Multiplicity of monochromatic solutions to $x + y < z$

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Abstract

For integers $n \geq 1$ and $k \geq 0$, let $M_k(n)$ represent the minimum number of monochromatic solutions to $x + y < z$ over all 2-colorings of $\{k+1, k+2, \ldots, k+n\}$. We show that for any $k \geq 0$, $M_k(n) = Cn^3(1 + o_k(1))$, where $C = \frac{1}{12(1+2\sqrt{2})^2} \approx .005686$. A structural result is also proven, which can be used to determine the exact value of $M_k(n)$ for given k and n.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and let $[a, b]$ denote the interval $\{n \in \mathbb{N} : a \leq n \leq b\}$. A function $\Delta : [a, b] \to [0, t-1]$ is referred to as a t-coloring of the set $[a, b]$. Given a t-coloring Δ and a system of linear equations or inequalities in m variables, a solution $(x_1, x_2, \ldots, x_m)$ to the system is monochromatic if and only if $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m)$.

In 1916, I. Schur [14] proved that for every $t \geq 2$, there exists a least integer $n = S(t)$, such that for every t-coloring of the set $[1, n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_3. \tag{1}$$

The integers $S(t)$ are called *Schur numbers*. It is known that $S(2) = 5$, $S(3) = 14$, and $S(4) = 45$, but no other Schur number is known [16]. In 1933, R. Rado generalized the concept of Schur numbers to arbitrary systems of linear equations. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution under every t-coloring of the natural numbers [4], [9], [10], [11]. For a given system of linear equations or inequalities E, the least integer n, provided that it exists, such that for every t-coloring of the set $[1, n]$ there exists a monochromatic solution to E is called the *t-color Rado number* (also referred to as the *t-color generalized Schur number*) for the system E. If such an integer n does not exist, then the t-color Rado "number" for the system is said to be ∞.

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In recent years the exact Rado numbers for several families of equations and inequalities have been found, but almost entirely for 2-colorings [1], [2], [7], [13]. Several other problems related to Schur number and Rado numbers have also been considered [6], [8]. In 1995, Graham, Rödl and Ruciński [5] proposed the following problem: Find (asymptotically) the least number of monochromatic solutions to Equation (1) that must occur in a 2-coloring of the set $[1,n]$. A problem of this nature – where the number of monochromatic solutions is to be determined – is called a multiplicity problem. This problem was solved by Robertson and Zeilberger [12], and independently by Schoen [15], with a nice proof given later by Datskovsky [3]. The answer was found to be $\frac{n^2}{22}(1+o(1))$.

In this paper we modify the problem of Graham, Rödl and Ruciński by changing Equation (1) to a system of inequalities:

$$\begin{array}{c} x_1 + x_2 < x_3 \\ x_1 \leq x_2 \leq x_3. \end{array} \tag{2}$$

Our original goal was to minimize the number of monochromatic solutions to (2) on the interval $[1,n]$. It became apparent that it was easier to prove a stronger result. To this end, for all integers $n \geq 1$ and $k \geq 0$, let $M_k(n)$ represent the least number of monochromatic solutions to (2) that can occur in a 2-coloring (consisting of zeros and ones) of the interval $[k+1, k+n]$. We determine the general structure of 2-colorings of the set $[k+1, k+n]$ that contain the minimum number of monochromatic solutions to (2).

In Section 2, we show that among all 2-colorings of $[k+1, k+n]$ for which the number of integers colored zero is fixed, the number of monochromatic solutions to system (2) is minimized by a 2-block coloring, with the integers on the left side of the interval colored the minority color. The proof is based on induction on n, which is why it was easier to establish a stronger result; allowing all possible nonnegative integer values of k leads to a stronger inductive hypothesis.

The main asymptotic result, Theorem 2, is presented in Section 3. We then consider intervals of the type $[cn+1, (1+c)n]$ (where we let $cn = \lfloor cn \rfloor$) for certain "small" values of c. We find the somewhat surprising result that the minimum number of monochromatic solutions, asymptotically, in these types of intervals (which also have length n) – where both endpoints tend toward infinity – is roughly the same as in the case when only the right endpoint tends to infinity.

