AVOIDING MONOCHROMATIC SEQUENCES WITH SPECIAL GAPS

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Abstract. For $S \subseteq \mathbb{Z}^+$ and k and r fixed positive integers, denote by $f(S,k;r)$ the least positive integer n (if it exists) such that within every r-coloring of $\{1,2,\ldots,n\}$ there must be a monochromatic sequence $\{x_1,x_2,\ldots,x_k\}$ with $x_i - x_{i-1} \in S$ for $2 \leq i \leq k$. We consider the existence of $f(S,k;r)$ for various choices of S, as well as upper and lower bounds on this function. In particular, we show that this function exists for all k if S is an odd translate of the set of primes and $r = 2$.

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1. Introduction. Van der Waerden's theorem on arithmetic progressions [10] states that for every partition of $\mathbb{Z}^+$ into r sets, at least one of the sets will contain arbitrarily long arithmetic progressions. An equivalent form of this theorem says that for all $k,r \in \mathbb{Z}^+$, there exists a least positive integer $n = w(k;r)$ such that within every r-coloring of $[1,n] = \{1,2,\ldots,n\}$ there must be a monochromatic k-term arithmetic progression. By replacing the family of arithmetic progressions, AP, with another family $\mathcal{F}$ of sets, one may ask if the corresponding theorem holds, i.e., is it true that for all $k,r \in \mathbb{Z}^+$, there exists a positive integer $n = f(k;r)$ such that for every r-coloring of $[1,n]$, there is a monochromatic k-term member of $\mathcal{F}$? Rado's theorem involving monochromatic solutions to systems of linear homogeneous equations illustrates one way of choosing $\mathcal{F}$. Other examples may be found in [3,4,6,7,8,9].

In [4], the authors considered replacing AP with the collection of those arithmetic progressions $\{x + id : 0 \leq i \leq k-1\}$ whose common differences, d, belong to a prescribed set. Specifically, for $r \in \mathbb{Z}^+$ and $A \subseteq \mathbb{Z}^+$, call A an *r-large* set if for every r-coloring of $\mathbb{Z}^+$ there exist arbitrarily long monochromatic arithmetic progressions whose common differences belong to A. Define A to be *large* if it is r-large for every r. They gave several sufficient conditions and some necessary conditions for largeness and 2-largeness. They also conjectured that any set that is 2-large must be large.

In this paper we consider a property related to largeness. We consider sequences where the differences between consecutive terms belong to a prescribed set S; however, we do not insist that the sequence be an arithmetic progression.

We begin with the following notation and definitions. For any string u and any $t \in \mathbb{Z}^+$, we denote by u^t the string of t consecutive u's. For $t = 0$, we let u^t represent the empty string. For $S \subseteq \mathbb{Z}^+$, a sequence of positive integers $\{x_1,\ldots,x_k\}$ is called a k-term *S-diffsequence* if $x_i - x_{i-1} \in S$ for $2 \leq i \leq k$. For $r \in \mathbb{Z}^+$, S is called *r-accessible* if whenever $\mathbb{Z}^+$ is r-colored, there are arbitrarily long monochromatic S-diffsequences. The set S is called *accessible* if it is r-accessible for all $r \in \mathbb{Z}^+$. If S is not accessible, the *degree of accessibility* of S, denoted $\mathrm{DA}(S)$, is the largest value of r such that S is r-accessible. Finally, we denote by $f(S,k;r)$ the least positive integer n (if it exists) such that for every r-coloring of $[1,n]$ there is a monochromatic k-term S-diffsequence. (Obviously, for S an r-accessible set, if $S \subseteq T$, then $f(S,k;r) \geq f(T,k;r)$.)

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Denote the family of all accessible sets by $\mathcal{A}$ and the family of all r-accessible sets by $\mathcal{A}_r$. Likewise, denote the families of large sets and r-large sets by $\mathcal{L}$ and $\mathcal{L}_r$. Clearly, $\mathcal{L} \subseteq \mathcal{A}$ and $\mathcal{L}_r \subseteq \mathcal{A}_r$ for all r. As stated before, it is conjectured that $\mathcal{L} = \mathcal{L}_2$. As we shall see, $\mathcal{A} \neq \mathcal{A}_2$ and $\mathcal{A}_2 \neq \mathcal{L}_2$. Moreover, Jungić [5] has proved that $\mathcal{A} \neq \mathcal{L}$.

