

# DOWN THE LARGE RABBIT HOLE

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ABSTRACT. This article documents my journey down the rabbit hole, chasing what I have come to know as a particularly unyielding problem in Ramsey theory on the integers: the 2-Large Conjecture. This conjecture states that if  $D \subseteq \mathbb{Z}^+$  has the property that every 2-coloring of  $\mathbb{Z}^+$  admits arbitrarily long monochromatic arithmetic progressions with common difference from  $D$  then the same property holds for any finite number of colors. We hope to provide a roadmap for future researchers and also provide some new results related to the 2-Large Conjecture.

**1. Prologue.** Mathematicians tend not to write of their failures. This is rather unfortunate as there are surely countless creative ideas that have never seen the light of day; I have long believed that a JOURNAL OF FAILED ATTEMPTS should exist. My goal with this article is 3-fold: (1) a chronicle of my battle with what I consider a particularly difficult conjecture; (2) to present my progress on this conjecture; and (3) to provide a roadmap to those who want to take on this challenging conjecture.

The majority of this work took place over the course of a year, circa 2010<sup>1</sup>. Since that time I have frequently revisited this intriguing problem, even though that year was mostly an exercise in banging my head against various brick walls. I wish I knew how to quit it. I love this conjecture, so much so that I've followed it down the rabbit hole. However, if we are to take away one message from Steinbeck's *Of Mice and Men*, it's that sometimes the rabbit doesn't love you back.

**2. What's Up, Doc?** Ramsey theory may best be summed up as “the study of the preservation of structures under set partitions”

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[17]. For this article, we will restrict our attention to the positive integers, and our investigation to the set of arithmetic progressions (our structure). As is common in Ramsey theory, we will use colors to denote set partition membership. Formally, for  $r \in \mathbb{Z}^+$ , an  $r$ -coloring of the positive integers is defined by  $\chi : \mathbb{Z}^+ \rightarrow \{0, 1, \dots, r-1\}$ . We say that  $S \subseteq \mathbb{Z}^+$  is *monochromatic* under  $\chi$  if  $|\chi(S)| = 1$ . In order to discuss the preservation of structure, and, subsequently, state the 2-Large Conjecture, we turn to a fundamental result in Ramsey theory on the integers: van der Waerden's Theorem [20].

**Theorem 1** (van der Waerden's Theorem). *For any fixed positive integers  $k$  and  $r$ , every  $r$ -coloring of  $\mathbb{Z}^+$  admits a monochromatic  $k$ -term arithmetic progression.*

In a certain sense, we cannot break the existence of arithmetic progressions via set partitioning since van der Waerden's Theorem proves that one of the partition classes *must* contain an arithmetic progression. If you don't believe me, try 2-coloring the first nine positive integers without creating a monochromatic 3-term arithmetic progression (I'll wait).

So now that we're all on board, the next attribute of arithmetic progressions to take note of is that they are closed under translation and dilation: if  $S = \{a, a + d, a + 2d, \dots, a + (k-1)d\}$  is a  $k$ -term arithmetic progression, and  $b$  and  $c$  are positive integers, then  $c + bS = \{(ab+c), (ab+c) + bd, (ab+c) + 2bd, \dots, (ab+c) + (k-1)bd\}$  is also a  $k$ -term arithmetic progression. It is this attribute that affords us a simple inductive argument when proving van der Waerden's Theorem. Specifically, assuming that the  $r = 2$  case of Theorem 1 is true (for all  $k$ ), we may prove that it is true for general  $r$  rather simply.

In order to proceed, we need a restatement of Theorem 1, which is often referred to as the finite version.

**Theorem 2** (van der Waerden's Theorem restatement). *For any fixed positive integers  $k$  and  $r$ , there exists a minimum integer  $w(k; r)$  such that every  $r$ -coloring of  $\{1, 2, \dots, w(k; r)\}$  admits a monochromatic  $k$ -term arithmetic progression.*

The proof of equivalence of Theorem 1 and Theorem 2 (at least the nontrivial direction) is given by *The Compactness Principle*, which, in this setting, could also be called *Cantor's Diagonal Principle*, as the proof is an application and slight modification of the diagonal argument Cantor used to prove that the set of real numbers is uncountable.

Now back to the induction argument. We may assume that  $w(k; s)$  exist for  $s = 2, 3, \dots, r - 1$  for any  $k \in \mathbb{Z}^+$ . Let  $m = w(k; r - 1)$  so that  $n = w(m; 2)$  exists. Consider  $\chi$ , an arbitrary  $r$ -coloring of  $\{1, 2, \dots, n\}$ . For ease of exposition, let the colors be red and  $r - 1$  different shades of blue. Consider someone who cannot distinguish between shades of blue so that the  $r$ -coloring looks like a 2-coloring to this person. By the definition of  $n$ , such a person would conclude that a monochromatic  $m$ -term arithmetic progression exists under  $\chi$ . If this monochromatic progression is red, we are done, so we assume that it is "blue." Let it be  $a + d, a + 2d, a + 3d, \dots, a + md$  and note that, since we can distinguish between shades of blue, we have an  $(r - 1)$ -colored  $m$ -term arithmetic progression. We have a one-to-one correspondence between  $(r - 1)$ -colorings of  $T = \{1, 2, \dots, m\}$  and  $a + dT = \{a + d, a + 2d, a + 3d, \dots, a + md\}$ . By the definition of  $m$  and because arithmetic progressions are closed under translation and dilation, we see that  $T$ , and hence  $a + dT$ , admits a monochromatic  $k$ -term arithmetic progression, thereby completing the inductive step.

Of course, the previous paragraph is only a partial proof since I made the significant assumption that Theorem 1 holds for two colors; however, we can state the following:

( $\star$ ) If every 2-coloring of  $\mathbb{Z}^+$  admits arbitrarily long monochromatic arithmetic progressions, then, for any  $r \in \mathbb{Z}^+$ , every  $r$ -coloring of  $\mathbb{Z}^+$  admits arbitrarily long monochromatic arithmetic progressions.

Having obtained this conditional result ( $\star$ ), the rabbit hole is starting to come into view.

Brown, Graham, and Landman [7] investigated a strengthening of Theorem 1 by restricting the set of allowable common differences.

**Definition** ( $r$ -large, large,  $D$ -ap). Let  $D \subseteq \mathbb{Z}^+$  and let  $r \in \mathbb{Z}^+$ . We refer to an arithmetic progression  $a, a + d, a + 2d, \dots, a + (k - 1)d$  with  $d \in D$  as a  $k$ -term  $D$ -ap. If for any  $k \in \mathbb{Z}^+$ , every  $r$ -coloring of  $\mathbb{Z}^+$

admits a monochromatic  $k$ -term  $D$ -ap, then we say that  $D$  is  $r$ -large. If  $D$  is  $r$ -large for all  $r \in \mathbb{Z}^+$ , then we say that  $D$  is large.

Using this definition, we would restate  $(\star)$  as:

$(\star)$  If  $\mathbb{Z}^+$  is 2-large, then  $\mathbb{Z}^+$  is large.