2. Structural Result

Definition 1 *For integers $k \geq -1, m \geq 0$, and $n \geq 1$, let $\mathcal{C}_m(k+1, k+n)$ be the set of all colorings of the form $\Delta : [k+1, k+n] \to [0,1]$ such that $|\Delta^{-1}(0)| = m$. Let $\Delta^m_{k+1,k+n} \in \mathcal{C}_m(k+1, k+n)$ be the coloring defined by*

$$\Delta^m_{k+1,k+n}(x) = \begin{cases} 0 & \text{if } k+1 \leq x \leq k+m \\ 1 & \text{if } k+m+1 \leq x \leq k+n. \end{cases}$$

For a given integer interval $[a,b]$, let us denote by $S(a,b)$ the number of solutions to (2), with $x_1, x_2, x_3 \in [a,b]$. For a given $\Delta \in \mathcal{C}_m(k+1, k+n)$, we denote the number of monochromatic solutions to (2) under Δ by $S(\Delta)$.

The following theorem states the fundamental result in this paper: A "two block" coloring with the "minority color" coming first will minimize the number of monochromatic solutions to (2) over all 2-colorings with a fixed number of integers having the minority color.

Theorem 1. *Let $k \geq 0, m \geq 0$, and $n \geq 1$ be integers, with $m \leq \frac{n}{2}$. If $\Delta \in \mathcal{C}_m(k+1, k+n)$, then $S(\Delta) \geq S\left(\Delta^m_{k+1,k+n}\right)$.*

Before proving Theorem 1, let us introduce some more terminology. For a given integer interval $[a, b]$, a solution $(x_1, x_2, x_3) \in [a, b]^3$ to (2) is called a *Type I solution* if $x_1, x_2 > a$, a *Type II solution* if $x_1 = a$, $x_2 > a$, and a *Type III solution* if $x_1 = x_2 = a$. For a given coloring Δ we will also denote the corresponding number of monochromatic solutions to (2) of a given type by $S_I(\Delta), S_{II}(\Delta)$, and $S_{III}(\Delta)$, with the obvious relation

$$S(\Delta) = S_I(\Delta) + S_{II}(\Delta) + S_{III}(\Delta). \tag{3}$$

Let us now collect a few facts that will be used in the proof of Theorem 1. We state the next two lemmas with the assumption that $\Delta(k+1) = 0$, but it is obvious that analogous facts are true when $\Delta(k+1) = 1$.

Before stating the first lemma, we make another definition.

Definition 2 *Let $k \geq -1$, $m \geq 1, n \geq 1$, and $s \geq 2$ be integers, with $s \leq n - m + 2$. We define ${}^s\Delta^m_{k+1,k+n} : [k+1, k+n] \to [0, 1]$ to be the 2-coloring*

$${}^s\Delta^m_{k+1,k+n}(x) = \begin{cases} 0 & \text{if } x = k+1 \\ 1 & \text{if } k+2 \leq x \leq k+s-1 \\ 0 & \text{if } k+s \leq x \leq k+s+m-2 \\ 1 & \text{if } k+s+m-1 \leq x \leq k+n. \end{cases}$$

Note that ${}^2\Delta^m_{k+1,k+n} = \Delta^m_{k+1,k+n}$.

Lemma 2. *For integers $k \geq -1, m \geq 0, n \geq 1$, and $s \geq 2$ with $s \leq n - m + 2$, if $\Delta \in \mathcal{C}_m(k+1, k+n)$ and $\Delta(k+1) = 0$, then $S_{II}(\Delta) \geq S_{II}\left({}^s\Delta^m_{k+1,k+n}\right)$.*

Proof. Let a_i represent the i^{th} integer colored zero by Δ. Consider solutions that are monochromatic under Δ and of the form $(k+1, a_i, x_3)$. For $k \geq 0$, given a_i, we see, by construction, that ${}^s\Delta^m_{k+1,k+n}$ maximizes the number of a_j, $j > i$, such that $a_j \leq k+1+a_i$, thereby minimizing the number of Type II solutions. This is because the integers $a_j > k+1$ that are of color 0 are consecutive. For $k = -1$ we have $S_{II}(\Delta) = \binom{m-1}{2}$ for any $\Delta \in \mathcal{C}_m(0, n-1)$ with $\Delta(0) = 0$, so the results holds. □

Remark. It is useful to note that the result in Lemma 2 is not dependent on the choice of s (the bounds given are only used with respect to Definition 2).