In Section 2 we give some basic lemmas and consider a few elementary examples; in particular, we show that for any $d \in \mathbb{Z}^+$, there is a set with degree of accessibility d (this is in contrast to what is conjectured about large sets). In Section 3 we prove that for each odd positive integer t there are arbitrarily long sequences of primes $p_1 < \cdots < p_k$ such that $p_i - p_{i-1} \in P + t$ for $2 \leq i \leq k$, where P is the set of primes. From this it follows that $P + t \in \mathcal{A}_2$. Section 4 contains some open questions and a table of computer-generated values of $f(S, k; 2)$ for a few sets S and values k.

2. A Few Simple Examples. We begin with two useful lemmas.

LEMMA 2.1. *Let $c \geq 0$ and $r \geq 2$, and let $S \subseteq \mathbb{Z}^+$. If every $(r-1)$-coloring of S yields arbitrarily long monochromatic $(S+c)$-diffsequences, then $S + c \in \mathcal{A}_r$.*

Proof. Let $S = \{s_i : i \in \mathbb{Z}^+\}$ and assume every $(r-1)$-coloring of S admits arbitrarily long monochromatic $(S+c)$-diffsequences. Let χ be an r-coloring of $\mathbb{Z}^+$. By induction on k, we show that, under χ, for all k there are k-term monochromatic $(S+c)$-diffsequences. Since there are trivially 1-term sequences, assume that under χ there is a monochromatic $(S+c)$-diffsequence $X = \{x_1, \ldots, x_k\}$. Say X is of color red. Consider $A = \{x_k + s_i + c : s_i \in S\}$. If some member of A is colored red, then we have a red $(k+1)$-term $(S+c)$-diffsequence. Otherwise we have an $(r-1)$-coloring of A and hence, A must contain arbitrarily long monochromatic $(S+c)$-diffsequences. □

Remark. The converse of Lemma 2.1 is false. For example, let $S = \{2\} \cup (2\mathbb{Z}^+ - 1)$. Let χ be the 2-coloring of S defined by $\chi(x) = 1$ if $x \equiv 1 \pmod 4$ or $x = 2$, and $\chi(x) = 0$ if $x \equiv 3 \pmod 4$. Then χ does not yield arbitrarily long monochromatic S-diffsequences (there are none of length four). On the other hand, $S \in \mathcal{A}_3$ [8, Remark (5)], and in fact $f(S, k; 3) \leq 6k^2 - 13k + 6$ (see Theorem 2.8); more generally, from this same reference it follows that if m is even, and $j \in \mathbb{Z}^+$, then the set $\{jm\} \cup \{x : x \equiv \frac{m}{2} \pmod{m}\}$ is 3-accessible.

We leave to the reader as an easy exercise the proof of the following result.

LEMMA 2.2. *Let S be a set of positive integers and let $k, r, j \in \mathbb{Z}^+$. If $f(S, k; r) = M$, then $f(jS, k; r) = j(M - 1) + 1$.*

Using Lemma 2.1, with $c = 0$ and $r = 2$, it is clear that the set $\{2^i : i \geq 0\}$ is 2-accessible. The following result tells us more.

THEOREM 2.3. *Let $a \in \mathbb{Z}^+ \setminus \{1, 3\}$, and define $S = \{(a-1)a^j : j = 0, 1, 2, \ldots\} \cup \{(a-1)^2 a^j : j = 0, 1, 2, \ldots\}$. Then $2 \leq DA(S) \leq a$ and $f(S, k; 2) \leq a^k - a + 1$.*

Proof. To show that $\mathrm{DA}(S) \leq a$, let χ be the $(a+1)$-coloring defined by $\chi(x) = x \bmod (a+1)$. Assume that $\chi(y) = \chi(z)$ and that $z - y \in S$. By the definition of χ, $a+1$ divides $z - y$, and therefore either $(a+1) | (a-1)a^j$ or $(a+1) | (a-1)^2 a^j$ for some $j \geq 0$. Since $a \neq 1, 3$, neither of these is possible.

