REFINED RESTRICTED PERMUTATIONS AVOIDING SUBSETS OF PATTERNS OF LENGTH THREE

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Abstract

Define $S_n^k(T)$ to be the set of permutations of $\{1, 2, \ldots, n\}$ with exactly k fixed points which avoid all patterns in $T \subseteq S_m$. We enumerate $S_n^k(T)$, $T \subseteq S_3$, for all $|T| \geq 2$ and $0 \leq k \leq n$.

1 Introduction

Let $\pi \in S_n$ be a permutation of $\{1, 2, \ldots, n\}$ written in one-line notation. Let $\alpha \in S_m$. We say that π *contains the pattern* α if there exist indices $i_1 < i_2 < \cdots < i_m$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_m}$ is equivalent to α, where we define equivalence as follows. Define $\overline{\pi}_{i_j} = |\{x : \pi_{i_x} \leq \pi_{i_j}, 1 \leq x \leq m\}|$. If $\alpha = \overline{\pi}_{i_1}\overline{\pi}_{i_2}\ldots\overline{\pi}_{i_m}$ then we say that α and $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_m}$ are equivalent.

Define $S_n^k(T)$, $T \subseteq S_m$, to be the set of permutations with k fixed points which avoid all patterns in T and let $s_n^k(T) = |S_n^k(T)|$.

In recent paper [RSZ], the authors began the study of $S_n^k(T)$ for $T \subseteq S_3$, $|T| = 1$. In this paper we enumerate $S_n^k(T)$ for all $T \subseteq S_3$ with $|T| \geq 2$. Thus, all that remains to enumerate are $S_n^k(123)$ and $S_n^k(231)$ for all $0 \leq k \leq n$.

We start by giving two bijections which preserve the parameter "number of fixed points." Let $I : S_n \to S_n$ be the group theoretical inverse. Let $R : S_n \to S_n$ be the reversal bijection, i.e. $R(\pi_1\pi_2\ldots\pi_n) = \pi_n\pi_{n-1}\ldots\pi_1$, and let $C : S_n \to S_n$ be the complement bijection, i.e. $C(\pi_1\pi_2\ldots\pi_n) = (n+1-\pi_1)(n+1-\pi_2)\ldots(n+1-\pi_n)$.

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It is easy to see that the bijections I and RC preserve the number of fixed points in a permutation. This reduces the number of cases that we have to consider.

In the sequel, we will be using the following notation. Let I_S be the characteristic function, i.e. $I_S = 1$ if S is true and $I_S = 0$ if S is false. We also define $s_n^j(T) = 0$ for $j < 0$ or $j > n$ for any $T \subseteq S_3$.

2 The Case $|T| = 2$

Using I and RC we see that we have the following cases.

(1) $\overline{\{123, 321\}} = \{\{123, 321\}\}$

(2) $\overline{\{123, 132\}} = \{\{123, 132\}, \{123, 213\}\}$

(3) $\overline{\{123, 231\}} = \{\{123, 231\}, \{123, 312\}\}$

(4) $\overline{\{132, 213\}} = \{\{132, 213\}\}$

(5) $\overline{\{132, 231\}} = \{\{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}\}$

(6) $\overline{\{132, 321\}} = \{\{132, 321\}, \{213, 321\}\}$

(7) $\overline{\{231, 312\}} = \{\{231, 312\}\}$

(8) $\overline{\{231, 321\}} = \{\{231, 321\}, \{312, 321\}\}$

Theorem 2.1 *$\{s_n^0(123, 321)\}_{n\geq 0} = 1, 0, 1, 2, 4, 0, 0, \ldots$, $\{s_n^1(123, 321)\}_{n\geq 0} = 0, 1, 0, 2, 0, 0, \ldots$, $\{s_n^2(123, 321)\}_{n\geq 0} = 0, 0, 1, 0, 0 \ldots$, and $s_n^k(123, 321) = 0$ for all $3 \leq k \leq n$.*

Proof. Obvious. □

Theorem 2.2 *For $n \geq 1$, $s_n^k(123, 132) = 0$ for $3 \leq k \leq n$,*

$$s_{2n+i}^2(123, 132) = \begin{cases} \frac{4^{n-1}+2}{3} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}, \quad s_{2n+i}^1(123, 132) = \begin{cases} \frac{2(4^{n-1}-1)}{3} & \text{if } i = 0 \\ \frac{4^n+2}{3} & \text{if } i = 1 \end{cases}, \text{and}$$

$$s_{2n+i}^0(123, 132) = \begin{cases} 4^{n-1} & \text{if } i = 0 \\ \frac{2(4^n-1)}{3} & \text{if } i = 1 \end{cases}.$$

Proof. Let $\pi \in S_n(123,132)$. It is easy to prove by induction that we must have $\pi = (\pi(1), \pi(2), \ldots, \pi(m))$ where $\pi(j) = (t_{j-1}-1, t_{j-1}-2, \ldots, t_j+1, t_{j-1})$ for some $n = t_0 > t_1 > t_2 > \cdots > t_m = 0$. We call $\pi(j)$ a j-block of π. (See [Man] for more details.)