Lemma 3. *For integers $k \geq -1, m \geq 0$, and $n \geq 1$, if $\Delta \in \mathcal{C}_m(k+1, k+n)$ and $\Delta(k+1) = 0$, then $S_{III}(\Delta) \geq S_{III}\left(\Delta^m_{k+1,k+n}\right)$.*

Proof. For $k = -1$ we have $S_{III}(\Delta) = m$ for any $\Delta \in \mathcal{C}_m(0, n-1)$ with $\gamma(0) = 0$. Hence, we assume $k \geq 0$ in the following. Since there are at most $k+2$ integers of color 0 under Δ that are less than $2k+3$, we see that

$$S_{III}(\Delta) = \left|\{x_3 \mid x_3 \in [2k+3, k+n] \text{ and } \Delta(x_3) = 0\}\right| \geq \max(0, m-k-2).$$

Since $S_{III}(\Delta^m_{k+1,k+n}) = \max(0, m-k-2)$, we are done. □

The next two lemmas compare the number of solutions to (2) in the intervals $[a, b]$, $[a+1, b]$, and $[a, b-1]$. Recall that $S(a, b)$ is the number of solutions to (2) with $x_1, x_2, x_3 \in [a, b]$.

Lemma 4. *Let a and b be nonnegative integers with $a < b$. Then*

$$S(a,b) - S(a+1,b) = \begin{cases} \frac{1}{2}(b-2a)(b-2a+1) & \text{if } 2a < b \\ 0 & \text{if } 2a \geq b. \end{cases}$$

Proof. We need to count the number of solutions of the form (a, x_2, x_3). In other words, we need to count the number of lattice points $(x_2, x_3) \in \mathbb{Z}^2$ that lie inside the triangular region defined by the conditions $x_3 > x_2 + a$, $x_2 \geq a$, and $x_3 \leq b$. These conditions define an empty set if $2a \geq b$. If $2a < b$, then there are exactly $b - 2a - i$ lattice points in this region and on the line $x_2 = a + i$ for $i = 0, 1, \ldots, b - 2a - 1$. Therefore, for $2a < b$ we have $S(a,b) - S(a+1,b) = 1 + 2 + \cdots + (b-2a) = \frac{1}{2}(b-2a)(b-2a+1)$. □

Lemma 5. *Let a and b be nonnegative integers with $a < b$. Then*

$$S(a,b) - S(a,b-1) = \begin{cases} \left\lfloor \left(\frac{b-2a+1}{2}\right)^2 \right\rfloor & \text{if } 2a < b \\ 0 & \text{if } 2a \geq b. \end{cases}$$

Proof. We enumerate all solutions of the form (x_1, x_2, b). Again, it is convenient to think in geometrical terms; that is, we count the number of lattice points $(x_1, x_2) \in \mathbb{Z}^2$ that satisfy the conditions $x_2 < b - x_1$, $x_2 \geq x_1$, and $x_1 \geq a$. If $2a \geq b$, then these conditions define an empty set. If $2a < b$, then there are exactly $b - 2a - 2i$ lattice points in this region and on the line $x_1 = a + i$, for $i = 0, 1, \ldots, \left\lfloor \frac{b-2a}{2} \right\rfloor$. Hence, the difference $S(a,b) - S(a,b-1)$ is the sum of all odd or all even integers up to $b - 2a$, depending on the parity of $b - 2a$. For completeness, we will perform the calculation: If $b - 2a$ is odd, then $S(a,b) - S(a,b-1) = 1 + 3 + \cdots + (b-2a) = \frac{1}{4}(b-2a+1)^2$, while, if $b - 2a$ is even, then $S(a,b) - S(a,b-1) = 2 + 4 + \cdots + (b-2a) = \frac{1}{4}(b-2a)(b-2a+2) = \frac{1}{4}(b-2a+1)^2 - \frac{1}{4}$. □

We have now established the results we will use to prove our main result.