We now can read the sign above that rabbit hole. It has the following conjecture, due to Brown, Graham, and Landman [7], scrawled on it:

**Conjecture** (2-Large Conjecture). Let  $D \subseteq \mathbb{Z}^+$ . If  $D$  is 2-large, then  $D$  is large.

All known 2-large sets are also large. Some 2-large sets are:  $m\mathbb{Z}^+$  for any positive integer  $m$  (in particular, the set of even positive integers); the range of any integer-valued polynomial  $p(x)$  with  $p(0) = 0$ ; any set  $\{[n\alpha] : n \in \mathbb{Z}^+\}$  with  $\alpha$  irrational. We will be visiting all of these sets on our journey.

As we move forward, you may think you have spotted the rabbit, but that rabbit is cunning. Beware of false promise, which comes to you in hare clothing.

**3. The Carrot.** So, what makes this conjecture so appealing? Firstly, the 2-Large Conjecture is so very natural given the proof of conditional statement  $(\star)$ . Secondly, there are several *a priori* disparate tools in Ramsey theory at our disposal. Thirdly, who doesn't like a challenge; the lure of the carrot is strong (but don't disregard the stick).

We can approach this problem:

- (1) purely measure-theoretically,
- (2) using measure-theoretic ergodic systems,
- (3) using discrete topological dynamical systems,
- (4) algebraically through the Stone-Ćech compactification of  $\mathbb{Z}^+$ , and
- (5) combinatorically/using other ad-hoc methods.

Even though I have described these approaches as disparate, there are connections between them that will become clear as we carry on the investigation.

**3.1. Measure-theoretic Approach.** On the measure-theoretic front, we must start with Szemerédi's [19] celebrated result. For  $A \subseteq \mathbb{Z}^+$ , let  $\bar{d}(A)$  denote the upper density of  $A$ :  $\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n}$ .

**Theorem 3** (Szemerédi's theorem). *Any subset  $S \subseteq \mathbb{Z}^+$  with  $\bar{d}(S) > 0$  contains arbitrarily long arithmetic progressions.*

Szemerédi's proof has been called elementary, but it is anything but easy, straightforward, or simple. In fact, contained within his proof is a logical flow chart on 24 vertices with 36 directed edges that furnishes the reader with an overview of the intricate web of logic used to prove the seminal result.

So, how do we mesh this result with 2-large sets? Since every  $2k$ -term arithmetic progression with common difference  $d$  contains a  $k$ -term arithmetic progression with common difference  $2d$ , we have large sets with positive density (the set of even positive integers). A result in [7] shows that  $\{10^n : n \in \mathbb{Z}^+\}$  is not 2-large, so we have sets with 0 density that are not 2-large. Perhaps there is a density condition that distinguishes large and non-large sets. Unfortunately, further exploration shows this is not true.

We can have sets with positive upper density that are not 2-large and we can have sets with zero upper density that are 2-large. To this end, first consider the set of odd integers  $D_1$ . Coloring  $\mathbb{Z}^+$  by alternating red and blue, we do not even have a monochromatic 2-term  $D_1$ -ap. Hence,  $D_1$  has positive density but is not 2-large. Now consider the set of squares  $D_2$ . As a very specific case of a far reaching extension of Szemerédi's result, Bergelson and Liebman [4] have shown that  $D_2$  is large. More generally (but still not as general as the full theorem), Bergelson and Liebman proved the following result.

**Theorem 4** (Bergelson and Liebman). *Let  $p(x) : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be a polynomial with  $p(0) = 0$ . Then the set  $D = \{p(i) : i \in \mathbb{Z}^+\}$  is large. More precisely, any subset of  $\mathbb{Z}^+$  of positive upper density contains arbitrarily long  $D$ -aps.*

In quick order we have seen that distinguishing large and non-large sets solely by their densities is not the correct approach. However, the proof of Theorem 4 leads us to our next approach.

**3.2. Measure-theoretic Ergodic Approach.** Closely related to the above approach is the use of ergodic systems. The connection between Szemerédi's Theorem and ergodic dynamical systems is provided by Furstenberg's correspondence principle [9], which uses the following notations. We remark here that we are specializing all results to the integers and that the stated results do not necessarily hold in different ambient spaces; see, e.g., [5].

**Notation.** For  $S \subseteq \mathbb{Z}^+$  and  $n \in \mathbb{Z}$ , we let  $S - n = \{s - n : s \in S\}$ . For the remainder of the article, we reserve the symbol  $T$  for the shift operator that acts on  $\mathcal{X}$ , the family of infinite sequences  $x = (x_i)_{i \in \mathbb{Z}}$ , by  $Tx_n = x_{n+1}$ .

**Theorem 5** (Furstenberg's Correspondence Principle). *Let  $E \subseteq \mathbb{Z}^+$  with  $\bar{d}(E) > 0$ . Then, for any  $k \in \mathbb{Z}^+$ , there exists a probability measure-preserving dynamical system  $(\mathcal{X}, \mathcal{B}, \mu, T)$  with a set  $A \in \mathcal{B}$  such that  $\mu(A) = \bar{d}(E)$  and*

$$\bar{d}\left(\bigcap_{i=0}^k (E - in)\right) \geq \mu\left(\bigcap_{i=0}^k T^{-in} A\right)$$

for any  $n \in \mathbb{Z}^+$ .

The above result can be viewed as the impetus for ergodic Ramsey theory as a field of research. Furstenberg proved that there exists  $d \in \mathbb{Z}^+$  such that, for any  $A$  with  $\mu(A) > 0$  we have

$$\mu(A \cap T^{-d}A \cap T^{-2d}A \cap \dots \cap T^{-kd}A) > 0.$$

By Theorem 5, we have  $E \cap (E - d) \cap (E - 2d) \cap \dots \cap (E - kd) \neq \emptyset$ . Hence, by taking  $a$  in this intersection, we have  $\{a, a+d, a+2d, \dots, a+kd\} \subseteq E$ . Consequently, Furstenberg provided an ergodic proof of Szemerédi's theorem<sup>2</sup>.

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<sup>2</sup>Furstenberg used Banach upper density and not upper density

Having followed this path it seems we have hit another dead end in our journey; there appears to be no mechanism for controlling the number of colors in these arguments. Perhaps a non-measure-theoretic dynamical system approach can help.

**3.3. Topological Dynamical Systems Approach.** As ergodic systems are specific types of dynamical systems, the 2-Large Conjecture may be susceptible to the use of a different breed of dynamical system, namely a topological one.

We will denote the space of infinite sequences  $(x_n)_{n \in \mathbb{Z}}$  with  $x_i \in \{0, 1, \dots, r-1\}$  by  $\mathcal{X}_r$  and let  $T$  remain the shift operator acting on  $\mathcal{X}_r$ . Specializing to our situation, we state Birkhoff's Multiple Recurrence Theorem due to Furstenberg and Weiss [11] (see also [6]).