Clearly $s_n^k(123,132) = 0$ for $3 \le k \le n$ since three fixed points yield a 123 pattern. Hence, we need only consider $k = 0,1,2$. However, using a result in [SiSc] and the fact that $s_n^k(123,132) = 0$ for $3 \le k \le n$, we immediately obtain obtain for $s_n^k(123,231)$ for one of $k = 0,1,2$, provided we know it for the other two values of k. Hence, we may prove the stated formulas by providing formulas for the following:

1. $s_n^2(123,132)$
2. $s_{2n+1}^1(123,132)$
3. $s_{2n}^0(123,132)$.

We start with $s_n^2(123,132)$. Consider the *graph of* π, which consists of the points $(i,\pi_i) \in \mathbb{Z}^2$. By the symmetries of the graph of π, we see that the number of intersections with the line $y = x$ (i.e. the number of fixed points of π) is two if and only if m is odd for $\pi = (\pi(1), \pi(2), \ldots, \pi(m))$ and the intersections are in the $\frac{m+1}{2}$-block of π. So, the number of elements in the $\frac{m+1}{2}$-block must be even, as well as the total number of elements of π. Hence, we have $s_n^2(123,132) = 0$ for n odd. For the case $2n$, let $2d$ be the number of elements in the $\frac{m+1}{2}$-block. Since the other blocks cannot contain a fixed point, we have $(s_{n-d}(123,132))^2$ permutations for $d = 1,2,\ldots,n$. Using a result in [SiSc], this gives $1 + \sum_{d=1}^{n-1}(2^{n-d-1})^2$ permutations. Hence, $s_{2n}^2(123,132) = \frac{4^{n-1}+2}{3}$.

Next, we look at $s_{2n+1}^1(123,132)$ and again consider the graph of π; we have a situation similar to the case above. We must have the middle block containing an odd number of elements. Let this middle block have $2d+1$ elements. Using a result in [SiSc], this gives $1 + \sum_{d=0}^{n-1}(2^{n-d-1})^2$ permutations. Hence, $s_{2n+1}^1(123,132) = \frac{4^n+2}{3}$.

Lastly, we consider $s_{2n}^0(123,132)$. From the block decomposition given at the beginning of this proof and the fact that we do not have a fixed point, for $\pi \in S_{2n}^0(123,132)$ we have $\pi = \pi(1)\pi(2)$ where $\pi(2) \in S_n(123,132)$ and $\pi(1)$ is a permutation of $n+1, n+2, \ldots, 2n$ which avoids both given patterns. Thus, using a result in [SiSc], $s_{2n}^0(123,132) = (s_n(123,132))^2 = 4^{n-1}$.

□

Theorem 2.3 *For $n \geq 2$, $s_n^k(123,231) = 0$ for $3 \leq k \leq n$,*

$$s_n^2(123,231) = \begin{cases} \frac{n(n+6)}{24} & \text{if } n \equiv 0\,(mod\ 6) \\ \frac{n^2-1}{24} & \text{if } n \equiv 1,5\,(mod\ 6) \\ \frac{(n+2)(n+4)}{24} & \text{if } n \equiv 2,4\,(mod\ 6) \\ \frac{n^2-9}{24} & \text{if } n \equiv 3\,(mod\ 6) \end{cases},$$

$$s_n^1(123,231) = \begin{cases} \frac{n^2}{6} & \text{if } n \equiv 0\,(mod\ 6) \\ \frac{7n^2-12n+29}{24} & \text{if } n \equiv 1,5\,(mod\ 6) \\ \frac{n^2-4}{6} & \text{if } n \equiv 2,4\,(mod\ 6) \\ \frac{7n^2-12n+45}{24} & \text{if } n \equiv 3\,(mod\ 6) \end{cases},$$

and $s_n^0(123,231) = \binom{n}{2} + 1 - s_n^1(123,231) - s_n^2(123,231)$.