Proof of Theorem 1. We use induction on n, the length of the interval being colored. When $n = 1$, the result is trivial. Let $n \in \mathbb{N}$ be given and assume that the conclusion of the theorem holds for every $k \geq 0$ and for every $m \in [0, \lfloor n/2 \rfloor]$ when the length of the interval to be colored is n. We will prove that the conclusion holds for every $k \geq 0$ and for every $m \in [0, \lfloor (n+1)/2 \rfloor]$ when the length of the interval to be colored is $n+1$.

Let $k \geq 0$ be given. For convenience we will consider the interval $[k, k+n]$ rather than the interval $[k+1, k+n+1]$. The case where $m = 0$ is trivial, so we may assume $m \geq 1$. Let $m \in [1, \lfloor (n+1)/2 \rfloor]$ be given and let $\Delta \in \mathcal{C}_m(k, k+n)$ be given. We will show that $S(\Delta) \geq S\left(\Delta^m_{k,k+n}\right)$. If $m = \frac{n+1}{2}$, then $|\Delta^{-1}(0)| = |\Delta^{-1}(1)|$. So without loss of generality we may assume that $\Delta(k) = 0$. Hence we may consider two cases, the case where $\Delta(k) = 0$ and the case where $\Delta(k) = 1$ and $m < \frac{n+1}{2}$.

`Case I.` $\Delta(k) = 0$. Consider the two colorings produced when Δ and $\Delta^m_{k,k+n}$ are restricted to the interval $[k+1, k+n]$, namely $\Delta|_{[k+1,k+n]}$ and $\Delta^m_{k,k+n}|_{[k+1,k+n]}$, and note that $\Delta^m_{k,k+n}|_{[k+1,k+n]} = \Delta^{m-1}_{k+1,k+n}$.

By the inductive hypothesis, $S\left(\Delta|_{[k+1,k+n]}\right) \geq S\left(\Delta^m_{k,k+n}|_{[k+1,k+n]}\right)$. Now, monochromatic solutions to (2) on the interval $[k+1, k+n]$ are precisely the Type I solutions on the interval $[k, k+n]$; that is, $S_I(\Delta) \geq S_I\left(\Delta^m_{k,k+n}\right)$. Recalling that ${}^2\Delta^m_{k,k+n} = \Delta^m_{k,k+n}$, it follows from Lemma 2 that $S_{II}(\Delta) \geq S_{II}\left(\Delta^m_{k,k+n}\right)$. From Lemma 3, we have $S_{III}(\Delta) \geq S_{III}\left(\Delta^m_{k,k+n}\right)$. Using the above three inequalities along with (3), it follows immediately that $S(\Delta) \geq S\left(\Delta^m_{k,k+n}\right)$ and Case I is complete. ⋄

`Case II.` $\Delta(k) = 1$ and $m < \frac{n+1}{2}$. For this case we define two additional colorings in $C_m(k, k+n)$. Let $\tilde{\Delta}$,

$\tilde{\tilde{\Delta}} \in C_m\,(k, k+n)$ be defined by

$$\tilde{\Delta} = \begin{cases} 1 & \text{if } x = k \\ 0 & \text{if } k+1 \le x \le k+m \\ 1 & \text{if } k+m+1 \le x \le k+n \end{cases} \quad \text{and} \quad \tilde{\tilde{\Delta}} = \begin{cases} 1 & \text{if } k \le x \le k+n-m \\ 0 & \text{if } k+n-m+1 \le x \le k+n. \end{cases}$$

As in Case I, we consider $\Delta\,|_{[k+1,k+n]}$ and $\Delta^m_{k,k+n}\,|_{[k+1,k+n]} = \Delta^{m-1}_{k+1,k+n}$. Since $m \le \frac{n}{2}$, from the inductive hypothesis, we have $S\left(\Delta\,|_{[k+1,k+n]}\right) \ge S\left(\tilde{\Delta}\,|_{[k+1,k+n]}\right)$. Hence,

$$S_I\,(\Delta) \ge S_I(\tilde{\Delta}). \tag{4}$$

By considering the analogous version of Lemma 2 where the first integer in the interval is instead colored 1, we obtain