**Theorem 6** (Birkhoff's Multiple Recurrence Theorem). *Let  $k, r \in \mathbb{Z}^+$ . Under the product topology, for any open set  $U \subseteq \mathcal{X}_r$  there exists  $d \in \mathbb{Z}^+$  so that  $U \cap T^{-d}U \cap T^{-2d} \cap \dots \cap T^{-kd}U \neq \emptyset$ .*

Furstenberg's and Weiss' result allowed them to give a new proof of van der Waerden's Theorem: Define a metric for  $x, y \in \mathcal{X}_r$  by

$$d(x, y) = \left( \min_{i \in \mathbb{Z}^+} x(i) \neq y(i) \right)^{-1},$$

where  $x(i)$  denotes the value/color of the  $i^{\text{th}}$  positive term in  $x$ . A small  $d(x, y)$  means we have value/color agreement in the initial terms of  $x$  and  $y$ . Let  $x \in \mathcal{X}_r$  be the sequence corresponding to any given arbitrary  $r$ -coloring of  $\mathbb{Z}^+$ . Theorem 6 helps prove that there exists  $y \in \{T^m x\}_{m \in \mathbb{Z}^+}$  such that all of  $d(y, T^d y), d(y, T^{2d} y), \dots, d(y, T^{kd} y)$  are less than 1 for some  $d$ . Hence,  $y, T^d y, T^{2d} y, \dots, T^{kd} y$  all have the same first value/color. Since  $y = T^a x$  for some  $a$  we have  $x_a, x_{a+d}, x_{a+2d}, \dots, x_{a+kd}$  all of the same value/color, meaning that  $a, a+d, \dots, a+kd$  is a monochromatic arithmetic progression.

**Remark 7.** We can actually have a guarantee that all of  $d(y, T^d y), d(y, T^{2d} y), \dots, d(y, T^{kd} y)$  are less than any  $\epsilon > 0$ ; however, this is not needed to prove van der Waerden's Theorem. It does provide for some very interesting results like arbitrary long progressions all with the same common difference each starting in a set of arbitrarily long consecutive

intervals. It should also be remarked that this latter result can be shown combinatorially, too.

So how can we use this to attack the 2-Large Conjecture? Given a 2-large set  $D$ , we have a guarantee that over the space  $\mathcal{X}_2$  there exists  $y \in \{T^m x\}_{m \in \mathbb{Z}^+}$  such that all of  $d(y, T^d y), d(y, T^{2d} y), \dots, d(y, T^{kd} y)$  are less than 1 for some  $d \in D$ . Our goal is to prove that this criterion implies the same over the space  $\mathcal{X}_r$ .

Although Remark 7 states that all of  $d(y, T^d y), d(y, T^{2d} y), \dots, d(y, T^{kd} y)$  can be arbitrarily small, we can only guarantee they are less than 1 (with our given metric) if we require  $d$  from a 2-large set. Hence, we could convert an  $r$ -coloring to a binary equivalent 2-coloring if we discovered a result that a long enough  $(r-1)$ -colored  $D$ -ap admits a monochromatic  $k$ -term  $D$ -ap (we have no such result, but this idea will prove fruitful in Section 6).

**4. Back Where We Started.** Presently, it seems we have ended up back where we started. Fittingly, this recurrence phenomenon is a key notion in dynamical systems. Momentarily, before getting to the Stone-Ćech compactification, we'll have a diagram to aid in visualizing how the different types of results based on the above approaches relate to each other.

In Figure 1, we give implications between the types of recurrence we have considered thus far, followed by their definitions.

All of the following definitions are given with respect to arithmetic progressions over the integers; as such, some of the definitions are specific cases of more general definitions. Some of the implications above fail in more general settings.

**Definitions.** Let  $r \in \mathbb{Z}^+$ . Denote the set of infinite sequences  $(x_n)_{n \in \mathbb{Z}}$  with  $x_i \in \{0, 1, \dots, r-1\}$  by  $\mathcal{X}_r$  and let  $T$  be the shift operator acting on  $\mathcal{X}_r$ . For  $D \subseteq \mathbb{Z}^+$ , we say that  $D$  is

- (i) *chromatically  $k$ -recurrent* if, for any  $r$ , every  $r$ -coloring of  $\mathbb{Z}^+$  admits a monochromatic  $k$ -term arithmetic progression  $a, a+d, \dots, a+(k-1)d$  with  $d \in D$ .



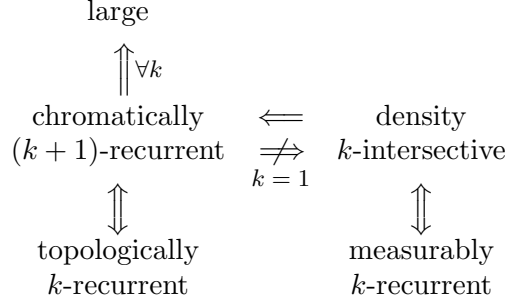


FIGURE 1. Relationship between types of recurrence considered thus far

(ii) *topologically  $k$ -recurrent* if, for any  $r$ , the dynamical system  $(\mathcal{X}_r, T)$  has the property that for every open set  $U \subseteq \mathcal{X}_r$  there exists  $d \in D$  such that

$$U \cap T^{-d}U \cap T^{-2d} \cap \dots \cap T^{-kd}U \neq \emptyset.$$

(iii) *density  $k$ -intersective* if for every  $A \subseteq \mathbb{Z}^+$  with  $\bar{d}(A) > 0$ , there exists  $d \in D$  such that

$$A \cap (A - d) \cap (A - 2d) \cap \dots \cap (A - kd) \neq \emptyset.$$

(iv) *measurably  $k$ -recurrent* if for any probability measure-preserving dynamical system  $(\mathcal{X}_r, \mathcal{B}, \mu, T)$  and any  $A \in \mathcal{B}$  with  $\mu(A) > 0$  there exists  $d \in D$  such that

$$\mu(A \cap T^{-d}A \cap T^{-2d}A \cap \dots \cap T^{-kd}A) > 0.$$

The fact that the double implications shown in Figure 1 are true has already been partially discussed; see [15] for details on the left double implication. The top-most implication is the definition of large. The negated implication was proved by Kříž [16] with nice write-ups by Jungić [15] and McCutcheon [18], while the remaining implication comes from the fact that any finite coloring of  $\mathbb{Z}^+$  contains a color class of positive upper density.

The negated implication offers a bit of insight – a tiny flashlight for our travels, if you will. The set used to prove this negation is the set

$C'$  given in [18]:

$$C' = \{c \in \{0, 1, \dots, M\} : c \pmod{p_i} = \pm 1 \text{ for at least } 2r \text{ indices } i\}$$

where  $r$  is sufficiently large,  $p_1, p_2, \dots, p_{2r+2}$  are sufficiently large primes, and  $M = \prod_{i=1}^{2r+2} p_i$ . We do not know if  $C'$  is 2-large/large or not (this seems to be a difficult problem in-and-of-itself), but if it is then there exists a set of positive upper density that does not contain a  $C'$ -ap. We should take this uncertainty as a warning that we have no guarantee a large set  $D$  has its  $D$ -aps lie inside a color class with positive upper density, even though Szemerédi's Theorem assures us that arbitrarily long arithmetic progressions are there.

**5. Stone-Čech Compactification Approach.** Although the three approaches in Section 3 all have nice links between them, the approach championed by Bergelson, Hindman, Strauss, and others is quite disparate from the previous methods we have seen. The approach is a blend of set theory, topology, and algebra. We'll start by describing the points in the Stone-Čech compactification on  $\mathbb{Z}^+$ , which requires the following definition (again, specialized to the positive integers).

**Definition** (filter, ultrafilter). Let  $p$  be a family of subsets of  $\mathbb{Z}^+$  (this lowercase  $p$  is the standard notation in this field). If  $p$  satisfies all of

- (i)  $\emptyset \notin p$ ;
- (ii)  $A \in p$  and  $A \subseteq B \Rightarrow B \in p$ ; and
- (iii)  $A, B \in p \Rightarrow A \cap B \in p$ ,

then we say that  $p$  is a *filter*. If, in addition,  $p$  satisfies

- (iv) for any  $C \subseteq \mathbb{Z}^+$  either  $C \in p$  or  $C^c = \mathbb{Z}^+ \setminus C \in p$

then we say that  $p$  is an *ultrafilter*. (Item (iv) means that  $p$  is not properly contained in any other filter.)