Proof. Clearly $s_n^k(123,231) = 0$ for $3 \leq k \leq n$ since three fixed points yield a 123 pattern. Hence, we need only consider $k = 0,1,2$. However, using a result in [SiSc] and the fact that $s_n^k(123,231) = 0$ for $3 \leq k \leq n$, we immediately obtain the stated equation for $s_n^0(123,231)$ upon proving the formulas for $k = 1,2$. Hence, we need only consider $k = 1,2$.

Let $\pi \in S_n(123,231)$ with $\pi_i = n$. If $i \neq 1$ then we must have $\pi = (i-1)(i-2)\cdots 21n(n-1)(n-2)\cdots i$ for any $2 \leq i \leq n$. This case contributes $n-1$ permutations to $S_n^1(123,231)$ for n odd and $\frac{n}{2}$ to $S_n^2(123,231)$ for n even. For $i = 1$, we have $\pi = n(n-1)\cdots(n-x+1)y(y-1)\cdots 1(n-x)(n-x-1)\cdots(y+1)$ for some $1 \leq x \leq n-2$ and $1 \leq y \leq n-x-1$, or we have $\pi = n(n-1)\cdots 21$.

We now consider the cases $k = 1,2$ for $i = 1$.

We have three potential intervals in which fixed points may reside:

(A) $n(n-1)\cdots(n-x+1)$,

(B) $y(y-1)\cdots 1$, or

(C) $(n-x)(n-x-1)\cdots(y+1)$

Hence, we have the following situations in which a fixed point appears.

A. For n odd we get one fixed point for any $\frac{n+1}{2} \leq x \leq n-2$ and $1 \leq y \leq n-x-1$, or if $\pi = n(n-1)\cdots 1$. The fixed point occurs at position $\frac{n+1}{2}$.

B. For $x+y$ odd we get one fixed point for any $1 \leq x \leq \frac{n-2}{2}$ and $x+1 \leq y \leq n-x-1$. The fixed point occurs at position $\frac{x+y+1}{2}$.

C. For $n+y$ odd we get one fixed point for any $1 \leq x \leq \frac{n-2}{2}$ and $1 \leq y \leq n-2x-1$. The fixed point occurs at position $\frac{n+y+1}{2}$.

We now consider the case $k = 2$ $(i = 1)$. To have two distinct fixed points we must have both B and C both occurring. Hence, we get

$$\sum_{x=1}^{\lfloor \frac{n-2}{3} \rfloor} \sum_{\substack{y=x+1 \\ n+y \ odd \\ x+y \ odd}}^{n-2x-1} 1 = \begin{cases} \frac{n(n-6)}{24} & if \ n \equiv 0 \, (mod \ 6) \\ \frac{(n-1)(n+1)}{24} & if \ n \equiv 1, 5 \, (mod \ 6) \\ \frac{(n-4)(n-2)}{24} & if \ n \equiv 2, 4 \, (mod \ 6) \\ \frac{(n-3)(n+3)}{24} & if \ n \equiv 3 \, (mod \ 6) \end{cases}$$

permutations with two fixed points in this case.

We now consider the case $k = 1$ $(i = 1)$. We may have only A occur, only B occur, or only C occur:

Only A occurs: In this case we require n to be odd and we get

$$1 + \sum_{x=\frac{n+1}{2}}^{n-2} \sum_{y=1}^{n-x-1} 1 = \frac{(n-3)(n-1)}{8} + 1$$

permutations with one fixed point.

Only B occurs: In this case we get

$$\sum_{x=1}^{\lfloor\frac{n-1}{3}\rfloor}\sum_{\substack{y=n-2x\\x+y\ odd}}^{n-x-1}1+\sum_{x=\lfloor\frac{n-1}{3}\rfloor+1}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{\substack{y=x\\x+y\ odd}}^{n-x-1}1+\sum_{x=1}^{\lfloor\frac{n-2}{3}\rfloor}\sum_{\substack{y=x+1\\x+y\ odd\\n+y\ even}}^{n-2x-1}1=\begin{cases}\frac{n(n-6)}{24}+3\binom{\frac{n+6}{6}}{2} & if\ \ n\equiv 0\,(mod\ 6)\\ \frac{(n-15)(n-1)}{24}+3\binom{\frac{n+5}{6}}{2} & if\ \ n\equiv 1\,(mod\ 6)\\ \frac{n(n-2)}{24}+3\binom{\frac{n+4}{6}}{2} & if\ \ n\equiv 2\,(mod\ 6)\\ \frac{(n-9)(n-3)}{24}+3\binom{\frac{n+3}{6}}{2} & if\ \ n\equiv 3\,(mod\ 6)\\ \frac{(n-12)(n+2)}{24}+3\binom{\frac{n+8}{6}}{2} & if\ \ n\equiv 4\,(mod\ 6)\\ \frac{(n-5)(n-3)}{24}+3\binom{\frac{n+1}{6}}{2} & if\ \ n\equiv 5\,(mod\ 6)\end{cases}$$

permutations with one fixed point.