Now let $\alpha : [1, a^k - a + 1] \to \{0, 1\}$. We will show, by induction on k, that under α there is a monochromatic k-term S-diffsequence. Obviously, it holds for $k = 1$. Now assume $k \geq 2$, and that it holds for $k-1$. Let $X = \{x_1, \ldots, x_{k-1}\}$ be a monochromatic S-diffsequence, say of color 0, that is contained in $[1, a^{k-1} - a + 1]$. Consider the set $A = \{x_{k-1} + (a-1)a^i : 0 \leq i \leq k-1\}$. Note that $A \subseteq [1, a^k - a + 1]$. If there exists $y \in A$ of color 0, then $X \cup \{y\}$ is a monochromatic k-term S-diffsequence. If, on the other hand, no such y exists, then A is a monochromatic k-term S-diffsequence. □

COROLLARY 2.4. *If $S = \{2^i : i \geq 0\}$, then $DA(S) = 2$ and $8(k-3) + 1 \leq f(S, k; 2) \leq 2^k - 1$ for all $k \geq 3$.*

Proof. The fact that $\mathrm{DA}(S) = 2$ and the upper bound follow from Theorem 2.3. For the lower bound, it is straightforward to show (by induction on k) that, for $k \geq 4$, the 2-coloring $\chi_k = (10010110)^{k-3}$ avoids monochromatic k-term S-diffsequences. □

The details of the proof of Corollary 2.4 are given in [9].

Remark. Note that $a = 2$ is the only value of a for which $\{a^i : i \geq 0\} \in \mathcal{A}_2$. This follows immediately from the observation that if $\gcd(i, m) = 1$ and $S = \{x \in \mathbb{Z}^+ : x \equiv i (\text{mod } m)\}$, then there exists a (fairly obvious) 2-coloring of $\mathbb{Z}^+$ that avoids monochromatic m-term S-diffsequences.

In [4] it was shown that if $A \notin \mathcal{L}_r$ and $B \notin \mathcal{L}_s$, then $A \cup B \notin \mathcal{L}_{rs}$ by using the canonical product coloring. The same simple argument can be used to prove the following lemma. We omit the details.

Lemma 2.5. *If $S \notin \mathcal{A}_r$ and $T \notin \mathcal{A}_s$ and either $S+T = \{s+t : s \in S, t \in T\} \subseteq S$ or $S + T \subseteq T$, then $S \cup T \notin \mathcal{A}_{rs}$.*

It is easy to see, using Lemma 2.1, that the set $S = \{2\} \cup (2\mathbb{Z}^+ - 1)$ is 2-accessible, since the set of odd numbers itself is an S-diffsequence. The next theorem tells us more about S. We omit the proof, which is given in [9].

Theorem 2.6. *If $S = \{2\} \cup (2\mathbb{Z}^+ - 1)$, then $DA(S) = 3$. Furthermore, $f(S, k; 3) \leq 6k^2 - 13k + 6$, $f(S, k; 2) = 3k - 3$ for k even, and $f(S, k; 2) = 3k - 4$ for k odd.*

The proof of the following example is an easy exercise and is left to the reader.

Proposition 2.7. *Let $F = \{F_1, F_2, F_3, F_4 \dots\} = \{1, 2, 3, 5 \dots\}$ be the set of Fibonacci numbers. Then $f(F, k; 2) \leq F_{k+2} - 2$ for $k \geq 1$.*

The following simple result provides us with examples of very sparse sets which are nonetheless accessible.

Proposition 2.8. *For $T \subseteq \mathbb{Z}^+$ infinite, $T - T = \{t - s : s < t \text{ and } s, t \in T\} \in \mathcal{A}$.*

Proof. Let $r \in \mathbb{Z}^+$ and let χ be an r-coloring of $\mathbb{Z}^+$. Let s be the minimum element in T and consider $T_s = \{t - s : t \in T\}$. Some color class must contain an infinite number of elements of T_s. This gives an infinitely long monochromatic $(T - T)$-diffsequence. Since this argument holds for all r, $T - T \in \mathcal{A}$. □

We now look at the accessibility of certain collections of congruence classes. In [4] it was proved that if $A \in \mathcal{L}_2$, then A must contain a multiple of every positive integer. We have seen that this is not true if we replace $\mathcal{L}_2$ with $\mathcal{A}_2$ (see, for example, Corollary 2.4 or Theorem 2.8). By the next proposition, we see that this condition is necessary in order for a set to be accessible. In addition to giving another example for which 2-accessible does not imply 2-large, it also shows that for all positive integers m, there exists a set having m as its degree of accessibility.