**Examples.** The set of subsets  $\mathcal{F} = \{A \subseteq \mathbb{Z}^+ : |\mathbb{Z}^+ \setminus A| < \infty\}$  is a filter but not an ultrafilter (it is known as the Fréchet filter). It is not an ultrafilter since, taking  $C$  from (iv) above to be the set of even positive integers we see that neither  $C$  nor its complement is in  $\mathcal{F}$ . The set  $\mathcal{G} = \{A \subseteq \mathbb{Z}^+ : x \in A\}$  for any fixed  $x \in \mathbb{Z}^+$  is an ultrafilter.

**Remark.** One hint that the ultrafilter direction may not prove useful is that the family of large sets is not a(n) (ultra)filter. Parts (i), (ii), and (iv) of the ultrafilter definition are satisfied, but part (iii) is not. To see this, consider  $A = \{i^3 : i \in \mathbb{Z}^+\}$  and  $B = \{i^3 + 8 : i \in \mathbb{Z}^+\}$ . These are both large sets (see [7]); however,  $A \cap B = \emptyset$  (Fermat's Last Theorem serves as a very useful result for counterexamples) and so cannot be large.

The Stone-Čech compactification of  $\mathbb{Z}^+$  is denoted by  $\beta\mathbb{Z}^+$ , and the points in  $\beta\mathbb{Z}^+$  are the ultrafilters, i.e.,  $\beta\mathbb{Z}^+ = \{p : p \text{ is an ultrafilter}\}$ . Having the space set, we need to define addition in  $\beta\mathbb{Z}^+$ .

**Definition** (addition in  $\beta\mathbb{Z}^+$ ). Let  $A \subseteq \mathbb{Z}^+$  and let  $p, q \in \beta\mathbb{Z}^+$ . As before,  $A - x = \{y \in \mathbb{Z}^+ : y + x \in A\}$ . We define the addition of two ultrafilters by  $A \in p + q \iff \{x \in \mathbb{Z}^+ : A - x \in p\} \in q$ .

The link between  $r$ -colorings and ultrafilters is provided by the following lemma.

**Lemma 8.** *Let  $r \in \mathbb{Z}^+$  and let  $p \in \beta\mathbb{Z}^+$ . For any  $r$ -coloring of  $\mathbb{Z}^+$ , one of the color classes is in  $p$ .*

*Proof.* Let  $\mathbb{Z}^+ = \cup_{i=1}^r C_i$ , where  $C_i$  is the  $i^{\text{th}}$  color class. By part (iv) of the definition of ultrafilter, if we assume that none of the  $C_i$ 's are in  $p$ , then each of their complements is in  $p$ . Applying part (iii) of the definition of a filter  $r-2$  times, we have  $\cap_{i=1}^{r-1} C_i^c \in p$ . But  $\cap_{i=1}^{r-1} C_i^c = C_r$  so  $C_r \in p$ , a contradiction.  $\square$

A number of results that use  $\beta\mathbb{Z}^+$  to give Ramsey-type results rely on the existence of an additive idempotent in  $\beta\mathbb{Z}^+$ . To this end, if  $p + p = p$  is such an element, then  $A \in p + p = p$  means that  $B = \{x \in \mathbb{Z}^+ : A - x \in p\}$  is in  $p$  as well. Thus, we have  $a \in A$  and  $d \in B$  such that  $a$  and  $a - d$  are both in  $A$ .

Hence, since  $p$  is a filter, by item (iii) above we have  $A \cap B \in p$ . Hence, appealing to Lemma 8, we can consider  $A$  to be a color class in an  $r$ -coloring so that we have a monochromatic solution to  $x + y = z$  with  $x = d$ ,  $y = a - d$ , and  $z = a$ . This result is known as Schur's Theorem.

In order to obtain van der Waerden's Theorem through the use of ultrafilters, quite a bit of algebra of  $\beta\mathbb{Z}^+$  is needed, so we will not present that here. The interested reader should consult the sublime book by Hindman and Strauss [14]. Unfortunately for our goal, while reading through this book it becomes quite clear that the number of colors used to state the Ramsey-type results is irrelevant and can be kept arbitrary in the arguments. So there does not seem a natural way to control for the number of colors. On the other hand, there are many types of *largeness* that can be proved using ultrafilters. A good description of largeness via ultrafilters is given by Bergelson and Downarowicz in [1]. An intuitive notion of largeness would be that a certain property  $P$  is large (in the general sense) if  $P$  itself is partition regular or if every set with property  $P$  is partition regular in some (possibly different) sense.

Perhaps one of these types of largeness will help achieve our goal. Figure 2, below, gives a summary of how these different types are related; most of the chart is due to Bergelson and Hindman [2].

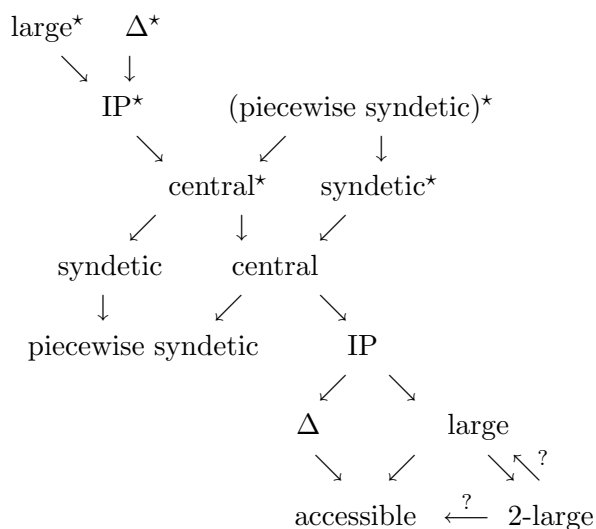


FIGURE 2. Type of largeness and implications. Missing implications are not true.

In defining the terms in Figure 2, we will start at the bottom and work our way up. As we move up the chart, the concepts are all types of largeness that increase in robustness.

**Definitions.** Let  $S \subseteq \mathbb{Z}^+$ . We say that  $S$  is

- (i) *accessible* if every  $r$ -coloring, for every  $r$ , admits arbitrarily long progressions  $x_1, x_2, \dots, x_n$  such that  $x_{i+1} - x_i \in S$  for  $i = 1, 2, \dots, n - 1$ .
- (ii) a  $\Delta$ -*set* if there exists  $T \subseteq \mathbb{Z}^+$  such that  $T - T \subseteq S$ .
- (iii) an *IP-set* if there exists  $T = \{t_i\} \subseteq \mathbb{Z}^+$  such that  $FS(T) = \{\sum_{f \in F} t_f : F \subseteq \mathbb{Z}^+ \text{ with } |F| < \infty\} \subseteq S$  (the notation  $FS(T)$  stands for the *finite sums* of  $T$ ).
- (iv) *piecewise syndetic* if there exists  $r \in \mathbb{Z}^+$  such that for any  $n \in \mathbb{Z}^+$  there exists  $\{t_1 < t_2 < \dots < t_n\} \subseteq S$  with  $t_{i+1} - t_i \leq r$  for  $1 \leq i \leq n - 1$ .
- (v) *syndetic* if there exists  $r \in \mathbb{Z}^+$  such that  $S = \{s_1 < s_2 < \dots\}$  satisfies  $s_{i+1} - s_i \leq r$  for all  $i \in \mathbb{Z}^+$ .
- (vi) a *central set* if  $S \in p$  where  $p$  is an idempotent ultrafilter with the property that for all  $A \in p$ , the set  $\{n \in \mathbb{Z}^+ : A + n \in p\}$  is syndetic. (The original, equivalent, definition comes from Furstenberg (see [10]) in the area of dynamical systems.)