Only C occurs: In this case we get

$$\sum_{x=1}^{\lfloor\frac{n-1}{3}\rfloor}\sum_{\substack{y=1\\n+y\ odd}}^{x}1+\sum_{x=\lfloor\frac{n-1}{3}\rfloor+1}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{\substack{y=1\\n+y\ odd}}^{n-2x-1}1+\sum_{x=1}^{\lfloor\frac{n-2}{3}\rfloor}\sum_{\substack{y=x+1\\x+y\ even\\n+y\ odd}}^{n-2x-1}1=\begin{cases}\frac{n(n-6)}{24}+3\binom{\frac{n+6}{6}}{2} & if\ \ n\equiv 0\,(mod\ 6)\\ \frac{(n-15)(n-1)}{24}+3\binom{\frac{n+5}{6}}{2} & if\ \ n\equiv 1\,(mod\ 6)\\ \frac{n(n-2)}{24}+3\binom{\frac{n+4}{6}}{2} & if\ \ n\equiv 2\,(mod\ 6)\\ \frac{(n-9)(n-3)}{24}+3\binom{\frac{n+3}{6}}{2} & if\ \ n\equiv 3\,(mod\ 6)\\ \frac{(n-12)(n+2)}{24}+3\binom{\frac{n+8}{6}}{2} & if\ \ n\equiv 4\,(mod\ 6)\\ \frac{(n-5)(n-3)}{24}+3\binom{\frac{n+1}{6}}{2} & if\ \ n\equiv 5\,(mod\ 6)\end{cases}$$

permutations with one fixed point.

Summing over all cases yields the stated formula. □

Theorem 2.4 *For $n \geq 1$, $s_n^{n-1}(213,132)=0$, $s_n^n(213,132)=1$,*

$$s_{2n+i}^0(213,132)=\begin{cases}\frac{5\cdot 4^{n-1}-2}{3} & if\ \ i=0\\ \frac{2(4^n-1)}{3} & if\ \ i=1\end{cases},$$

$$s_{2n+i}^{2k}(213,132) = \begin{cases} 4^{n-k-1} & \text{if } i=0 \\ \\ 0 & \text{if } i=1 \end{cases} \quad \text{for } 1 \le k \le n-1, \text{ and}$$

$$s_{2n+i}^{2k+1}(213,132) = \begin{cases} 0 & \text{if } i=0 \\ \\ 4^{n-k-1} & \text{if } i=1 \end{cases} \quad \text{for } 0 \le k \le n-1.$$

Proof. The proof is very similar to the proof of Theorem 2.2 and we provide a sketch of this proof.

It is easy to prove by induction that $\pi = (\pi(1), \pi(2), \ldots, \pi(m))$ where $\pi(j) = (t_j+1, t_j+2, \ldots, t_{j-1}-1, t_{j-1})$ for some $0 = t_0 < t_1 < t_2 < \cdots < t_m = n$. We call $\pi(j)$ a j-block of π. (See [Man] for more details.)

Recall that the graph of π consists of the points $(i, \pi_i) \in \mathbb{Z}^2$.

For $2k$ fixed points, we see that the middle block of $\pi = (\pi(1), \ldots, \pi(m))$ must be of length $2k$ and must contain all $2k$ fixed points and that n must be even. Hence, we have $s_{2n}^{2k} = (s_{n-k}(213,132))^2 = 4^{n-k-1}$ and $s_{2n+1}^{2k} = 0$ for $1 \le k \le n-1$.

Similarly, for $2k+1$ fixed points, we have $s_{2n+1}^{2k+1} = (s_{n-k}(213,132))^2 = 4^{n-k-1}$ and $s_{2n}^{2k+1} = 0$ for $0 \le k \le n-1$.