Proposition 2.9. *For $m \geq 2$, let $S_m = \{x \in \mathbb{Z}^+ : m \nmid x\}$. Then $DA(S_m) = m - 1$.*

Proof. That $\mathrm{DA}(S_m) \leq m - 1$ is easily seen by considering the m-coloring χ of $\mathbb{Z}^+$ defined by $\chi(x) = x \pmod m$. To prove the reverse inequality, let χ be any $(m-2)$-coloring of S_m. Then some color must contain an infinite number of elements from each of at least two of the residue classes $1 \pmod m$, $2 \pmod m$, $\dots, (m-1) \pmod m$. Thus, some color contains an infinite S_m-diffsequence and the proof is complete. □

For some results about the values of $f(S_m, k; 2)$ for specific choices of m, we refer the reader to [9].

3. Translations of the Set of Primes. In [4], the question was raised as to whether there exists a translation of P, the set of primes, that is large, or for that matter 2-large. Since a 2-large set must contain a multiple of every integer, $P + e \notin \mathcal{L}_2$ if e is even. Likewise, by Proposition 2.9, if e is even, then $P + e \notin \mathcal{A}$, and P itself is not 4-accessible. In fact, $P \notin \mathcal{A}_3$. To see this, color the multiples of 9 green, the remaining even numbers red, and the remaining odd numbers blue. It is easy to see that any sequences of 9 reds, 9 blues, or 2 greens must have numbers that differ by a non-prime. We do not know if P is 2-accessible or if some even translation of P is 2-accessible. On the other hand, as we shall see in this section, all odd translations of P are 2-accessible. In fact, we prove the following stronger result.

THEOREM 3.1. *Let $t \in \mathbb{Z}^+$ be odd. For any $k \geq 2$, there exist $p_1, p_2, \ldots, p_k \in P$ such that $p_i - p_{i-1} \in P + t$ for $2 \leq i \leq k$.*

Combining Theorem 3.1 with Lemma 2.1, we immediately get the following.

COROLLARY 3.2. *If t is odd, then $P + t \in \mathcal{A}_2$.*

We prove Theorem 3.1 as an application of a powerful number theoretic result due to Balog [1]. The application is stated as Theorem 3.3 below. Before stating the theorem, we mention some notation.

Let $\mathbf{b} = (b_1, b_2, \ldots, b_k) \in \mathbb{Z}^k$, $p \in P$, and $x \in \mathbb{R}^+$. We define:

$\pi(x; \mathbf{b}) = |\{n : 1 < n + b_i \leq x \text{ is prime for every } 1 \leq i \leq k\}|$;
$\rho(p) = \rho(p; \mathbf{b}) = |\{b_i \pmod p) : 1 \leq i \leq k\}|$;
$\sigma(\mathbf{b}) = \prod_{p \in P} \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\rho(p)}{p}\right)$;
$T(x; \mathbf{b}) = \sum\limits_{R(x)} \frac{1}{\log(n+b_1)\log(n+b_2)\ldots\log(n+b_k)}$, where $R(x) = \{n : 1 < n + b_i \leq x$ for all i, $1 \leq i \leq k\}$.

Finally, we remind the reader of the following standard notation: For functions $f(x)$ and $g(x)$, we write $f(x) \gg g(x)$ if there exists a positive constant c such that $\liminf_{x\to\infty} \frac{f(x)}{g(x)} \geq c$. Furthermore, for a parameter k, we write $f(x) \gg_k g(x)$ if the constant c is dependent upon k.

THEOREM 3.3. *(Balog). Let $k \in \mathbb{Z}^+$, $x \in \mathbb{R}^+$, and $t \in \mathbb{Z}^+ \cup \{0\}$. Define $B = \{(0, q_1 + t, \ldots, \sum_{i=1}^{k-1}(q_i + t)) : q_i \in P, k < q_i \leq x/2k, 1 \leq i \leq k-1\}$. Then*

$$\sum_{\mathbf{b} \in B} |\pi(x; \mathbf{b}) - \sigma(\mathbf{b})T(x; \mathbf{b})| \ll_k \frac{x^k}{\log^{2k} x}.$$

Remark. This is a special case of Balog's theorem ([1], p.49) (where we use $A = 2k$, $c = 0$, $D = 1$, and $a_i = 1$ for $1 \leq i \leq k$).

