The Determination of 2-color Zero-sum Generalized Schur Numbers

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Abstract

Consider the equation $\mathcal{E} : x_1 + \cdots + x_{k-1} = x_k$ and let k and r be positive integers such that $r \mid k$. The number $S_{\mathfrak{z},2}(k;r)$ is defined to be the least positive integer t such that for any 2-coloring $\chi : [1,t] \to \{0,1\}$ there exists a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$ satisfying $\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$. In a recent paper, the first author posed the question of determining the exact value of $S_{\mathfrak{z},2}(k;4)$. In this article, we solve this problem and show, more generally, that $S_{\mathfrak{z},2}(k,r) = kr - 2r + 1$ for all positive integers k and r with $r \mid k$ and $k \geq 2r$.

1. Introduction

For $r \in \mathbb{Z}^+$, there exists a least positive integer $S(r)$, called a *Schur number*, such that within every r-coloring of $[1, S(r)]$ there is a monochromatic solution to the linear equation $x_1 + x_2 = x_3$.

In 1933, Rado [10] generalized the work of Schur to arbitrary systems of linear equations. For any integer $k \geq 2$ and $r \in \mathbb{Z}^+$, there exists a least positive integer $S(k;r)$, called a *generalized Schur number*, such that every r-coloring of $[1, S(k;r)]$ admits a monochromatic solution to the equation $\mathcal{E} : x_1 + \cdots + x_{k-1} = x_k$. Indeed, Rado's result proves, in particular, that the number $S(k,r)$ exists (is finite). In [2], Beutelspacher and Brestovansky proved the exact value $S(k;2) = k^2 - k - 1$.

Before we analogize the above number, we need the following definition.

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Definition 1. Let $r \in \mathbb{Z}^+$. We say that a set of integers $\{a_1, a_2, \ldots, a_n\}$ is *r-zero-sum* if $\sum_{i=1}^{n} a_i \equiv 0 \pmod{r}$.

The Erdős-Ginzburg-Ziv Theorem [5] is one of the cornerstones of zero-sum theory (see, for instance, [1] and [9]). It states that any sequence of $2n-1$ integers must contain an n-zero-sum subsequence of n integers. In recent times, zero-sum theory has made remarkable progress (see, for instance, [3], [4], [6], [7], [8]).

In [11], the first author replaced the "monochromatic property" of the generalized Schur number by the "zero-sum property" and introduced the following new number which is called a *zero-sum generalized Schur number*.

Definition 2. Let k and r be positive integers such that $r \mid k$. We define $S_{\mathfrak{z}}(k;r)$ to be the least positive integer t such that for any r-coloring $\chi : [1,t] \to \{0, \ldots, r-1\}$ there exists a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to $x_1 + \cdots + x_{k-1} = x_k$ satisfying $\displaystyle\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

Notation. Throughout the article, we represent the equation $x_1 + \cdots + x_{k-1} = x_k$ by $\mathcal{E}$.

Since $r \mid k$, note that if $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ is a monochromatic solution to equation $\mathcal{E}$, then clearly it is an r-zero-sum solution. Hence, we get, $S_{\mathfrak{z}}(k;r) \le S(k;r)$ and therefore, $S_{\mathfrak{z}}(k;r)$ is finite.

In [11], the first author calculated lower bounds of this number for some r. In particular, he proved the following result.

Theorem 1. *[11] Let k and r be positive integers such that $r \mid k$. Then,*

$$S_{\mathfrak{z}}(k;r) \ge \begin{cases} 3k-3 & \text{when } r = 3; \\ 4k-4 & \text{when } r = 4; \\ 2(k^2-k-1) & \text{when } r = k \text{ is an odd positive integer.} \end{cases}$$

In the same article, he introduced another number meant only for 2-colorings, but keeping the r-zero-sum notion.

Definition 3. Let k and r be positive integers such that $r \mid k$. We denote by $S_{\mathfrak{z},2}(k;r)$ the least positive integer such that every 2-coloring of $\chi : [1, S_{\mathfrak{z},2}(k;r)] \to \{0,1\}$ admits a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$ satisfying $\displaystyle\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

Since any 2-coloring of $[1, S_{\mathfrak{z}}(k;r)]$ is also an r-coloring (for $r \geq 2$), we see that $S_{\mathfrak{z},2}(k;r) \leq S_{\mathfrak{z}}(k;r)$ and hence $S_{\mathfrak{z},2}(k;r)$ is finite.

In [11], the first author proved the following theorem related to these 2-color zero-sum generalized Schur numbers.