$$S_{II}\,(\Delta) \ge S_{II}(\tilde{\Delta}). \tag{5}$$

By considering the similarly analogous version of Lemma 3, we get

$$S_{III}\,(\Delta) \ge S_{III}(\tilde{\tilde{\Delta}}).$$

Adding and subtracting $S_{III}(\tilde{\Delta})$ on the right-side of the last inequality gives us

$$S_{III}\,(\Delta) \ge S_{III}(\tilde{\Delta}) - \left(S_{III}(\tilde{\Delta}) - S_{III}(\tilde{\tilde{\Delta}})\right). \tag{6}$$

By adding inequalities (4), (5), and (6) and using relationship (3), we obtain

$$S\,(\Delta) \ge S(\tilde{\Delta}) - \left(S_{III}(\tilde{\Delta}) - S_{III}(\tilde{\tilde{\Delta}})\right). \tag{7}$$

In this situation, $S_{III}(\Delta) = \left|\{x_3 \in [k, k+n] \mid 2k < x_3 \text{ and } \Delta\,(x_3) = 1\}\right|$ for any given 2-coloring Δ. It follows that

$$S_{III}(\tilde{\Delta}) = \begin{cases} n-m & \text{if } 0 \le k \le m \\ n-k & \text{if } m+1 \le k \le n \\ 0 & \text{if } n < k \end{cases} \tag{8}$$

and

$$S_{III}(\tilde{\tilde{\Delta}}) = \begin{cases} n-m-k & \text{if } 0 \le k \le n-m \\ 0 & \text{if } n-m < k \end{cases}$$

so that

$$S_{III}(\tilde{\Delta}) - S_{III}(\tilde{\tilde{\Delta}}) = \begin{cases} k & \text{if } 0 \le k \le m \\ m & \text{if } m+1 \le k \le n-m \\ n-k & \text{if } n-m+1 \le k \le n \\ 0 & \text{if } n < k. \end{cases}$$

We will now calculate $S(\tilde{\Delta}) - S\left(\Delta^m_{k,k+n}\right)$. Let $S(\tilde{\Delta})\,|_{x_1=k}$ represent the number of solutions to (2) with $x_1 = k$ that are monochromatic under $\tilde{\Delta}$. Define $S(\tilde{\Delta})\,|_{x_3=k+m}$, $S(\Delta^m_{k,k+n})\,|_{x_1=k}$, and $S(\Delta^m_{k,k+n})\,|_{x_1=k+m}$ in a similar manner.

Since $\Delta^m_{k,k+n}$ and $\tilde{\Delta}$ differ only in the color of integers k and $k+m$, we have

$$S(\tilde{\Delta}) - S\left(\Delta^m_{k,k+n}\right) = S(\tilde{\Delta})\,|_{x_1=k} - S(\Delta^m_{k,k+n})\,|_{x_1=k} + S(\tilde{\Delta})\,|_{x_3=k+m} - S(\Delta^m_{k,k+n})\,|_{x_1=k+m}.$$

It is clear that $S(\tilde{\Delta})\,|_{x_1=k} = S_{II}(\tilde{\Delta}) + S_{III}(\tilde{\Delta})$. The value of $S_{III}(\tilde{\Delta})$ is given in (8). As for $S_{II}(\tilde{\Delta})$, we have

$$\begin{aligned} S_{II}(\tilde{\Delta}) &= \left|\{x_2, x_3 \in [k+m+1, k+n] : k + x_2 < x_3\right| \\ &= \begin{cases} \frac{1}{2}\,(n-k-m-1)\,(n-k-m) & \text{if } 0 \le k \le n-m \\ 0 & \text{if } n-m+1 \le k. \end{cases} \end{aligned}$$

Since $S(\Delta^m_{k,k+n})\,|_{x_1=k} = S(k, k+m-1) - S(k+1, k+m-1)$, from Lemma 4 we have

$$S(\Delta^m_{k,k+n})\,|_{x_1=k} = \begin{cases} \frac{1}{2}(m-k-1)(m-k) & \text{if } 0 \le k \le m-2 \\ 0 & \text{if } m-1 \le k. \end{cases}$$