We will need the following technical lemma.

LEMMA 3.4. *Let $k \geq 2$ and let $t \geq 1$ be odd. For $\mathbf{q} = (q_1, \ldots, q_{k-1}) \in P^{k-1}$, let*

$$\mathbf{b}(\mathbf{q}) = \mathbf{b}(\mathbf{q}, t) = \left(0, q_1 + t, \ldots, \sum_{i=1}^{k-1}(q_i + t)\right).$$

Let

$$M(x) = \left\{\mathbf{q} \in P^{k-1} : \mathbf{q} \in \left(k, \frac{x}{2k}\right]^{k-1} \text{ and } \rho(p, \mathbf{b}(\mathbf{q})) < p \text{ for all } p \in P\right\}.$$

Then $|M(x)| \gg_k (\frac{x}{\log x})^{k-1}$.

Proof. Our goal is to show that "most" $\mathbf{q} \in P^{k-1}$ fail to define a complete set of residue classes modulo p, for all $p \in P$. First of all, it is clear that $\rho(p, \mathbf{b}(\mathbf{q})) < p$ for any prime $p > k$ since $\rho(p, \mathbf{b}(\mathbf{q})) \le k$. Hence, we need only to consider those primes $r_1, r_2, \ldots, r_d \le k$; let $m = \prod_{i=1}^{d} r_i$. We will obtain a lower bound for the number of $\mathbf{q}$ such that $\rho(r_i, \mathbf{b}(\mathbf{q})) < r_i$ for all $1 \le i \le d$.

It suffices to show that for some integer h, all entries of $(h, h, \ldots, h) + \mathbf{b}(\mathbf{q})$ are not divisible by r_i, for all i, $1 \le i \le d$. To this end, choose h such that $\gcd(h, m) = 1$ (note that h is odd). Obviously, this condition holds for the first entry.

In order to have that each r_i, $1 \le i \le d$, does not divide $h+q_1+t$ (the second entry in $(h, h, \ldots, h) + \mathbf{b}(\mathbf{q})$), it is sufficient that q_1 belond to some particular congruence class $c_1 \pmod m$, where $\gcd(c_1, m) = 1$. Since t and h are both odd, such a c_1 exists, and by Dirichlet's theorem for primes in arithmetic progressions, there are $\gg_k \frac{x}{\log x}$ choices for q_1.

Similarly, once $h, q_1, q_2, \ldots, q_{j-1}$ have been chosen, we need only consider those q_j so that for each r_i, q_j does not belong to any of the residue classes $-(h + q_1 + q_2 + \cdots q_{j-1} + jt) \pmod{r_i}$, $1 \le i \le d$. So it suffices to have q_j belong to one specific congruence class $c_j (\mathrm{mod}\ m)$, with $\gcd(c_j, m) = 1$.

Using the above criteria, we have at least $\prod_{i=2}^{d}(r_i - 2)$ reduced residue classes modulo $m/2$ in which the entries of $(h, h, \ldots, h) + \mathbf{b}(\mathbf{q})$ may reside. To see this, note that for $2 \le j \le d$, for each prime r_j, we cannot have $c_j = 0 \pmod{r_j}$ or $-(h + q_1 + \cdots + q_{j-1} + jt) \pmod{r_j}$, giving $r_j - 2$ choices. Now, by Dirichlet's theorem we have $\gg_k \frac{x}{\log x}$ choices for each q_i, and thus $\gg_k (\frac{x}{\log x})^{k-1}$ valid choices for the $(k-1)$-tuple of primes $(q_1, q_2, \ldots, q_{k-1})$ that belong to $M(x)$. ☐

Using Theorem 3.3 and Lemma 3.4, we have the following result.

Lemma 3.5. *Let $k \ge 2$, $t \ge 1$ and odd, and $x \in \mathbb{R}^+$. If*

$$W(x) = \left\{(p, q_1, \ldots, q_{k-1}) : p, q_1, \ldots, q_{k-1} \text{ are primes and } k < q_1, q_2, \ldots, q_{k-1} \le \frac{x}{2k}\right\}$$

and

$$S(x) = \left\{(p, q_1, \ldots, q_{k-1}) \in W : p + \sum_{j=1}^{i}(q_j + t) \le x \text{ is prime for all } i\, (0 \le i \le k)\right\},$$

then $|S(x)| \gg_k \frac{x^k}{\log^{2k-1} x}$.

