BOUNDS ON SOME VAN DER WAERDEN NUMBERS

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Abstract

For positive integers s and $k_1, k_2, \ldots, k_s$, the van der Waerden number $w(k_1, k_2, \ldots, k_s; s)$ is the minimum integer n such that for every s-coloring of $\{1, 2, \ldots, n\}$, with colors $1, 2, \ldots, s$, there is a k_i-term arithmetic progression of color i for some i. We give an asymptotic lower bound for $w(k, m; 2)$ for fixed m. We include a table of values of $w(k, 3; 2)$ that are very close to this lower bound for $m = 3$. We also give a lower bound for $w(k, k, \ldots, k; s)$ that slightly improves previously-known bounds. Upper bounds for $w(k, 4; 2)$ and $w(4, 4, \ldots, 4; s)$ are also provided.

1. Introduction

Two fundamental theorems in combinatorics are van der Waerden's Theorem [18] and Ramsey's Theorem [16]. The theorem of van der Waerden says that for all positive integers s and $k_1, k_2, \ldots, k_s$, there exists a least positive integer $n = w(k_1, k_2, \ldots, k_s; s)$ such that whenever $[1, n] = \{1, 2, \ldots, n\}$ is s-colored (i.e., partitioned into s sets), there is a k_i-term arithmetic progression with color i for some i, $1 \leq i \leq s$.

Similarly, Ramsey's Theorem has an associated "threshold" function $R(k_1, k_2, \ldots, k_s; s)$ (which we will not define here). The order of magnitude of $R(k, 3; 2)$ is known to be $\frac{k^2}{\log k}$ [11], while the best known upper bound on $R(k, m; 2)$ is fairly close to the best known lower bound. In contrast, the order of magnitude of $w(k, 3; 2)$ is not known, and the best known lower and upper bounds on $w(k, k; 2)$ are

$$(k-1)2^{(k-1)} \leq w(k, k; 2) < 2^{2^{2^{2^{2^{(k+9)}}}}},$$

the lower bound known only when $k-1$ is prime. The lower bound is due to Berlekamp [1] and the upper bound is a celebrated result of Gowers [6], which answered a long-standing question of Ron Graham. Graham currently offers 1000 USD for a proof or disproof of $w(k,k;2) < 2^{k^2}$ [2]. Several other open problems are stated in [14].

Recently, there have been some further breakthroughs in the study of the van der Waerden function $w(k,m;2)$. One was the amazing calculation that $w(6,6;2) = 1132$ by Kouril [12], extending the list of previously known values $w(3,3;2) = 9$, $w(4,4;2) = 35$, and $w(5,5;2) = 178$. A list of other known exact values of $w(k,m;2)$ appears in [15]. Improved lower bounds on several specific values of $w(k,k;s)$ are given in [3] and [10].

In another direction, Graham [7] gives an elegant proof that if one defines $w_1(k,3)$ to be the least n such that every 2-coloring of $[1,n]$ gives either k consecutive integers in the first color or a 3-term arithmetic progression in the second color, then

$$k^{c\log k} < w_1(k,3) < k^{dk^2},$$

for suitable constants $c, d > 0$. This immediately gives $w(k,3;2) < k^{dk^2}$ since we trivially have $w(k,3;2) \le w_1(k,3)$. In view of Graham's bounds on $w_1(k,3)$, it would be desirable to obtain improved bounds on $w(k,3;2)$. Of particular interest is the question of whether or not there is a non-polynomial lower bound for $w(k,3;2)$.

In this note we give a lower bound of $w(k,3;2) > k^{(2-o(1))}$. Although this may seem weak, we do know that $w(k,3;2) < k^2$ for $5 \le k \le 16$ (i.e., for all known values of $w(k,3;2)$ with $k \ge 5$; see Table 1). More generally, we give a lower bound on $w(k,m;2)$ for arbitrary fixed m. We also present a lower bound for the classical van der Waerden numbers $w(k,k,\ldots,k;s)$ that is a slight improvement over previously published bounds. In addition, we present an upper bound for $w(k,4;2)$ and an upper bound for $w(4,4,\ldots,4;s)$.

2. Upper and Lower Bounds for Certain van der Waerden Functions

We shall need several definitions, which we collect here.

For positive integers k and n,

$$r_k(n) = \max_{S\subseteq[1,n]} \{|S| : S \text{ contains no } k\text{-term arithmetic progression}\}.$$

For positive integers k and m, denote by $\chi_k(m)$ the minimum number of colors required to color $[1,m]$ so that there is no monochromatic k-term arithmetic progression.

