in R$ we must have $2 \in B$ to avoid a red solution. Hence $2(l+1) \in R$. This implies that $l+1 \in B$, which in turn implies that $3l+1 \in R$. From this we deduce that $3l, 3l+2 \in B$. Since $3l \in B$ we must have $l \in R$ or the $(l+2)$-tuple $(2, 2, \ldots, 2, l, 3l)$ would be a blue solution. Using $l, 2(l+1) \in R$ we must have $l+2 \in B$. But this implies that $3l+2 \in R$ (else the $(l+2)$-tuple $(2, 2, \ldots, 2, l+2, 3l+2)$ would be a blue solution), contradicting the above deduction that $3l+2 \in B$.

Subcase B: $1 \in B$

Since $1 \in B$ we must have $l+1 \in R$. Thus $2(l+1) \in B$. From $1, 2(l+1) \in B$ we must have $3l+2 \in R$ to avoid the blue solution given by the $(l+2)$-tuple $(1, 1, \ldots, 1, 2l+2, 3l+2)$. Since $3l+2 \in R$ we must have $2l+1 \in B$ (else $(l+1, 2l+1, 3l+2)$ would be a red solution). Now $1, 2l+1 \in B$ implies that $3l+1 \in R$ (else the $(l+2)$-tuple $(1, 1, \ldots, 1, 2l+1, 3l+1)$ would be a blue solution). Since $3l+1 \in R$ we must have $2l \in B$ (otherwise $(l+1, 2l, 3l+1)$ would be a red solution). From here we conclude that $3l \in R$.

Now consider the color of 2. If $2 \in R$, then the triple $(2, 3l, 3l+2)$ is a red solution, a contradiction. If $2 \in B$, then the $(l+2)$-tuple $(1, 2, 2, \ldots, 2, 2l+1)$ is a blue solution, again a contradiction.

`Case II:` $l \geq 4$ and even.

We prove the equivalent statement: $S(3, l+2) \leq 3l+1$ for $l \geq 2$ even. In this case assume, for a contradiction, that there exists a 2-coloring of $[3l+1]$ which avoids both a red and a blue solution.

Subcase A: $1 \in R$

Using the same deductions as in `Case I.A` we see that $2, l+1, 3l \in B$ and $2(l+1), 3l+1 \in R$.

We proceed by a series of easy implications. To avoid the blue solution given by the $(l+2)$-tuple $(2, 2, \ldots, 2, l, 3l)$ we must have $l \in R$, and hence $l-1 \in B$ (if $l = 2$ this is our contradiction and we are done, so we may assume that $l \geq 4$). This implies that $3l-1 \in R$ (or the $(l+2)$-tuple $(2, 2, \ldots, 2, l-1, 3l-1)$ would be a blue solution). To avoid the red solution $(l-3, 2(l+1), 3l-1)$ we must have $l-3 \in B$ (if $l = 4$ this is our contradiction and we are done, so we may assume that $l \geq 6$). To avoid the blue solution given by the $(l+2)$-tuple $(2, 2, \ldots, 2, l-3, 3l-3)$ we must have $3l-3 \in R$, which in turn implies that $l-5 \in B$. Since l is even, continuing this process will imply that $1 \in B$, a contradiction.

Subcase B: $1 \in B$

From `Case I.B` we have $l+1 \in R$ and $2(l+1) \in B$. From these we deduce that $l+2 \in R$ or else the $(l+2)$-tuple $(1, 1, \ldots, 1, l+2, 2l+2)$ would be a blue solution. This implies that $2l+3 \in B$, and hence $l+3 \in R$. Continuing this line of reasoning we see that $l+j \in R$ and $2l+j \in B$ for $j=2,3,\ldots,l+1$. Since $l+1, l+4 \in R$ we must have $3 \in B$. But now we have the blue solution given by the $(l+2)$-tuple $(1,3,3,\ldots,3,3l+1)$, a contradiction.

`Case III:` $4 \leq k \leq l$.

We prove the equivalent statement: $S(k+1,l+1) \leq kl+k-1$ for $3 \leq k \leq l$. In this case we assume, for a contradiction, that there exists a 2-coloring of $[kl+k-1]$ which avoids both a red and a blue solution.

We assume that $1 \in R$, since the proof for $1 \in B$ may be obtained by interchanging the colors and interchanging k and l. Since $1 \in R$ we must have $k \in B$, and hence $kl \in R$, which in turn implies that $l \in B$. We then see that $1, kl \in R$ implies that $kl+k-1 \in B$. We deduce from this that $2k-1 \in R$ (else we would have the blue solution given by the $(l+1)$-tuple $(k,k,\ldots,k,2k-1,kl+k-1)$. Since $2k-1 \in R$, we must have $2 \in B$ or we would have the red solution given by the $(k+1)$-tuple $(1,2,2,\ldots,2,2k-1)$. From this we must have $3l-2 \in R$ to avoid the blue solution given by the $(l+1)$-tuple $(2,2,\ldots,2,l,3l-2)$. This in turn shows that $3l+k-3 \in B$, or the $(k+1)$-tuple $(1,1,\ldots,1,3l-2,3l+k-3)$ would be a red solution.

We next show that $l+1 \in B$. To this end, we first show that $k+2 \in B$. To deduce this we show that $3 \in R$. If $3 \in B$, then we would have the blue solution given by the $(l+1)$-tuple $(3,3,\ldots,3,k,3l+k-3)$. Hence, $3 \in R$. Since $1,3 \in R$ we see that $k+2 \in B$. We next show that $l+1 \in B$. Assume, for a contradiction, that $l+1 \in R$. Then we must have $2l+k \in B$ to avoid the red solution given by the $(k+1)$-tuple $(1,1,\ldots,1,l+1,l+1,2l+k)$. But this leads to the blue solution given by the $(l+1)$-tuple $(2,2,\ldots,2,k+2,2l+k)$, the desired contradiction.

Since $2,k,l+1$ and $3l+k-3$ are all blue, the $(l+1)$-tuple $(2,2,\ldots,2,k,l+1,3l+k-3)$ gives a blue solution, a contradiction.

Appendix

In the original proofs of the upper bounds the Maple package `AUTOISSAI` (which can be downloaded at `http://math.colgate.edu/~aaron/`) was used to automatically deduce the color of many elements.

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