Proof. We use the notation from Theorem 3.3 and Lemma 3.4. In order to apply Theorem 3.3, we first obtain effective bounds for ρ, σ, and T.

We first show that $0 < \sigma(\mathbf{b}(\mathbf{q})) < \infty$. The fact that $\sigma(\mathbf{b}(\mathbf{q})) < \infty$ is shown in [2]. We see that for all $\mathbf{q} \in M(x)$, by the defintion of $M(x)$ we have $\rho(p; \mathbf{b}(\mathbf{q})) < p$ for all primes p. Since it is also true that $\rho(p; \mathbf{b}(\mathbf{q})) \le k$ for any prime p, we have

$$\sigma(\mathbf{b}(\mathbf{q})) \ge \prod_{p \le k}\left(1 - \frac{1}{p}\right)^{-k}\left(1 - \frac{p-1}{p}\right)\prod_{p > k}\left(1 - \frac{1}{p}\right)^{-k}\left(1 - \frac{k}{p}\right) = \sigma_k, \qquad (3.1)$$

a constant dependent upon only k. We next show that $\sigma_k > 0$.

Clearly, we have the finite product in (3.1) positive, so we must show that the infinite product in (3.1) converges to a positive constant. To this end, let $1 + a_p = (1 - 1/p)^{-k}(1 - k/p)$. By the binomial theorem, we have $a_p = \frac{-\sum_{i=2}^{k}(-1)^{k-i}\binom{k}{i}p^{-i}}{(1-1/p)^k}$. Since $|a_p| \le \frac{\sum_{i=2}^{k}\binom{k}{i}p^{-i}}{(1-1/p)^k} \le \frac{\sum_{i=2}^{k}\binom{k}{i}p^{-2}}{(1-1/p)^k} \le \frac{\sum_{i=2}^{k}\binom{k}{i}p^{-2}}{1/2^k} = 2^k(2^k - k - 1)p^{-2}$, we see that

$\sum_{p\in P} a_p$ converges absolutely. It follows that $\prod_{p\in P}(1+a_p)$ converges to a positive number. Thus, from (3.1),

$$\text{for all } \mathbf{q}\in M(x),\ \sigma(\mathbf{b}(\mathbf{q}))\geq \sigma_k>0. \tag{3.2}$$

We next bound $T(x;\mathbf{b}(\mathbf{q}))$ by using

$$\begin{aligned}|\{n:1<n+b_i\leq x, 1\leq i\leq k\}| &= (x-b_k)+O(1)\\ &= x-\textstyle\sum_{i=1}^{k-1}(q_i+t)+O(1)\\ &> x-\textstyle\sum_{i=1}^{k-1} q_i - kt\\ &\gg_{k,t} x-k\left(\tfrac{x}{2k}\right)\\ &= \tfrac{x}{2}.\end{aligned}$$

This gives us

$$T(x;\mathbf{b}(\mathbf{q}))\gg_{k,t}\frac{x}{2\log^k x}. \tag{3.3}$$

We now apply our above bounds to complete the proof. Noting that $\{\mathbf{b}(\mathbf{q}):\mathbf{q}\in M(x)\}\subseteq B$ (where B is as defined in Theorem 3.3), Theorem 3.3 implies that

$$\sum_{\mathbf{q}\in M(x)}\Big|\pi(x;\mathbf{b}(\mathbf{q}))-\sigma(\mathbf{b}(\mathbf{q}))T(x;\mathbf{b}(\mathbf{q}))\Big|\ll_k\frac{x^k}{\log^{2k}x}. \tag{3.4}$$

Let $N(x)=P^{k-1}\cap(k,\frac{x}{2k}]^{k-1}$. Since we can write

$$W(x)=\bigcup_{p\in P}\ \bigcup_{\mathbf{q}\in N(x)}\{(p,q_1,q_2,\ldots,q_{k-1})\},$$