The remaining categories in Figure 2 all have a  $*$  on them. This is the designation for the *dual* property. If  $X$  is one of the non-starred properties in Figure 2, then we say that a set  $S$  is in  $X^*$  if  $S$  intersects every set that has property  $X$ .

All implications and non-implications that do not involve any property in  $\{2\text{-large, large, accessible, large}^*\}$  are from [2]. The fact that accessible does not imply large was first shown by Jungić [15], who provided a non-explicit accessible set that is not 7-large. Recently, Guerreiro, Ruzsa, and Silva [12] provided an explicit accessible set that is not 3-large. In the next section, we give an explicit accessible set that is not 2-large, explaining why accessible does not imply 2-large. The fact that a  $\Delta$ -set is also accessible is from [17, Th. 10.27], while the fact that large implies accessible is straightforward. The set of cubes is large (and, hence, accessible) but not a  $\Delta$ -set. To see this, assume otherwise and let  $\{s_i\}$  be an increasing sequence of positive integers

such that  $s_j - s_i$  is a cube for all  $j > i$ . Then there exists  $t < u < v$  such that  $s_v - s_t, s_v - s_u$ , and  $s_u - s_t$  are all cubes. But then we have  $s_v - s_t = (s_v - s_u) + (s_u - s_t)$  as an integer solution to  $z^3 = x^3 + y^3$ , a contradiction. Large\* implying IP\* comes from the fact that all IP sets are large sets. To see that IP\* does not imply large\*, let  $D$  be the set of integer cubes. Then  $D \notin \text{IP}$  (in fact, for any  $x, y \in D$  we have  $x + y \notin D$ )<sup>3</sup>. Hence, any  $A \in \text{IP}$  cannot be a subset of  $D$ , meaning that  $A \cap D^c \neq \emptyset$ . Thus,  $D^c$  intersects all IP-sets; in other words,  $D^c \in \text{IP}^*$ . By Theorem 4, we know that  $D$  is large. Now, because  $D^c$  does not intersect the large set  $D$  we see that  $D^c \notin \text{large}^*$ .

Investigating some of the properties in Figure 2, we have some interesting results that could aid in proving or disproving the 2-large conjecture.

**Theorem 9** (Furstenberg and Weiss [11]). *Let  $k, r \in \mathbb{Z}^+$ . For any  $r$ -coloring of  $\mathbb{Z}^+$ , for some color  $i$ , the set of common differences of monochromatic  $k$ -term arithmetic progressions of color  $i$  is in  $\text{IP}^*$ .*

Theorem 9 gives a very strong property for the set of common differences in monochromatic arithmetic progressions. Along these same lines, we have the following result.

**Theorem 10** (Bergelson and Hindman [3]). *For any  $k \in \mathbb{Z}^+$  and any infinite subset of positive integers  $A$ , under any finite coloring of  $\mathbb{Z}^+$  there exists a monochromatic  $k$ -term arithmetic progression with common difference in  $FS(A)$ .*

Unfortunately, neither of these last two theorems helps guide us out of the rabbit hole, as generic 2-large sets don't necessarily have any obvious structures. So we now find ourselves moving into the darkest corner of the hole: the combinatorics encampment.

**6. Some Combinatorial Results.** We start this section by providing an accessible set that is not 2-large. Had this result not been possible, we would have disproved the 2-Large Conjecture (since there

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<sup>3</sup>I have a cute, short proof of this fact, but, like Fermat, there is not enough room in the bottom margin here, especially given the length of this lengthy, and unnecessary, footnote.

exists an accessible set that is not large) and been saved from the depths of the rabbit hole. But, alas, it was not meant to be.

**Theorem 11.** *There exists an accessible set that is not 2-large.*

*Proof.* This proof takes as inspiration the proof from [12], which provides an accessible set that is not 3-large. Let  $S = \{2^{4i} : i \geq 0\}$ . From [17, Th. 10.27], we know that  $S - S = \{2^{4j} - 2^{4i} : 0 \leq i < j\}$  is an accessible set. We will provide a 2-coloring of  $\mathbb{Z}^+$  that avoids monochromatic 25-term  $(S - S)$ -aps.

For each  $n \in \mathbb{Z}^+$ , write  $n$  in its binary representation so that  $\sum_{i \geq 0} b_i(n)2^i = n$ . Next, partition the  $b_i = b_i(n)$  into intervals of length 4:

$$[\dots b_i b_{i-1} \dots b_2 b_1 b_0] = \bigcup_{j \geq 0} I_j(b),$$

where  $I_j(b) = [b_{4j+3}, b_{4j+2}, b_{4j+1}, b_{4j}]$ . Each of these  $I_j(b)$ 's is one of 16 possible binary sequences (if some or all of  $b_{4j+3}$ ,  $b_{4j+2}$ , and  $b_{4j+1}$  are missing for the largest  $j$ , we take the missing terms to be 0). Apply the mapping  $m$  to each  $I_j(b)$ :

$$m : \begin{cases} [0, 0, 1, 1], [0, 1, a, b], [1, 0, 0, 0], [1, 0, 1, 0], [1, 0, 1, 1] \longrightarrow 0 \\ [0, 0, 0, 0], [0, 0, 0, 1], [0, 0, 1, 0], [1, 0, 0, 1], [1, 1, c, d] \longrightarrow 1, \end{cases}$$

where  $a, b, c, d$  may be, independently, either 0 or 1.

Let  $m_j(b) = m(I_j(b))$ . We color the integer  $n$  by  $\chi(n) = \sum_j m_j(b) \pmod{2}$ . We will now show that  $\chi$  is a 2-coloring of  $\mathbb{Z}^+$  that does not admit a monochromatic 25-term arithmetic progression with common difference from  $S - S$ .

Let  $x_1 < x_2 < \dots < x_{25}$  be an arithmetic progression with common difference  $d = 2^{4s} - 2^{4t}$ . We will use the shorthand  $I_j(\ell)$  to represent  $I_j(x_\ell)$  and  $m_j(\ell)$  to represent  $m(I_j(\ell))$ . First, consider  $U = \{I_t(\ell) : 8 \leq \ell \leq 23\}$ . By definition of  $d$ , we see that  $I_t(\ell)$  and  $I_t(\ell + 1)$  will differ; moreover, by the definition of  $d$ , the set  $U$  will contain all 16 possible binary strings of length 4. In particular, there exists  $r \in \{8, 9, \dots, 23\}$  such that  $I_t(r) = [0, 0, 1, 0]$ . Thus, by

adding/subtracting multiples of  $d$ , we can conclude the following:

$j$	$I_t(j)$	$m(I_t(j))$
$r - 7$	$[1, 0, 0, 1]$	1
$r - 6$	$[1, 0, 0, 0]$	0
$r - 5$	$[0, 1, 1, 1]$	0
$r - 4$	$[0, 1, 1, 0]$	0
$r - 3$	$[0, 1, 0, 1]$	0
$r - 2$	$[0, 1, 0, 0]$	0
$r - 1$	$[0, 0, 1, 1]$	0
$r$	$[0, 0, 1, 0]$	1
$r + 1$	$[0, 0, 0, 1]$	1
$r + 2$	$[0, 0, 0, 0]$	1

(we do not need  $j \in \{r + 3, r + 4, \dots, r + 8\}$  and these need not exist).