The function $w_1(k,3)$ has been defined in Section 1. Similarly, we define $w_1(k,4)$ to be the least n such that every 2-coloring of $[1,n]$ has either k consecutive integers in the first color or a 4-term arithmetic progression in the second color.

We begin with an upper bound for $w_1(k,4)$. The proof is essentially the same as the proof given by Graham [7] of an upper bound for $w_1(k,3)$. For completeness, we include the proof here. We will make use of a recent result of Green and Tao [9], who showed that for some constant $c>0$,

$$r_4(n) < ne^{-c\sqrt{\log\log n}} \tag{1}$$

for all $n \geq 3$.

Proposition 2.1 There exists a constant $c>0$ such that $w_1(k,4) < e^{k^{c\log k}}$ for all $k \geq 2$.

Proof. Suppose we have a 2-coloring of $[1,n]$ (assume $n \geq 4$) with no 4-term arithmetic progression of the second color and no k consecutive integers of the first color. Let $t_1 < t_2 < \cdots < t_m$ be the integers of the second color. Hence, $m < r_4(n)$. Let us define $t_0 = 0$ and $t_{m+1} = n$. Then there must be some i, $1 \leq i \leq m$, such that

$$t_{i+1} - t_i > \frac{n}{2r_4(n)}.$$

(Otherwise, using $r_4(n) \geq 3$, we would have $n = \sum_{i=0}^{m}(t_{i+1}-t_i) \leq \frac{n(m+1)}{2r_4(n)} \leq \frac{n(r_4(n)+1)}{2r_4(n)} \leq \frac{n}{2} + \frac{n}{6}$.)

Using (1), we now have an i with

$$t_{i+1} - t_i > \frac{n}{2r_4(n)} > \frac{1}{2}e^{c\sqrt{\log\log n}}.$$

If $n \geq e^{k^{d\log k}}$, $d = c^{-2}$, then $\frac{1}{2}e^{c\sqrt{\log\log n}} \geq k$ and we have k consecutive integers of the first color, a contradiction. Hence, $n < e^{k^{d\log k}}$ and we are done. □

Clearly $w(k,4;2) \leq w_1(k,4)$. Consequently, we have the following result.

Corollary 2.2 There exists a constant $d>0$ such that $w(k,4;2) < e^{k^{d\log k}}$ for all $k \geq 2$.

Using Green and Tao's result, it is not difficult to obtain an upper bound for $w(4,4,\ldots,4;s)$.

Proposition 2.3 There exists a constant $d>0$ such that $w(4,4,\ldots,4;s) < e^{s^{d\log s}}$ for all $s \geq 2$.

Proof. Consider a $\chi_4(m)$-coloring of $[1,m]$ for which there is no monochromatic 4-term arithmetic progression. Some color must be used at least $\frac{m}{\chi_4(m)}$ times, and hence $\frac{m}{\chi_4(m)} \leq r_4(m)$ so that $\frac{m}{r_4(m)} \leq \chi_4(m)$. Let $c>0$ be such that (1) holds for all $n \geq 3$, and let $m = e^{s^{d\log s}}$, where $d = c^{-2}$. Then $\chi_4(m) \geq \frac{m}{r_4(m)} > e^{c\sqrt{\log\log m}} = s$. This means that every s-coloring of $[1,m]$ has a monochromatic 4-term arithmetic progression. Since $m = e^{s^{d\log s}}$, the proof is complete. □

It is interesting that the bounds in Corollary 2.2 and Proposition 2.3 have the same form.

The following theorem gives a lower bound on $w(k,k,\ldots,k;s)$. It is deduced without too much difficulty from the Symmetric Hypergraph Theorem as it appears in [8], combined with an old result of Rankin [17]. To the best of our knowledge it has not appeared in print before, even though it is better, for large s, than the standard lower bound $\frac{cs^k}{k}(1+o(1))$ (see [8]), as well as the lower bounds $s^{k+1}-\sqrt{c(k+1)\log(k+1)}$ and $\frac{ks^k}{e(k+1)^2}$ due to Erdős and Rado [4], and Everts [5], respectively. We give the proof in some detail. The proof makes use of the following facts:

$$\chi_k(n) < \frac{2n\log n}{r_k(n)}(1+o(1)), \tag{2}$$

which appears in [8] as a consequence of the Symmetric Hypergraph Theorem; and

$$r_k(n) > ne^{-c(\log n)^{\frac{1}{\lfloor \log_2 k \rfloor + 1}}}, \tag{3}$$

which, for some constant $c>0$, holds for all $n \geq 3$ (this appears in [17]).