Theorem 2. *[11] Let k and r be two positive integers such that $r \mid k$. Then,*

$$S_{\mathfrak{z},2}(k;r) = \begin{cases} 2k-3; & \text{if } r = 2 \\ 3k-5; & \text{if } r = 3 \text{ and } k \neq 3 \\ k^2-k-1; & \text{if } r = k \end{cases}$$

One notes that the exact values of $S_{\mathfrak{z},2}(k;r)$ for $r = 2, 3$ and $S_{\mathfrak{z},2}(r,r)$ do not show any obvious generalization to $S_{\mathfrak{z},2}(k;r)$ for any k which is a multiple of r. However, the computations given in [11] when $r = 4$ and $k = 4, 8, 12$, and when $r = 5$ and $k = 5, 10, 15$, were enough for us to conjecture a general formula, which turns out to hold. To this end, by Theorem 3 below, we answer a question posed by the first author in [11] and, more generally, determine the exact values of $S_{\mathfrak{z},2}(k;r)$.

Theorem 3. *Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Then,*

$$S_{\mathfrak{z},2}(k;r) = rk - 2r + 1.$$

2. Preliminaries

We start by presenting a pair of lemmas useful for proving our upper bounds.

Lemma 1. *Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Let $\chi : [1, rk-2r+1] \to \{0,1\}$ be a 2-coloring such that $\chi(1) = \chi(r-1) = 0$. Then there exists an r-zero-sum solution to equation $\mathcal{E}$ under χ.*

Proof. Consider the solution $(1, 1, \ldots, 1, k-1)$. If $\chi(k-1) = 0$, then, since $\chi(1) = 0$, we are done. Hence, we shall assume that $\chi(k-1) = 1$.

Next, we look at the solution

$$(\underbrace{1, \ldots, 1}_{k-r}, \underbrace{k-1, k-1, k-1}_{r-1}, rk-2r+1).$$

Since $\chi(1) = 0$ and $\chi(k-1) = 1$, we can assume that $\chi(rk-2r+1) = 0$; otherwise, we have exactly r integers of color 1 and so the solution is r-zero-sum.

Since $(1, r, r, \ldots, r, rk-2r+1)$ is a solution to $\mathcal{E}$, we can assume that $\chi(r) = 1$.

Finally, consider

$$(\underbrace{r-1, \ldots, r-1}_{r-1}, \underbrace{r, \ldots, r}_{k-r}, rk-2r+1).$$

Since $\chi(r-1) = 0, \chi(r) = 1$, $\chi(rk-2r+1) = 0$, and $r \mid k$, this solution is r-zero-sum, thereby proving the lemma. $\square$

Lemma 2. *Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Let $\chi : [1, rk-2r+1] \to \{0,1\}$ be a coloring such that $\chi(1) = 0$ and $\chi(r-1) = 1$. If one of the following holds, then there exists an r-zero-sum solution to equation $\mathcal{E}$:*

(a) $\chi(r) = 0$*;*
(b) $\chi(k-2) = 1$*;*
(c) $\chi(k-1) = 0$*;*
(d) $\chi(k) = 0$
(e) $\chi(rk-2r-1) = 0$*;*
(f) $\chi(rk-2r+1) = 1$.

Proof. We will prove each possibility separately; however, the order in which we do so matters so we will not be proving them in the order listed.

(c) Consider $(1, 1, \ldots, 1, k-1)$. If $\chi(k-1) = 0$ then this solution is r-zero sum.

(d) By considering the solution $(r-1, \ldots, r-1, (r-1)(k-1))$, we can assume that $\chi((r-1)(k-1)) = 0$. Using this in $(\underbrace{1, \ldots, 1}_{k-r+1}, \underbrace{k, \ldots, k}_{r-2}, (r-1)(k-1))$ along with the assumption that $\chi(k) = 0$, we have an r-zero-sum solution.

(f) From part (c), we may assume that $\chi(k-1) = 1$. Looking at $(r-1, \ldots, r-1, k-1, rk-2r+1)$, since $\chi(k-1) = \chi(r-1) = 1$, and we assume that $\chi(rk-2r+1) = 1$, we have an r-zero sum solution.

(a) From part (f), we may assume that $\chi(rk-2r+1) = 0$. With this assumption, we see that $(1, r, \ldots, r, rk-2r+1)$ is r-zero-sum when $\chi(r) = 0$.

(b) From parts (a) and (f), we may assume $\chi(r) = 1$ and $\chi(rk-2r+1) = 0$. Under these assumptions, we find that $(\underbrace{1, \ldots, 1}_{k-r-1}, r, \underbrace{k-2, \ldots, k-2}_{r-1}, rk-2r+1)$ is an r-zero-sum solution with $\chi(k-2) = 1$.

(e) By considering $(\underbrace{1,\ldots,1}_{r-1},\underbrace{r,\ldots,r}_{k-2r},\underbrace{2r-3,\ldots,2r-3}_{r},rk-2r-1)$ and using $r \mid k$, we have an r-zero-sum solution when $\chi(rk-2r-1)=0$. $\square$

3. Proof of the Main Result

First, we restate the main result.