Next, we consider how  $d$  affects  $I_s(\ell)$  for  $\ell \in \{r - 7, r - 6, \dots, r - 1, r + 1, r + 2\}$  for all possible cases of  $I_s(r)$ . If  $I_s(r) = [1, 0, 0, 0]$  we have  $m_s(r) = 0$  so that  $m_s(r) + m_t(r) = 1$ . This gives us  $I_s(r + 1) = [1, 0, 0, 1]$  so that  $m_s(r + 1) + m_t(r + 1) = 2$ . Since all other  $I_j(r)$  are unaffected by the addition of  $d$  (to get to  $I_j(r + 1)$ ); there are no carries with addition by  $d$ ). Hence,  $\chi(x_r) \neq \chi(x_{r+1})$ .

This same analysis (perhaps switching the values of the sums  $m_s(j) + m_t(j)$ ,  $j = r, r + 1$ ) holds when  $I_s(r) \in \{[1, 0, 0, 1], [1, 0, 1, 1]\}$ . If  $I_s(r)$  is in  $\{[1, 1, 0, 1], [1, 1, 1, 0], [1, 1, 1, 1], [0, 0, 0, 1], [0, 0, 1, 0], [0, 1, 0, 0], [0, 1, 0, 1], [0, 1, 1, 0], [0, 1, 1, 1]\}$  then we have  $\chi(x_r) \neq \chi(x_{r-1})$  easily since there are no carry issues. If  $I_s(r) = [1, 0, 1, 0]$  then  $\chi(x_r) \neq \chi(x_{r+2})$ ; if  $I_s(r) = [1, 1, 0, 0]$  then  $\chi(r) \neq \chi(x_{r-3})$ . The remaining two cases  $I_s(r) \in \{[0, 0, 0, 0], [0, 0, 1, 1]\}$  both involve carries and need extra attention.

If  $I_s(r) = [0, 0, 0, 0]$  then either  $\chi(r) \neq \chi(r - 4)$  or  $\chi(r) \neq \chi(r - 5)$  depending on whether or not any carries change the value of  $\sum_{j>s} m_j(r - 1)$ . Lastly, if  $I_s(r) = [0, 0, 1, 1]$ , then either  $\chi(r) \neq \chi(r - 6)$  or  $\chi(r) \neq \chi(r - 7)$  depending on whether or not any carries change the value of  $\sum_{j>s} m_j(r - 1)$ .

We have shown that for any possible  $I_s(r)$ , the 25-term  $(S - S)$ -ap cannot be monochromatic, thereby proving the theorem.  $\square$



We will now present some positive results. But first, some definitions.

**Definition** (*r*-syndetic). Let  $S = \{s_1 < s_2 < \dots\}$  be a syndetic set with  $s_{i+1} - s_i \leq r$  for all  $i \in \mathbb{Z}^+$ . Then we say that  $S$  is *r-syndetic*

**Definition** (anastomotic). Let  $D \subseteq \mathbb{Z}^+$ . If every syndetic set admits a  $k$ -term  $D$ -ap then we say that  $D$  is *k-anastomotic*. If  $D$  is  $k$ -anastomotic for all  $k \in \mathbb{Z}^+$ , then we say that  $D$  is *anastomotic*.

**Remark.** The term syndetic has been defined by other authors and is an adjective meaning serving to connect. The term anastomotic is new and I chose it since its meaning is: serving to communicate between parts of a branching system.

**Theorem 12.** *Let  $D \subseteq \mathbb{Z}^+$ . If  $D$  is  $r$ -large, then every  $r$ -syndetic set admits arbitrarily long  $D$ -aps. Conversely, if every  $(2r + 1)$ -syndetic set admits arbitrarily long  $D$ -aps, then  $D$  is  $r$ -large.*

*Proof.* The first statement is straightforward: for any  $r$ -syndetic set  $S$ , we define an  $r$ -coloring  $\chi : \mathbb{Z}^+ \rightarrow \{1, \dots, r\}$  by  $\chi(i) = \min(\{s - i : s \in S \text{ with } i \leq s\} + 1)$ . Since  $D$  is  $r$ -large we have arbitrarily long  $D$ -aps under  $\chi$ . Since arithmetic progressions are translation invariant, by the definition of  $\chi$  we see that  $S$  admits arbitrarily long  $D$ -aps.

To prove the second statement, consider  $\sigma$ , an arbitrary  $r$ -coloring of  $\mathbb{Z}^+$  using the colors  $1, 2, \dots, r$ . For every color  $i$ , replace each occurrence of the color  $i$  in  $\sigma$  by the string of length  $r$  with a 1 in the  $i$ th position and 0 in all others. This process gives us a 2-coloring  $\hat{\sigma}$  of  $\mathbb{Z}$ . Let  $S$  be the set of positions of all 1s under  $\hat{\sigma}$ . Note that  $S$  is a  $(2r + 1)$ -syndetic set. By assumption,  $S$  admits arbitrarily long monochromatic  $D$ -aps. In particular, it admits arbitrarily long  $(rD)$ -aps (by taking every  $r$ th term in a sufficiently long  $D$ -ap). Since we now have an arbitrarily long  $(rD)$ -ap, in the original  $r$ -coloring  $\sigma$  we have a monochromatic  $D$ -ap.  $\square$

**Remark.** Recently, Host, Kra, and Maass [13] have independently proven a result similar to Theorem 12; their result is slightly stronger

in that they prove that “ $(2r+1)$ -syndetic” can be replaced by “ $(2r-1)$ -syndetic.”

An immediate consequence of Theorem 12 is the following:

**Corollary 13.** *Let  $D \subseteq \mathbb{Z}^+$ . Then  $D$  is large if and only if  $D$  is anastomotic.*

It would be nice if every syndetic set contained an infinite arithmetic progression as we would be done: since every 2-large set contains a multiple of every integer, every 2-large set would be anastomotic and, by Corollary 13 we would be done. Unfortunately, this is not true.

Let  $\alpha$  be irrational and let  $\chi : \mathbb{Z}^+ \rightarrow \{0, 1\}$  be  $\chi(x) = \lfloor \alpha n \rfloor - \lfloor \alpha(n-1) \rfloor$ . Each color class under  $\chi$  corresponds to a syndetic set (e.g., if  $\alpha$  is the golden ratio, then each color class is 4-syndetic). Consider  $S = \{i : \chi(i) = 0\}$ .

Assume, for a contradiction, that there exist integers  $a$  and  $d$  such that  $S$  contains  $a + dn$  for all  $n \in \mathbb{Z}^+$ . Then we have  $\lfloor \alpha(a + dn) \rfloor = \lfloor \alpha(a + d(n-1)) \rfloor$  for all positive integers  $n \geq 2$ . Let  $\beta = \alpha d$  and note that  $\beta$  is also irrational. Let  $\{x\}$  be the fractional part of  $x$ . Then  $\{\{\beta n\} : n \in \mathbb{Z}^+\}$  is dense in  $[0, 1)$ .