Theorem 2.4 Let $k \geq 3$ be fixed, and let $z = \lfloor \log_2 k \rfloor$. There exists a constant $d > 0$ such that $w(k,k,\ldots,k;s) > s^{d(\log s)^z}$ for all sufficiently large s.

Proof. We make use of the observation that for positive integers s and m, if $s \geq \chi_k(m)$, then $w(k,k,\ldots,k;s) > m$, which is clear from the definitions. For large enough m, (2) gives

$$\chi_k(m) < \frac{2m\log m}{r_k(m)}\left(1+\frac{1}{2}\right) = \frac{3m\log m}{r_k(m)}. \tag{4}$$

Now let $d = \left(\frac{1}{2c}\right)^{z+1}$, where c is from (3), and let $m = s^{d(\log s)^z}$, where s is large enough so that (4) holds. By (3), noting that $\log m = d(\log s)^{z+1} = \left(\frac{\log s}{2c}\right)^{z+1}$, we have

$$\frac{m}{r_k(m)} < e^{c(\log m)^{\frac{1}{z+1}}} = e^{c\cdot\frac{\log s}{2c}} = \sqrt{s}.$$

Therefore,

$$\frac{3m\log m}{r_k(m)} < 3d\sqrt{s}(\log s)^{z+1} < s$$

for sufficiently large s. Thus, for sufficiently large s,

$$\chi_k(m) < \frac{3m\log m}{r_k(m)} < s.$$

According to the observation at the beginning of the proof, this implies that $w(k,k,\ldots,k;s) > m = s^{d(\log s)^z}$, as required. □

We now give a lower bound on $w(k,m;2)$. We make use of the Lovász Local Lemma (see [8] for a proof), which will be implicitly stated in the proof.

Theorem 2.5 Let $m \geq 3$ be fixed. Then for all sufficiently large k,

$$w(k, m; 2) > k^{m-1-\frac{1}{\log\log k}}.$$

Proof. Given m, choose $k > m$ large enough so that

$$k^{\frac{1}{2m \log\log k}} > \left(m - \frac{1}{2\log\log k}\right) \log k \tag{5}$$

and

$$6 < \frac{\log k}{\log\log k}. \tag{6}$$

Next, let $n = \lfloor k^{m-1-\frac{1}{\log\log k}} \rfloor$. To prove the theorem, we will show that there exists a (red, blue)-coloring of $[1, n]$ for which there is no red k-term arithmetic progression and no blue m-term arithmetic progression.

For the purpose of using the Lovász Local Lemma, randomly color $[1, n]$ in the following way. For each $i \in [1, n]$, color i red with probability $p = 1 - k^{\alpha - 1}$ where

$$\alpha \stackrel{\text{def}}{=} \frac{1}{2m \log\log k},$$

and color it blue with probability $1 - p$.

Let $\mathcal{P}$ be any k-term arithmetic progression. Then, since $1 + x \leq e^x$, the probability that $\mathcal{P}$ is red is

$$p^k = \left(1 - k^{\alpha-1}\right)^k \leq (e^{-k^{\alpha-1}})^k = e^{-k^{\alpha}}.$$

Hence, applying (5), we have

$$p^k < \left(\frac{1}{e}\right)^{\left(m - \frac{1}{2\log\log k}\right)\log k} = \frac{1}{k^{m-\frac{1}{2\log\log k}}}.$$

Also, for any m-term arithmetic progression $\mathcal{Q}$, the probability that $\mathcal{Q}$ is blue is

$$(1-p)^m = (k^{\alpha-1})^m = \frac{1}{k^{m-\frac{1}{2\log\log k}}}.$$

Now let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$ be all of the arithmetic progressions in $[1, n]$ with length k or m. So that we may apply the Lovász Local Lemma, we form the "dependency graph" G by setting $V(G) = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t\}$ and $E(G) = \{\{\mathcal{P}_i, \mathcal{P}_j\} : i \neq j, \mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset\}$. For each $\mathcal{P}_i \in V(G)$, let $d(\mathcal{P}_i)$ denote the degree of the vertex $\mathcal{P}_i$ in G, i.e., $|\{e \in E(G) : \mathcal{P}_i \in e\}|$. We now estimate $d(P_i)$ from above. Let $x \in [1, n]$. The number of k-term arithmetic progressions $\mathcal{P}$ in $[1, n]$ that contain x is bounded above by $k \cdot \frac{n}{k-1}$, since there are k positions that x may occupy in $\mathcal{P}$ and since the gap size of $\mathcal{P}$ cannot exceed $\frac{n}{k-1}$. Similarly, the number of m-term arithmetic progressions $\mathcal{Q}$ in $[1, n]$ that contain x is bounded above by $m \cdot \frac{n}{m-1}$.