Using a result in [SiSc] gives the stated formula for $s_n^0(213,132)$. □

Theorem 2.5 *For $n \ge 3$, $s_n^0(132,231) = \frac{1}{3}(2^{n-1} + (-1)^n)$, $s_n^k(132,231) = \frac{2}{3}(2^{n-k-1} + (-1)^{n-k})$ for $1 \le k \le n-2$, $s_n^{n-1}(132,231) = 0$, and $s_n^n(132,231) = 1$.*

Proof. Let $\pi \in S_n(132,231)$. Clearly, we must have either $\pi_1 = n$ or $\pi_n = n$. For the case $\pi_n = n$ we have $s_{n-1}^{k-1}(132,231)$ permutations. Hence, we need only consider $\pi_1 = n$. Write $\pi = (n, \pi(1), 1, \pi(2))$. We must have $\pi(1)$ be a decreasing sequence and $\pi(2)$ be an increasing sequence. Hence, we cannot have a fixed point in $\pi(2)$ and we can have at most one fixed point in $\pi(1)$. Thus, for $k \ge 2$ the case $\pi_1 = n$ is empty giving $s_n^k(132,231) = s_{n-1}^{k-1}(132,231)$ for $k \ge 2$.

It remains to consider $k = 0, 1$ for $\pi = (n, \pi(1), 1, \pi(2))$. We consider $k = 0$ first (the case $k = 1$ will follow immediately). We may place the elements $2, 3, \ldots, n-1$ in 2^{n-2} ways such that $\pi(1)$ is decreasing and $\pi(2)$ is increasing. To see this, insert the elements $2, 3, \ldots, n-1$ in reverse order at 1's position and slide the element all the way to the left or all the way to right. However, we must subtract those permutations which contain a fixed point, which may only occur in $\pi(1)$.

Let f be a fixed point and write $\pi(1) = (\sigma(1), f, \sigma(2))$. We have $\binom{n-f-1}{f-2}$ choices for the elements in $\sigma(1)$. Once these are chosen, there are then 2^{f-2} ways to place the elements $2, 3, \ldots, f-1$ into $\sigma(2)$ and $\pi(2)$ such that $\sigma(2)$ is decreasing and $\pi(2)$ is increasing. Thus, we have

$$s_n^0(132, 231) = 2^{n-2} - \sum_{f=2}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-f-1}{f-2} 2^{f-2}.$$

From here, it is easy to check that $s_n^0(132, 231) = s_{n-1}^0(132, 231) + 2s_{n-2}^0(132, 231)$, which gives $s_n^0(132, 231) = \frac{1}{3}(2^{n-1} + (-1)^n)$ for $s_1^0(132, 231) = 0$ and $s_2^0(132, 231) = 1$.

For the case $k = 1$, the above argument shows that

$$s_n^1(132, 231) = s_{n-1}^0(132, 231) + \sum_{f=2}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-f-1}{f-2} 2^{f-2}.$$

From here, it is easy to check that $s_n^1(132, 231) = s_{n-1}^1(132, 231) + 2s_{n-2}^1(132, 231)$, which gives $s_n^1(132, 231) = \frac{2}{3}(2^{n-2} + (-1)^{n-1})$ for $s_1^1(132, 231) = 1$ and $s_2^1(132, 231) = 0$. □

Remark. $s_n^0(132, 231) = J_{n-2}$ where J_n is the n^{th} Jacobsthal number.

Theorem 2.6 *For* $n \geq 1$, $s_n^k(132, 321) = n - k - 1$ *for* $0 \leq k \leq n-1$ *and* $s_n^n(132, 321) = 1$.

Proof. Let $\pi \in S_n(132, 321)$. It is easy to see that in order to avoid both 132 and 321 we have either $\pi = (\pi', n)$ where $\pi' \in S_{n-1}(132, 321)$ or $\pi = (j+1)(j+2)\cdots(n-1)n12\ldots j$ for some $1 \leq j \leq n-1$.

Adjusting for the number of fixed points we see that

$$s_n^k(132, 321) = s_{n-1}^{k-1}(132, 321) + \sum_{j=1}^{n-1} I_{k=0}.$$

A straightforward induction on k for $0 \leq k < n$ finishes the proof. □

As a corollary of Theorem 2.6 we rederive a result given in [SiSc].

Corollary 2.7 $s_n(132, 321) = \binom{n}{2} + 1$

Proof. Summing over $0 \leq k \leq n$ we obtain the result immediately.