Consider  $y \in (\{-\alpha a\}, \{\alpha - \alpha a\})$ . We claim that we cannot have  $\{\beta n\} = y$  for any  $n \in \mathbb{Z}^+$ . Assume to the contrary that  $\{\beta j\} = y$  so that  $\beta j = \ell + y$  for some integer  $\ell$ . Then  $\beta j - y$  is an integer. By choice of  $y$ , we have  $\beta j - y$  strictly between  $\beta j + \alpha a - \alpha$  and  $\beta j + \alpha a$ . But this is not possible since  $\lfloor \beta j + \alpha a - \alpha \rfloor = \lfloor \beta j + \alpha a \rfloor$ . Hence,  $\{\{\beta n\} : n \in \mathbb{Z}^+\} \cap (\{-\alpha a\}, \{\alpha - \alpha a\}) = \emptyset$ . But  $\{\{\beta n\} : n \in \mathbb{Z}^+\}$  is dense in  $[0, 1)$ , a contradiction.<sup>4</sup>

Hence, we have a syndetic set without an infinite arithmetic progression. The same analysis shows that  $\{i : \chi(i) = 1\}$  does not contain one either. Hence, we can cover the positive integers with two syndetic sets, neither of which contain an infinite arithmetic progression.

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<sup>4</sup>The preceding argument is based on an answer given by Mario Carneiro on [math.stackexchange.com](https://math.stackexchange.com) for question 1487778; I was unable to find a published reference.

**6.1. A Brief Detour.** We take a brief side trip back to the dynamical system setting in order to expand on Figure 1 to include the new notions just introduced so that we have an overview of how the different types of recurrence are related. Furthermore, we define two other types of recurrence to display the relationships between a fixed number of colors and an arbitrary number of colors.

**Definitions.** Let  $r \in \mathbb{Z}^+$  be fixed and consider  $D \subseteq \mathbb{Z}^+$ . We say  $D$  is *r-chromatically k-recurrent* if every finite  $r$ -coloring of  $\mathbb{Z}^+$  admits a monochromatic  $k$ -term  $D$ -ap; we say  $D$  is *r-syndetically k-recurrent* if every  $r$ -syndetic set contains a  $k$ -term  $D$ -ap.

In Figure 3 below, we assume that the same restrictions as those in Figure 1 are still in place. Missing implications are unknown.

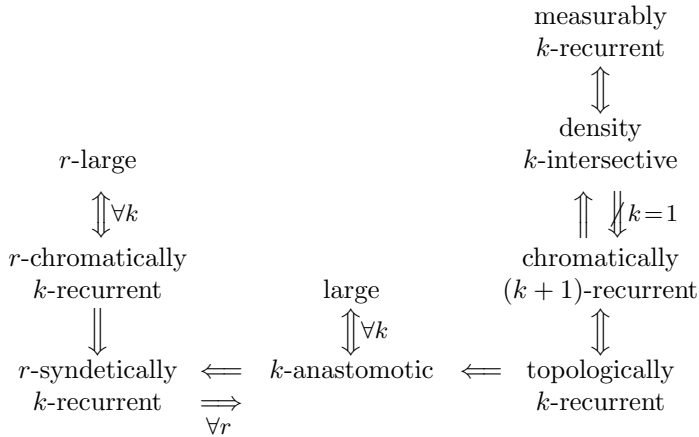


FIGURE 3. Relationship between types of recurrence considered in this article

**6.2. Back to the Combinatorics Encampment.** You are surely asking yourself, if you’ve traveled with me this far: do you have any positive results? Well, to help aid you in keeping a sanguine outlook, I’ll now offer a few items that offer a glimmer of hope for escape from our current residence in the dark depths of the large rabbit hole (pun intended).

We start with a strong condition for a set to be large.

**Definition** (bounded multiples condition). We say that  $D \subseteq \mathbb{Z}^+$  satisfies the *bounded multiples condition* if there exists  $M \in \mathbb{Z}^+$  such that for every  $i \in \mathbb{Z}^+$ , there exists  $m \leq M$  such that  $im \in D$ . In other words, for every positive integer  $i$ , at least one element of  $\{i, 2i, 3i, \dots, Mi\}$  is in  $D$ .

Using this definition, we have the following characterization.

**Theorem 14.** *If  $D \subseteq \mathbb{Z}^+$  satisfies the bounded multiples condition, then  $D$  is large.*

*Proof.* Let  $M$  be the constant that exists by the bounded multiples condition. We proceed by showing that  $D$  is  $k$ -anastomotic for all  $k$ . Let  $A \subseteq \mathbb{Z}^+$  be syndetic. If  $A^c$  is not syndetic, then it has arbitrarily long gaps. This means that  $A$  contains arbitrarily long intervals. In this situation,  $A$  contains arbitrarily long  $D$ -aps.

Now let both of the sets  $A = \{a_i\}_{i \in \mathbb{Z}^+}$  and  $A^c = B = \{b_i\}_{i \in \mathbb{Z}^+}$  be  $g$ -syndetic with  $g = \max_{i \in \mathbb{Z}^+} \{a_{i+1} - a_i, b_{i+1} - b_i\}$ . Let  $\chi(n)$  equal 1 if  $n \in A$  and 0 if  $n \in B$ . Define the  $2^{g+1}$ -coloring  $\gamma : \mathbb{Z}^+ \rightarrow \{0, 1\}^{g+1}$  by  $\gamma(n) = (\chi(n), \chi(n+1), \dots, \chi(n+g))$ . Note that  $\gamma(n)$  cannot consist of all 0s or all 1s by the definition of  $g$ .

Since  $\gamma$  is a finite coloring of  $\mathbb{Z}^+$ , by van der Waerden's Theorem there exists a monochromatic  $Mk$ -term arithmetic progression under  $\gamma$ . By the definition of  $\gamma$  both  $A$  and  $B$  contain  $Mk$ -term arithmetic progressions, each with the same common difference. Let  $d$  be this common difference.

Since  $D$  satisfies the bounded multiples condition, there exists  $m \leq M$  such that  $md \in D$ . By taking every  $m^{\text{th}}$  term of the  $Mk$ -term arithmetic progressions, we see that both  $A$  and  $B$  have  $k$ -term  $D$ -aps.  $\square$

**Remark.** The converse of Theorem 14 is not true. We know that the set of perfect squares is large; however, it does not satisfy the bounded multiples condition. To see this, consider a prime  $p$ . Then the smallest multiple of  $p$  in the set of perfect squares is  $p^2$ . Since  $p$

may be arbitrarily large, we do not have the existence of  $M$  needed in the definition above.

As was done at the beginning of this article, we will now present a “finite version” of the 2-large definition. Instead of appealing to the Compactness Principle, we will offer a terse proof of equivalence.