Since $S(\tilde{\Delta})\,|_{x_3=k+m} = S(k+1, k+m) - S(k+1, k+m-1)$, from Lemma 5 we have

$$S(\tilde{\Delta})\,|_{x_3=k+m} = \begin{cases} \left\lfloor \left(\frac{m-k-1}{2}\right)^2 \right\rfloor & \text{if } 0 \le k \le m-3 \\ 0 & \text{if } m-2 \le k. \end{cases}$$

Lemma 4 and the fact that $S(\Delta^m_{k,k+n})\,|_{x_1=k+m} = S(k+m, k+n) - S(k+m+1, k+n)$ also gives us

$$S(\Delta^m_{k,k+n})\,|_{x_1=k+m} = \begin{cases} \frac{1}{2}(n-k-2m)(n-k-2m+1) & \text{if } 0 \le k \le n-2m-1 \\ 0 & \text{if } n-2m \le k. \end{cases}$$

We will now show that $S(\tilde{\Delta}) - S\left(\Delta^m_{k,k+n}\right) \ge S_{III}(\tilde{\Delta}) - S_{III}(\tilde{\tilde{\Delta}})$. In other words, if we let

$$E = S_{III}(\tilde{\Delta}) + S_{II}(\tilde{\Delta}) - S(\Delta^m_{k,k+n})\,|_{x_1=k} + S(\tilde{\Delta})\,|_{x_3=k+m} - S(\Delta^m_{k,k+n})\,|_{x_1=k+m}$$

and let $G= S_{III}(\tilde{\Delta}) - S_{III}(\tilde{\tilde{\Delta}})$, we will show that $E \ge G$ for every $k \ge 0$.

Assume that $0 \le k \le m-2$. In this situation $G = k$. If $k \ge n-2m$, then

$$E = n - m + \frac{(n-k-m-1)(n-k-m)}{2} - \frac{(m-k-1)(m-k)}{2} + \left\lfloor \left(\frac{m-k-1}{2}\right)^2 \right\rfloor - 0.$$

Now let $s = n-2m$ and $t = m-k-1$. Note that both s and t are nonnegative. Our equation for E can be rewritten as

$$E = t + k + s + 1 + \frac{(t+s)(t+s+1)}{2} - \frac{t(t+1)}{2} + \left\lfloor \left(\frac{t}{2}\right)^2 \right\rfloor.$$

From here, it is clear that $E \ge k = G$. If, on the other hand, $k < n-2m$, then

$$E = n - m + \frac{(n-k-m-1)(n-k-m)}{2} - \frac{(m-k-1)(m-k)}{2} + \left\lfloor \left(\frac{m-k-1}{2}\right)^2 \right\rfloor - \frac{(n-k-2m)(n-k-2m+1)}{2}.$$

Let $x = n-2m-k-1$ and $w = m-k-1$. Note that both x and w are nonnegative and that $m > w$. Also, $n-m = k+m+x+1$, $(n-k-2m)(n-k-2m+1) = (x+1)(x+2)$, and $x+1-\frac{(x+1)(x+2)}{2} = -\frac{x(x+1)}{2}$. This allows us to rewrite E as

$$E = k + m + \frac{(x+m)(x+m+1) - w(w+1)}{2} + \left\lfloor \left(\frac{w}{2}\right)^2 \right\rfloor - \frac{x(x+1)}{2}.$$

Since $0 \le w < m$,, we have $(x+m)(x+m+1) \ge x(x+1) + m(m+1) > x(x+1) + w(w+1)$, so that $E \ge k + m + \left\lfloor \left(\frac{w}{2}\right)^2 \right\rfloor > k = G$.

For the remaining cases we have $k \ge m-1$. Hence, $S(\Delta^m_{k,k+n})\,|_{x_1=k} = 0$, $S(\tilde{\Delta})\,|_{x_3=k+m} = 0$, and $S_{II}(\tilde{\Delta}) \ge S(\Delta^m_{k,k+n})\,|_{x_1=k+m}$. Dropping these from E we get $E \ge S_{III}(\tilde{\Delta})$. By definition, we have $G \le S_{III}(\tilde{\Delta})$. Hence, $E \ge G$ for $k \ge m-1$.