Theorem 2.8 *For $n \geq 1$,*

$$s_n^k(231,312) = \begin{cases} 2^{\frac{n-k-2}{2}} \left(\binom{\frac{n+k}{2}}{\frac{n-k}{2}} + \binom{\frac{n+k-2}{2}}{\frac{n-k}{2}} \right) & \text{for } n+k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Proof. For $\pi \in S_n(231,312)$ it is easy to see that we must have $\pi = \pi(1)n(n-1)\cdots j$ with $\pi(1) \in S_{j-1}^k(231,312) \cup S_{j-1}^{k-1}(231,312)$ (depending on the parity of $n+j$) for some $1 \leq j \leq n$. Hence, we have

$$s_n^k(231,312) = \sum_{\substack{j=1 \\ n-j \text{ odd}}}^{n-1} s_{j-1}^k(231,312) + \sum_{\substack{j=1 \\ n-j \text{ even}}}^{n} s_{j-1}^{k-1}(231,312).$$

From here we see that $s_n^k(231,312) = 2s_{n-2}^k(231,312) + s_{n-1}^{k-1}(231,312)$. Hence, using initial conditions it is easy to check that the formula given is correct. $\square$

Theorem 2.9 *Let $G_k(x)$ be the generating function for $\{s_n^k(231,321)\}_{n\geq 0}$. Then $G_k(x) = \frac{x^k(1-x)^{k+1}}{(1-x-x^2)^{k+1}}$. In particular, $s_n^0(231,321) = F_{n-2}$, for $n \geq 2$, where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence initialized by $F_0 = F_1 = 1$.*

Proof. Let $\pi \in S_n(231,321)$. It is easy to see that we must have $\pi = (\pi', \sigma)$ where $\pi \in S_{n-j}^{k-I_{j=1}}(231,321)$ and $\sigma = n(n-j+1)(n-j+2)\cdots(n-1)$ for some $1 \leq j \leq n$.

Hence, we have $s_n^k(231,321) = s_{n-1}^{k-1}(231,321) + \sum_{j=2}^n s_{n-j}^k(231,321)$ for $0 \leq k \leq n$. This implies that $s_n^k(231,321) = s_{n-1}^{k-1}(231,321) + s_{n-2}^k(231,321) + s_{n-1}^k(231,321) - s_{n-2}^{k-1}(231,321)$.

For $k = 0$ we have $s_n^0(231,321) = s_{n-1}^0(231,321) + s_{n-2}^0(231,321)$, from which it follows that $G_0(x) = \frac{1-x}{1-x-x^2}$. For $k > 0$ we have $G_k(x) = \frac{x(1-x)}{1-x-x^2} G_{k-1}(x)$. From here all of the results follow. $\square$

The following corollary rederives a result in [SiSc] .

Corollary 2.10 $\sum_{n\geq 0} s_n(231,321)x^n = \frac{1-x}{1-2x}$*, i.e.* $s_n(231,321) = 2^{n-1}$ *for* $n \geq 1$*.*

Proof. Summing $\sum_{k\geq 0} G_k(x)$ gives the desired result. $\square$

We now make the following definition.

Definition 2.1 *Let $S_1, S_2 \subseteq S_m$. If $s_n^k(S_1) = s_n^k(S_2)$ for all $0 \leq k \leq n$ for all $n \geq m$ we say that S_1 and S_2 are Super-Wilf equivalent. Furthermore, we say that S_1 and S_2 are in the same Super-Wilf class.*

Remark. Using the above definition and results from [RSZ] we see that 321, 132, and 213 are Super-Wilf equivalent, 231 and 312 are Super-Wilf equivalent, and the eight cases given at the beginning of Section 2 provide eight Super-Wilf classes.

3 The Case $|T| = 3$

Using I and RC we see that we have the following cases, except that case (1) is grouped together because both 123 and 321 are to be avoided, which is not possible for $n \geq 5$.

(1) $\overline{\{123, 132, 321\}} = \{\{123, 132, 321\}, \{123, 213, 321\}, \{123, 231, 321\}, \{123, 312, 321\}\}$

(2) $\overline{\{123, 132, 213\}} = \{\{123, 132, 213\}\}$

(3) $\overline{\{123, 132, 231\}} = \{\{123, 132, 231\}, \{123, 132, 312\}, \{123, 213, 231\}, \{123, 213, 312\}\}$

(4) $\overline{\{123, 231, 312\}} = \{\{123, 231, 312\}\}$

(5) $\overline{\{132, 213, 231\}} = \{\{132, 213, 231\}, \{132, 213, 312\}\}$

(6) $\overline{\{132, 213, 321\}} = \{\{132, 213, 321\}\}$

(7) $\overline{\{132, 231, 312\}} = \{\{132, 231, 312\}, \{213, 231, 312\}\}$

(8) $\overline{\{132, 231, 321\}} = \{\{132, 231, 321\}, \{132, 312, 321\}, \{213, 231, 321\}, \{213, 312, 321\}\}$