**Lemma 15.** *Let  $D$  be 2-large. For each  $k \in \mathbb{Z}^+$ , there exists an integer  $N = N(k, D)$  such that every 2-coloring of  $\{1, 2, \dots, N\}$  admits a monochromatic  $k$ -term  $D$ -ap.*

*Proof.* Assume not and, for each  $i \in \mathbb{Z}^+$ , let  $\chi_i$  be a 2-coloring of  $\{1, 2, \dots, i\}$  with no monochromatic  $k$ -term  $D$ -ap. Define  $\gamma$ , inductively, by  $\gamma(j) = c_j$  where  $\chi_i(j) = c_j$  occurs infinitely often among those  $\chi_i$  where  $\chi_i(\ell) = \gamma(\ell)$  for  $\ell < j$ . Now note that  $\gamma$  is a 2-coloring of  $\mathbb{Z}^+$  with no monochromatic  $k$ -term  $D$ -ap, a contradiction.  $\square$

We will have use for the following notation in the remainder of this section.

**Notation.** Let  $m \in \mathbb{Z}^+$  and  $D \subseteq \mathbb{Z}^+$ . Then  $mD = \{md : d \in D\}$  and  $D^m = \left\{ \prod_{f \in F} d_f : d_i \in D, F \subseteq \mathbb{Z}^+ \text{ with } |F| = m \right\}$ .

We can now present two easy lemmas. We will provide a proof for the first and leave the very similar proof of the second to the reader.

**Lemma 16.** *Let  $D$  be 2-large and let  $m \in \mathbb{Z}^+$ . Then  $N(k, mD) \leq mN(k, D)$ .*

*Proof.* Consider any 2-coloring of the first  $mN(k, D)$  positive integers. We will show that there exists a monochromatic  $k$ -term  $mD$ -ap. Given our coloring, consider only those integers divisible by  $m$ . Via the obvious one-to-one correspondence between  $\{m, 2m, \dots, mN(k, D)\}$  and  $\{1, 2, \dots, N(k, D)\}$ , we have a monochromatic  $k$ -term  $D$ -ap in the latter interval, meaning that we have a monochromatic  $k$ -term  $mD$ -ap in the former interval.  $\square$

**Lemma 17.** *Let  $m \in \mathbb{Z}^+$ . Then  $D$  is  $r$ -large if and only if  $mD$  is  $r$ -large.*

We also will use the following definition.

**Definition.** Let  $D$  be 2-large. Define  $M(k, D; 2) = N(k, D)$  and, for  $r \geq 3$ ,

$$M(k, D; r) = N(M(k, D; r-1), D),$$

where  $N(k, D)$  is the integer from Lemma 15.

As  $N(k, D)$  exists for all  $k$  (by Lemma 15), we see that  $M(k, D; r)$  is well-defined for all  $k$  and  $r$ .

We are now ready to finally see a ray of hope in the next theorem. For comparison, Brown, Graham, and Landman [7] were able to show that if  $D$  is 2-large then  $D^m$  is  $2^m$ -large for  $m \in \mathbb{Z}^+$ .

**Theorem 18.** *Let  $D \subseteq \mathbb{Z}^+$  be 2-large. Then  $D^2$  is large.*

*Proof.* We will prove this by showing that, for  $r \geq 2$ , any  $r$ -coloring of an  $M(k, D; r)$ -term  $D$ -ap admits a monochromatic  $k$ -term  $D^2$ -ap. We let  $k$  be arbitrary and induct on  $r$ . We start with  $r = 2$  and let  $\gamma$  be a 2-coloring of  $a + d, a + 2d, \dots, a + M(k, D; 2)d$ , with  $d \in D$ . By definition of  $M(k, D; 2)$  and the translation and dilation invariance of arithmetic progressions, we have a monochromatic  $k$ -term  $dD$ -ap. As  $dD \subseteq D^2$ , the base case is done.

We now assume the statement holds for  $r - 1$  colors and will show it holds for  $r$  colors.

Determine  $t \in \mathbb{Z}^+$  so that  $2^t \leq r < 2^{t+1}$ . Using the colors  $0, 1, \dots, r - 1$ , for a given  $r$ -coloring of the first  $M(k, D; r)$  positive integers, we consider the associated 2-coloring defined by the morphism writing  $i$  in binary. For example, with  $r = 8$

$$0237 \rightarrow 000\ 010\ 011\ 111$$

(the spaces are for clarity). Hence we have a 2-coloring of  $[1, tM(k, D; r)]$ . By Lemma 16 and the definition of  $M$  we have an  $M(k, D; r - 1)$ -term monochromatic  $tD$ -ap under this 2-coloring.

Now, by the definition of  $t$ , note that in a list of the binary representations of  $0, 1, \dots, r-1$ , e.g., (with  $r = 6$ ):

$$000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 110,$$

each place (associated with  $2^i$ ) has both the bits 0 and 1 occurring.

This is crucial, since by having an  $M(k, D; r-1)$ -term monochromatic  $tD$ -ap under the binary coloring, in the original coloring we have an  $(r-1)$ -coloring of an  $M(k, D; r-1)$ -term  $D$ -ap. To see this, note that our monochromatic  $tD$ -ap means that along some congruence class modulo  $t$  we have either only 0 bits or only 1 bits. In either situation, by Lemma 17, this implies that the  $M(k, D; r-1)$ -term  $D$ -ap is void of at least 1 color. By the induction hypothesis, this admits a monochromatic  $k$ -term  $D^2$ -ap.  $\square$

**Remark.** The lynchpin of the above proof holding us back from proving that the 2-Large Conjecture is true (which I believe it is) is in the base case. If we could show that any 2-coloring of a long-enough  $D$ -ap admits a monochromatic  $D$ -ap the rest of the proof could remain unchanged and we would have a proof that the 2-Large Conjecture is true. We would need an integer  $S = S(D)$  such that for any  $d \in D$  there exists  $s < S$  such that  $sd \in D$  (this is less restrictive than  $D$  satisfying the bounded multiples condition).

**7. Epilogue.** Having attained Theorem 18, a hidden door in the rabbit hole has opened, leading us back to the Stone-Ćech locale; by Theorem 18, if  $D$  is a 2-large subsemigroup of  $(\mathbb{Z}^+, \cdot)$ , then  $D$  is large. For example, this result gives us, for any monomial  $x^n$ : if  $S = \{x^n : x \in \mathbb{Z}^+\}$  is 2-large, then it is large. This holds since  $i^n \cdot j^n = (ij)^n$  gives us that  $S$  is a semigroup (of course, we already know  $S$  is large via other means). However, the set of odd positive integers is also a semigroup (under multiplication) but is not 2-large since, from [7], a 2-large set must have a multiple of every positive integer. Combining the range of polynomials and multiples of every positive integer, Frantzikinakis [8] has shown that if  $p(n) : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  is an integer-valued polynomial then  $D = \{p(n) : n \in \mathbb{Z}^+\}$  is measurably  $k$ -recurrent for all  $k$  if and only if it contains multiples of every positive integer.

And now we once again find ourselves wading in the dynamical systems pool after traveling through the Stone-Čech locale.

Note that Frantzikinakis' result is stronger than  $D$  being large (see Figure 3) and relies heavily on polynomials but also suggests that, perhaps, if  $D$  contains multiples of every positive integer, then  $D$  is large. If this were true, then the 2-Large Conjecture is true. But yet again, we are foiled: the set  $\{n! : n \in \mathbb{Z}^+\}$  clearly contains a multiple of every positive integer, but is not 2-large [7].

And now we are back in the combinatorics encampment.

Okay silly rabbit, enough tricks; I surrender.

For now.

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