Since we have shown that $E \ge G$ for all $k \ge 0$, as desired, the proof of this case is complete. ⋄

Since Cases I and II exhaust all possibilities, the proof of Theorem 1 is complete. □

3. Asymptotic Results

Before stating and proving the main result of this section, we will give an asymptotic formula for the number of solutions to (2) in an arbitrary interval (regardless of whether or not the solutions are monochromatic).

Lemma 6. *Let $a < b$ be positive integers. The number of solutions in $[a, b]$ to $x_1 + x_2 < x_3$ with $x_1 < x_2$ is*

$$\begin{cases} \frac{(b-2a)^3}{12} + O((b-a)^2) & \text{if } 2a < b \\ 0 & \text{if } 2a \geq b. \end{cases}$$

Proof. The case $2a \geq b$ is trivial so we consider the case when $2a < b$. Let $i \in [a, b]$ and consider pairs (x_1, x_2) as the following sets: $S_i = \{(i, j) : j \in [a, b - i - 1]\}$. Note that every pair in every S_i provides a solution to $x_1 + x_2 < x_3$, but that (x_1, x_2) and (x_2, x_1) are counted as distinct. Moreover, there is no pair $(x_1, x_2) \in [a, b]^2$ such that $x_1 + x_2 < x_3$ with $x_3 \leq b$ that is not a member of some S_i, $a \leq i \leq b - a - 1$. Lastly, for fixed i and j, there are $b - (i + j)$ possible solutions for x_3, provided $i + j \leq b - 1$. Hence, the number of solutions to $x_1 + x_2 < x_3$ in $[a, b]$ is given by

$$\begin{aligned} \frac{1}{2} \sum_{i=a}^{b-a-1} \sum_{j=a}^{b-i-1} (b - i - j) &= \frac{1}{2} \sum_{i=a}^{b-a-1} \frac{(b-a-i)(b-a-i+1)}{2} = \frac{1}{4} \sum_{i=a}^{b-a-1} \left((b-a-i)^2 + O(b-a)\right) \\ &= \frac{(b-2a)(b-2a+1)(2(b-2a)+1)}{24} + O((b-a)(b-2a)) \\ &= \frac{(b-2a)^3}{12} + O((b-a)^2). \end{aligned}$$

□

Having Lemma 6 behind us, we now present and prove one of our main results.

Theorem 2 *For any given integer $k \geq 0$, the minimum number of monochromatic solutions to (2) that can occur in any 2-coloring of $[k + 1, k + n]$ is*

$$M_k(n) = Cn^3(1 + o_k(1)),$$

where $C = \frac{1}{12(1+2\sqrt{2})^2} \approx .005685622025$.

Proof. As in the rest of this paper, we let m be the number of integers of color 0. We may assume $m \leq \frac{n}{2}$. Using Theorem 1, we need only enumerate the monochromatic solutions under $\Delta^m_{k+1,k+n}$. Furthermore, we need only enumerate those solutions with $x_1 < x_2$, since solutions with $x_1 = x_2$ account for $O_k(n^2)$ of the total number of monochromatic solutions, and this can be incorporated into the error term. We suppress all other error terms for ease of reading. We separate the proof into 3 cases as a consequence of Lemma 6.

`Case I.` $k + 3 \leq m \leq \frac{n-k}{2}$. We start our enumeration with color 0. The elements of color 0 are those in $[k + 1, k + m]$. From Lemma 6, we have $\frac{(m-k)^3}{12}$ solutions to $x_1 + x_2 < x_3$ of color 0 (note that $k + 2 < m$ is needed here). For color 1, we consider the interval $[k + m + 1, k + n]$. Applying Lemma 6 again (which is valid by our assumption on m), we have $\frac{(n-k-2m)^3}{12}$ solutions of color 1.