(9) $\overline{\{231, 312, 321\}} = \{\{231, 312, 321\}\}$

Theorem 3.1 *Let $\alpha \in \{132, 213, 231, 312\}$. Then*

$$\begin{aligned}
&\{s_n^0(123, 321)\}_{n\geq 0} = 1, 0, 1, 2, 1, 0, 0, \ldots \text{ for } \alpha \neq 231, 312 \\
&\{s_n^0(123, 321)\}_{n\geq 0} = 1, 0, 1, 1, 1, 0, 0, \ldots \text{ for } \alpha = 231 \text{ or } \alpha = 312 \\
&\{s_n^1(123, \alpha, 321)\}_{n\geq 0} = 0, 1, 0, 2, 0, 0, \ldots \text{ for } \alpha \neq 132, 213 \\
&\{s_n^1(123, \alpha, 321)\}_{n\geq 0} = 0, 1, 0, 1, 0, 0, \ldots \text{ for } \alpha = 132 \text{ or } \alpha = 213 \\
&\{s_n^2(123, \alpha, 321)\}_{n\geq 0} = 0, 0, 1, 0, 0 \ldots, \text{ for any } \alpha, \text{ and}
\end{aligned}$$

$s_n^k(123, \alpha, 321) = 0$ *for any α, for all $3 \leq k \leq n$.*

Proof. Obvious. □

Theorem 3.2 *For $n \geq 3$, $s_n^k(123,132,213) = 0$ for $3 \leq k \leq n$, $s_n^2(123,132,213) = F^2_{\frac{n-2}{2}} I_{n\ even}$, $s_n^1(123,132,213) = F^2_{\frac{n-1}{2}} I_{n\ odd}$, and*

$$s_n^0(123,132,213) = \begin{cases} F_n - F^2_{\frac{n-2}{2}} & \text{if } n \text{ is even} \\ \\ F_n - F^2_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases},$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence initialized by $F_0 = F_1 = 1$.

Proof. Clearly, $s_n^k(123,132,213) = 0$ for $k \geq 3$ since three fixed points are a 123 pattern. We may then use a result from from [SiSc] to get $s_n^0(123,132,213) = s_n(123,132,213) - s_n^1(123,132,213) - s_n^2(123,132,213) = F_n - s_n^1(123,132,213) - s_n^2(123,132,213)$. Hence, we only need consider $k = 1,2$.

We start with $k = 2$. The fixed points must be adjacent in order to avoid all of $123, 132, 213$. Let the fixed points be j and $j+1$. Write $\pi = (\pi(1), j, j+1, \pi(2))$. Then $\pi(1)$ is on the elements $\{n-j+2, \ldots, n\}$. To avoid the 123 pattern, we require $n-j+2 \geq j+2$ and $n-j+1 \leq j+1$. Hence, $j = \frac{n}{2}$ (so that n must be even). Thus, since it is not possible to have fixed points in $\pi(1)$ or $\pi(2)$ we have $s_n^2(123,132,213) = (s_{\frac{n-2}{2}}(123,132,213))^2 I_{n\ even}$, which, using a result from [SiSc], gives the stated formula.

Now consider $k = 1$. Let j be the fixed point. Note that $j \neq 1, n$ in order to avoid all of $123, 132, 213$. Write $\pi = (\pi(1), j, \pi(2))$. In order to avoid 123 and 213, there exists at most one $x \in \pi(1)$ with $x < j$. Assume, for a contradiction, that such an x exists. Then we must have $x = \pi_{j-1}$ in order to avoid the 132 pattern. Next, for $z \in \pi(2)$ we must have $z < j$ else xjz is a 123 pattern. This forces $x = j-1$ to be a fixed point, a contradiction. Hence, no such x exists so that for all $i \in \pi(1)$ we have $i > j$. In order to avoid 123 and 132, there exists at most one $y \in \pi(2)$ with $y > j$. Assume, for a contradiction, that such a y exists. Then we must have $y = \pi_{j+1}$ in order to avoid the 213 pattern. Furthermore, for all $i \in \pi(1)$ we must have $i > j$ so as not to have a 213 pattern with ijy. This forces $y = j+1$ to be a fixed point, a contradiction. Hence, no such y exists. Since we have $i \in \pi(1)$, $i > j$ and $k \in \pi(2)$, $k < j$, and for j to be a fixed point we require n to be odd so that $j = \frac{n+1}{2}$, we get $s_n^1(123,132,213) = (s_{\frac{n-1}{2}}(123,132,213))^2 I_{n\ odd}$, which, using a result from [SiSc], gives the stated formula. $\square$