we have

$$|S(x)|\ \geq \sum_{\mathbf{q}\in M(x)}\pi(x;\mathbf{b}(\mathbf{q})).$$

Using (3.2) and (3.3) along with Lemma 3.4, inequality (3.4) yields

$$\begin{aligned}\sum_{\mathbf{q}\in M(x)}\pi(x;\mathbf{b}(\mathbf{q})) &\gg_k \sum_{\mathbf{q}\in M(x)}\sigma(\mathbf{b}(\mathbf{q}))T(x;\mathbf{b}(\mathbf{q}))-O\left(\tfrac{x^k}{\log^{2k}x}\right)\\ &\gg_{k,t}\sigma_k|M(x)|\left(\tfrac{x}{2}+O(1)\right)\left(\tfrac{1}{\log^k x}\right)-O\left(\tfrac{x^k}{\log^{2k}x}\right)\\ &\gg_{k,t}\sigma_k\left(\tfrac{x}{\log x}\right)^{k-1}\left(\tfrac{x}{\log^k x}\right)-O\left(\tfrac{x^k}{\log^{2k}x}\right)\\ &\gg_{k,t}\tfrac{x^k}{\log^{2k-1}x}.\end{aligned}$$

□

Having Lemma 3.5, we are now in a position to complete the proof of Theorem 3.1: We choose primes $p_1,\ q_1,\ldots,q_{k-1}$ so that $p_i=p_1+\sum_{j=1}^{i-1}(q_j+t)\in P$, for $2\leq i\leq k$. Since $p_i-p_{i-1}=q_{i-1}+t$ for $2\leq i\leq k$, we are done.

4. Open Questions And Some Exact Values. There are many interesting questions left unanswered about accessibility. Here is a list of some.

1. Let $T = \{2^i : i \geq 0\}$. What is the exact value of $f(T, k; 2)$? In Table 1 (below) we give the first few values of this function.
2. What is the exact value of $f(S, k; 3)$, where $S = \{2\} \cup (2\mathbb{Z}^+ - 1)$?
3. What is a formula for $f(S_m, k; 2)$? Calculations for the case $m = 6$ support our conjecture that for $k \geq 2$,

$$f(S_6, k; 2) = \begin{cases} (5k-4)/2 & \text{if } k \equiv 2 (\text{mod } 4) \\ (5k-5)/2 & \text{if } k \equiv 3 (\text{mod } 4) \\ (5k-6)/2 & \text{if } k \equiv 0 (\text{mod } 4) \\ (5k-7)/2 & \text{if } k \equiv 1 (\text{mod } 4) \end{cases}.$$

Also note that (see [9]) if we let $am \leq k < (a+1)m$, a a nonnegative integer, we have $2k + 2a - 1 \leq f(S_m, k; 2)$, with equality when $a = 0$. We believe that this inequality is an equality for all $a \in \mathbb{Z}^+$.

4. If t is an odd positive integer, what is $\mathrm{DA}(P+t)$? Moreover, is it true that for every 2-coloring of P, there exist arbitrarily long monochromatic $(P+t)$-diffsequences? If the answer to the latter question is true, then by Lemma 2.1, $P + t \in \mathcal{A}_3$.
5. What is the order of magnitude of $f(P + t, k; 2)$ for a fixed odd positive integer t? Table 1 below includes some specific values of this function.
6. As stated earlier, $P \notin \mathcal{A}_3$. Is $P \in \mathcal{A}_2$? If so, what is the magnitude of $f(P, k; 2)$? We have calculated the first several values of $f(P, k; 2)$ (see Table 1).
7. Let F be the set of Fibonacci numbers. What is $\mathrm{DA}(F)$? What is the order of magnitude of $f(F, k; 2)$?
8. For $k, m \geq 2$, what can we say about $\mathrm{DA}(S)$ and $f(S, k; 2)$ where S is the union of more than one congruence class modulo m?

The following table gives the exact value of $f(S, k; 2)$ for various S and k. The symbols T, F, and P denote $\{2^i : i \geq 0\}$, the set of Fibonacci numbers, and the set of primes, respectively.

$S \setminus k$	2	3	4	5	6	7	8
T	3	7	11	17	25	35	51
F	3	5	9	11	15	19	21
P	5	9	13	21	25	33	?
$P+1$	7	13	21	27	35	?	?
$P+2$	9	17	25	33	?	?	?
$P+3$	11	21	31	42	?	?	?
$P+4$	13	25	37	?	?	?	?
$P+5$	15	29	?	?	?	?	?
$P+6$	17	33	?	?	?	?	?
$P+7$	19	37	?	?	?	?	?
S_5	3	5	7	11	13	15	19
S_6	3	5	7	9	13	15	17

Table 1

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