Theorem 3. Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Then,

$$S_{\mathfrak{z},2}(k;r) = rk - 2r + 1.$$

Proof. We start with the lower bound. To prove that $S_{\mathfrak{z},2}(k;r) > rk-2r$, we consider the 2-coloring χ of $[1, rk-2r]$ defined by $\chi(i)=0$ for $1 \leq i \leq k-2$ and $\chi(i)=1$ for $k-1 \leq i \leq rk-2r$. Assume, for a contradiction, that χ admits an r-zero-sum solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$. Then $\chi(\hat{x}_i)=1$ for some $i \in \{1,2,\ldots,k\}$; otherwise the solution is monochromatic of color 0, but $\sum_{i=1}^{k-1} \hat{x}_i \geq k-1$, meaning that $\hat{x}_k$ cannot be of color 0.

Assuming that $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ is r-zero-sum and not monochromatic of color 0, we must have $\chi(x_j)=1$ for at least r of the x_j's. Since the minimum integer under χ that is of color 1 is $k-1$, this gives us

$$\sum_{i=1}^{k-1} x_i \geq (r-1)(k-1) + 1(k-r) = (rk-2r+1) > rk-2r,$$

which is out of bounds, a contradiction. Hence, χ does not admit an r-zero-sum solution to $\mathcal{E}$ and we conclude that $S_{\mathfrak{z},2}(k;r) \geq rk-2r+1$.

We now move on to the upper bound. We let $\chi : [1, rk-2r+1] \to \{0,1\}$ be an arbitrary 2-coloring. We may assume that $\chi(1)=0$ since χ admits an r-zero-sum solution if and only if the induced coloring $\overline{\chi}$ defined by $\overline{\chi}(i) = 1-\chi(i)$ also does so.

The cases $r=2,3$ have been done by Theorem 2. Hence, we may assume that $r \geq 4$. We must handle the case $r=4$ separately; we start with this case.

We will show that $4k-7$ serves as an upper bound for $S_{\mathfrak{z}}(k;r)$. Consider the following solution to $\mathcal{E}$:

$$(1,1,1,\underbrace{2,\ldots,2}_{k-8},3,3,k,k,4k-7).$$

Noting that $r-1=3$ and $rk-2r+1=4k-7$, by Lemmas 1 and 2, we may assume $\chi(3)=1$, $\chi(k)=1$, and $\chi(4k-7)=0$. Since k is a multiple of 4 and $k\geq 8$, we see that $k-8$ is also a multiple of 4. Hence, the color of 2 does not affect whether or not this solution is 4-zero-sum. Of the integers not equal to 2, we have exactly four of them of color 1. Hence, this solution is 4-zero-sum. This, along with the lower bound above, proves that $S_{\mathfrak{z},2}(k;4)=4k-7$.

We now move on to the cases where $r\geq 5$. We proceed by assuming that no r-zero-sum solution occurs under an arbitrary 2-coloring $\chi:[1,rk-2r+1]\to\{0,1\}$. From Lemmas 1 and 2, we may assume the following table of colors holds.

color 0	color 1
1	$r-1$
$k-2$	r
$rk-2r+1$	$k-1$
	k
	$rk-2r-1.$

In order for the solution

$$(\underbrace{1,\ldots,1}_{k-r},\underbrace{k-2,\ldots,k-2}_{r-2},k+r-3,rk-2r+1)$$

not to be r-zero-sum, we deduce that $\chi(k+r-3)=1$. Using this in the solution

$$(\underbrace{1,\ldots,1}_{k-r},\underbrace{k,\ldots,k}_{r-2},k+r-3,rk-3)$$

we may assume that $\chi(rk-3)=0$. In turn, we use this in

$$(\underbrace{2,\ldots,2}_{k-r-1},r,r-1,\underbrace{k,\ldots,k}_{r-2},rk-3)$$

to deduce that $\chi(2)=1$. Modifying this last solution slightly, we consider

$$(\underbrace{3,\ldots,3}_{k-r-1},r,r,r,\underbrace{k,\ldots,k}_{r-3},rk-3)$$

to deduce that $\chi(3)=1$. Finally, since $r\geq 5$, we can consider

$$(\underbrace{2,\ldots,2}_{k-2r+6},\underbrace{3,\ldots,3}_{r-5},\underbrace{k-1,\ldots,k-1}_{r-2},rk-2r-1).$$

We see that this solution is monochromatic (of color 1), and, hence, is r-zero-sum. This proves that $S_{\mathfrak{z},2}(k;r)\leq rk-2r+1$ for $r\geq 5$, which, together with the lower bound at the beginning of the proof, gives us $S_{\mathfrak{z},2}(k;r) = rk - 2r + 1$, thereby completing the proof. $\square$

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