Theorem 3.3 *For $n \geq 3$, $s_n^0(123,132,231) = \lfloor \frac{n}{2} \rfloor$, $s_n^1(123,132,231) = \lfloor \frac{n}{2} \rfloor + (-1)^{n+1}$, $s_n^2(123,132,231) = \frac{1}{2}(1+(-1)^n)$, and $s_n^k(123,132,231) = 0$ for $3 \leq k \leq n$.*

Proof. Let $\pi \in S_n(123,132,231)$. It is easy to see that we must have $\pi = n(n-1)\cdots(n-j+1)(n-j-1)(n-j-2)\cdots 21(n-j)$ for some $0 \leq j \leq n-1$. From here the results follow easily. $\square$

Theorem 3.4 *For $n \geq 3$, $s_n^0(123,231,312) = s_n^2(123,231,312) = \frac{n}{2}(1 - I_{n\ odd})$, $s_n^1(123,231,312) = n(1 - I_{n\ even})$, and $s_n^k(123,231,312) = 0$ for $3 \leq k \leq n$.*

Proof. Let $\pi \in S_n(123,231,312)$. It is easy to see that we must have $\pi = j(j-1)\cdots 1n(n-1)\cdots(j+1)$. From here the results follow easily. □

Theorem 3.5 *For $n \geq 3$, $s_n^0(132,213,231) = \lfloor \frac{n}{2} \rfloor I_{n\ odd} + (\frac{n}{2}+1)I_{n\ even}$, $s_n^1(132,213,231) = \lfloor \frac{n}{2} \rfloor I_{n\ odd}$, and $s_n^k(132,213,231) = I_{n=k}$ for $2 \leq k \leq n$.*

Proof. Let $\pi \in S_n(132,213,231)$. It is easy to see that we must have $\pi = n(n-1)\cdots(n-j+1)12\cdots(n-j)$ for some $1 \leq j \leq n$. From here the results follow easily. □

Theorem 3.6 *For $n \geq 3$, $s_n^0(132,213,321) = n-1$ and $s_n^k(132,213,321) = I_{n=k}$ for $1 \leq k \leq n$.*

Proof. Let $\pi \in S_n(132,213,321)$. Then we must have $\pi = j(j+1)\cdots n12\cdots(j-1)$ for some $1 \leq j \leq n$. From here the results follow easily. □

Theorem 3.7 *For $n \geq 2$, $s_n^0(132,231,312) = \frac{1}{2}(1+(-1)^n)$, and for $k \geq 1$,*

$$s_n^{2k}(132,231,312) = \begin{cases} 1+(-1)^n & \text{if } n > 2k \\ 1 & \text{if } n = 2k \end{cases}, \text{and}$$

$$s_n^{2k-1}(132,231,312) = \begin{cases} 1+(-1)^{n+1} & \text{if } n > 2k-1 \\ 1 & \text{if } n = 2k-1 \end{cases}.$$

Proof. Let $\pi \in S_n(132,231,312)$. It is easy to see that we must have $\pi = j(j-1)\cdots 1(j+1)(j+2)\cdots n$ for some $1 \leq j \leq n$. From here the results follow easily. □

Theorem 3.8 *For $n \geq 3$,*

$$s_n^k(132,231,321) = \begin{cases} 1 & \text{if } 0 \leq k \leq n-2 \\ 0 & \text{if } k = n-1 \\ 1 & \text{if } k = n \end{cases}.$$

Proof. Let $\pi \in S_n(132,231,321)$. It is easy to see that we must have $\pi = j12\cdots(j-1)(j+1)\cdots n$ for some $1 \leq j \leq n$. From here the results follow easily. □

Theorem 3.9 *For $n \geq 3$ and $0 \leq k \leq n$, $s_n^k(231,312,321) = \binom{\frac{n+k}{2}}{k} I_{n+k\ even}$.*

Proof. Let $\pi \in S_n^k(231,312,321)$. Then π must be of the form $(1,\pi')$ or $(2,1,\pi'')$ where $\pi' \in S_{n-1}^{k-1}(231,312,321)$ and $\pi'' \in S_{n-1}^k(231,312,321)$. This gives us $s_n^k(231,312,321) = s_{n-1}^{k-1}(231,312,321) + s_{n-2}^k(231,312,321)$. A straightforward induction on $n+k$ finishes the proof. □

4 The Cases $|T| \geq 4$

The cases $|T| \geq 4$ are easy and in fact $s_n^k(T) \in \{0,1,2\}$ for all $T \subseteq S_3$, $|T| \geq 4$, for all $n \geq 1$ and $0 \leq k \leq n$.

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