Let $\mathcal{P}_i$ be any k-term arithmetic progression contained in $[1,n]$. The total number of k-term arithmetic progressions $\mathcal{P}$ and m-term arithmetic progressions $\mathcal{Q}$ in $[1,n]$ that may have non-empty intersection with $\mathcal{P}_i$ is bounded above by

$$k\left(k\cdot\frac{n}{k-1}+m\cdot\frac{n}{m-1}\right)<kn\left(2+\frac{2}{m-1}\right), \tag{7}$$

since $k>m$. Thus, $d(\mathcal{P}_i)<kn\left(2+\frac{2}{m-1}\right)$ when $|\mathcal{P}_i|=k$. Likewise, $d(\mathcal{P}_i)<mn\left(2+\frac{2}{m-1}\right)$ when $|\mathcal{P}_i|=m$. Thus, for all vertices $\mathcal{P}_i$ of G, we have $d(\mathcal{P}_i)<kn\left(2+\frac{2}{m-1}\right)$.

To finish setting up the hypotheses for the Lovász Local Lemma, we let X_i denote the event that the arithmetic progression $\mathcal{P}_i$ is red if $|\mathcal{P}_i|=k$, or blue if $|\mathcal{P}_i|=m$, and we let $d=\max\limits_{1\le i\le t} d(\mathcal{P}_i)$. We have seen above that for all i, $1\le i\le t$, the probability that X_i occurs is at most

$$q\stackrel{\text{def}}{=}\frac{1}{k^{m-\frac{1}{2\log\log k}}},$$

while from (7) we have $d<2kn\left(1+\frac{1}{m-1}\right)$.

We are now ready to apply the Lovász Local Lemma, which says that in these circumstances, if the condition $eq(d+1)<1$ is satisfied, then there is a (red, blue)-coloring of $[1,n]$ such that no event X_i occurs, i.e., such that there is no red k-term arithmetic progression and no blue m-term arithmetic progression. This will imply

$$w(k,m;2)>n=k^{m-1-\frac{1}{\log\log k}},$$

as desired. Thus, the proof will be complete when we verify that $eq(d+1)<1$. Using $m\ge 3$, we have $d<3kn$, so that $d+1<3kn+1<e^2kn$. Hence, it is sufficient to verify that

$$e^3qkn<1. \tag{8}$$

Since $q=\frac{1}{k^{m-\frac{1}{2\log\log k}}}$ and $n\le k^{m-1-\frac{1}{\log\log k}}$, inequality (8) may be reduced to (6), and the proof is now complete. □

Remark. As long as $k>e^{m^6}$, the inequality of Theorem 2.5 holds. To show this, we need only show that conditions (5) and (6) hold if $k>e^{m^6}$. That (6) holds is obvious. For (5), it suffices to have $k^{\frac{1}{2m\log\log k}}>m\log k$; that is $\log k>2m\log\log k(\log m+\log\log k)$. When $k\ge e^{m^6}$, we have $2m\log\log k(\log m+\log\log k)\le 2(\log k)^{1/6}\log\log k(\frac{1}{6}\log\log k+\log\log k)=\frac{7}{3}(\log k)^{1/6}(\log\log k)^2$. Since $(\log\log k)<(\log k)^{7/20}$ for $k\ge e^{m^6}$ we have $2m\log\log k(\log m+\log\log k)\le\frac{7}{3}(\log k)^{13/15}$. Finally, since $(\log k)^{2/15}\ge\frac{7}{3}$ for $k\ge e^{m^6}$, condition (5) is satisfied.

We end with a table of computed values. These were all computed with a standard backtrack algorithm except for $w(14,3;2)$, $w(15,3;2)$, and $w(16,3;2)$, which are due to Michal Kouril [13]. The values $w(k,3;2)$, $k\le 12$, appeared previously in [15].

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$w(k,3;2)$	6	9	18	22	32	46	58	77	97	114	135	160	186	218	238
$w_1(k,3)$	9	23	34	73	113	193	?	?	?	?	?	?	?	?	?

Table 1: Small values of $w(k,3)$ and $w_1(k,3)$

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