Hence, our goal is to minimize, over $m \in \left[k + 3, \frac{n-k}{2}\right]$, the function

$$f_k(m) = \frac{(m-k)^3 + (n-k-2m)^3}{12}.$$

We have $4f_k'(m) = (m-k)^2 - \left(\sqrt{2}(n-k-2m)\right)^2$, so that our minimum occurs when $m - k = \sqrt{2}(n - k - 2m)$, i.e., at

$$\hat{m} = \frac{\sqrt{2}}{1+2\sqrt{2}}n - \frac{\sqrt{2}-1}{1+2\sqrt{2}}k. \tag{9}$$

This gives us

$$f_k(\hat{m}) = \frac{1}{12}\left(1 + \frac{1}{2\sqrt{2}}\right)(\hat{m}-k)^3 = \frac{1}{12}\left(\frac{1}{(1+2\sqrt{2})^2}\right)(n-3k)^3.$$

Letting $n \to \infty$ while k is fixed, we see that

$$\lim_{n\to\infty} \frac{f_k(\hat{m})}{n^3} = \frac{1}{12(1+2\sqrt{2})^2}.$$

It now remains to check this value against the endpoints $m = k+3$ and $m = \frac{n-k}{2}$. This is a routine calculation for which we obtain values for $\lim_{n\to\infty} \frac{f_k(\hat{m})}{n^3}$ of $\frac{1}{12}$ and $\frac{1}{96}$, respectively. Since $\frac{1}{12(2+2\sqrt{2})^2}$ is less than both of these values, the minimum in this case occurs at $\hat{m}$. ⋄

`Case II.` $\frac{n-k}{2} + 1 \leq m \leq \frac{n}{2}$. This is similar to Case I, except that there is no monochromatic solution of color 1 to consider since $2(k+m+1) \geq k+n$ in this case. Hence, we have only $\frac{(m-k)^3}{12}$ monochromatic solutions to $x + y < z$ (all of color 0). Thus, we want to minimize, over $m \in \left[\frac{n-k}{2}+1, \frac{n}{2}\right]$, the function

$$h_k(m) = \frac{(m-k)^3}{12}.$$

We clearly have $m > k$ for large n, so that $h(m)$ is an increasing function on $\left[\frac{n-k}{2}+1, \frac{n}{2}\right]$. Hence, it takes its minimum value at $\hat{m} = \frac{n-k}{2} + 1$. This gives us

$$h_k(\hat{m}) = \frac{(n-3k+2)^3}{96}.$$

Letting $n \to \infty$ with k fixed, we see that $\lim_{n\to\infty} \frac{h_k(\hat{m})}{n^3} = \frac{1}{96}$. ⋄

`Case III.` $1 \leq m \leq k+2$. In this case, the only monochromatic solutions are of color 1. Hence, we have $\frac{(n-k-2m)^3}{12}$ monochromatic solutions. Defining

$$j_k(m) = \frac{(n-k-2m)^3}{12},$$

we want to minimize $j_k(m)$ over $[1, k+2]$. Since $1 \leq m \leq k+2$, we have $\frac{(n-3k-4)^3}{12} \leq j_k(m) \leq \frac{(n-k-2)^3}{12}$. As k is fixed, $j_k(m)$ clearly has a rate of growth of $\frac{n^3}{12}$. ⋄

Considering the minima in all 3 cases, we see that $\min\left(\frac{1}{12(2+2\sqrt{2})^2}, \frac{1}{16}, \frac{1}{12}\right) = \frac{1}{12(2+2\sqrt{2})^2}$, thereby completing the proof. □

4. Concluding Remarks

As mentioned in the Introduction, the exact values of $M_k(n)$ can be determined for specific n and k, using Theorem 1. Also, applying the affine transformation $\phi(x) = x - k$ to the interval $[k+1, k+n]$, we have that $M_k(n)$ is also the number of monochromatic solutions to the inequality

$$x_1 + x_2 < x_3 - k, \tag{10}$$

with $x_1 \leq x_2$, on the interval $[1, n]$. In this paper, k was only permitted to assume nonnegative values (as well as -1 in certain of the lemmas). For the sake of clarity and simplicity, we did not strive to achieve the most general result possible and we did not prove all the results for a general integer value of k. In particular, for $k = -1$, inequality (10) becomes $x_1 + x_2 \leq x_3$ and no dramatic changes in the proofs are required. Other negative values of k would need to be examined more carefully. While we do not expect any big surprises, we leave the cases $k \leq -2$ open.

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