ZERO-SUM GENERALIZED SCHUR NUMBERS

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Abstract

Let r and k be positive integers with $r \mid k$. Denote by $S_{\mathfrak{z}}(k;r)$ the minimum integer n such that every coloring $\chi : [1,n] \to \{0,1,\ldots,r-1\}$ admits a solution to $\sum_{i=1}^{k-1} x_i = x_k$ with $\sum_{i=1}^{k} \chi(x_i) \equiv 0 \pmod{r}$. We give some formulas and lower bounds for various instances.

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1. Introduction

We start with the definition of the standard generalized Schur numbers. For any positive integers k and r, there exists a minimal integer $S(k;r)$ such that any r-coloring χ of $[1, S(k;r)]$ admits a monochromatic solution to $\sum_{i=1}^{k-1} x_i = x_k$. This follows directly from Ramsey's theorem by defining the coloring of each edge ij of the complete graph K_n to be $\chi(|j-i|)$. A monochromatic K_k under this coloring with vertices $i_1 < i_2 < \cdots < i_k$ means that $i_{j+1} - i_j$ and $i_k - i_1$ are all the same color under χ. Letting $x_j = i_{j+1} - i_j$ for $j = 1, 2, \ldots, k-1$ and $x_k = i_k - i_1$ finishes the proof. This is a generalization of the Schur numbers, which are the special case $k = 3$. (Note that these definitions do not agree with those found in [4].) An alternative method of showing that $S(k;r)$ exists would be to provide an upper bound for it. Beutelspacher and Brestovansky [4] showed that $S(k;2) = k^2 - k - 1$ thereby providing the independent existence of $S(k;r)$ for $r = 2$.

In this article we change the monochromatic property to a zero-sum property.

Definition 1.1. Let $a_1, a_2, \ldots, a_n$ be a sequence of non-negative integers and let $m \in \mathbb{Z}^+$. We say that the sequence is *m-zero-sum* if $\sum_{i=1}^{n} a_i \equiv 0 \pmod{m}$.

The foundational zero-sum result is the Erdős-Ginzberg-Ziv theorem [12], which states that any sequence of $2n-1$ integers contains an n-zero-sum subsequence of n integers. Since around 1990, research activity concerning zero-sum results has flourished, through both the lens of additive number theory and Ramsey theory. An important extension of the Erdős-Ginzberg-Ziv theorem is the weighted Erdős-Ginzberg-Ziv theorem

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due to Grynkiewicz [14]. It allows us to multiply the integers in the Erdős-Ginzberg-Ziv theorem by weights; in particular, if $w_1, w_2, \ldots, w_n$ is an n-zero-sum sequence and $a_1, a_2, \ldots, a_{2n-1}$ is a sequence of $2n-1$ integers, then there exists an n-term subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ and a permutation π of $\{i_1, i_2, \ldots, i_n\}$ such that $\sum_{j=1}^{n} w_j a_{\pi(i_j)} \equiv 0 \pmod{n}$. Further recent results can be found in [1], [3], and [13] among many others.

Most investigations of zero-sum sequences do not have a structure imposed on them. This is in contrast to zero-sum results on edgewise colored graphs, which have been around for many years (see, e.g., [2], [5], [8], and [11]). Some notable exceptions are found in works of Bialostocki, such as [7] and [9] where the zero-sum sequence $x_1, x_2, \ldots, x_n$ satisfies $\sum_{i=1}^{n-1} x_i < x_n$ and in [6] where $x_{i+1} - x_i \leq x_i - x_{i-1}$ for $1 \leq i \leq n-1$. These exceptions, however, do not have a rigid structure imposed on them due to the use of inequality. Very recently, a rigid structure similar to what we are investigating in this article was investigated in [10]. This article continues investigation of zero-sum sequences with an imposed rigid structure.

Throughout the paper we let $\mathcal{E}$ represent the equation $\sum_{i=1}^{k-1} x_i = x_k$.

Definition 1.2. Let $k, r \in \mathbb{Z}^+$ with $r \mid k$. We denote by $S_{\mathfrak{z}}(k;r)$ the minimum integer such that every coloring of $[1, S_{\mathfrak{z}}(k;r)]$ with the colors $0, 1, \ldots, r-1$ admits an r-zero-sum solution to $\mathcal{E}$. We denote by $S_{\mathfrak{z},2}(k;r)$ the minimum integer such that every coloring of $[1, S_{\mathfrak{z},2}(k;r)]$ with the colors 0 and 1 admits an r-zero-sum solution to $\mathcal{E}$.

The above definition assumes the existence of the respective minimum numbers. Existence follows directly from the existence of the generalized Schur numbers $S(k;r)$. Note that we need only prove the existence of $S_{\mathfrak{z}}(k;r)$ since we easily have $S_{\mathfrak{z},2}(k;r) \leq S_{\mathfrak{z}}(k;r)$ as $\mathbb{Z}_2 \subseteq \mathbb{Z}_r$. The existence of $S_{\mathfrak{z}}(k;r)$ comes from $S_{\mathfrak{z}}(k;r) \leq S(k;r)$ as any monochromatic solution to $\mathcal{E}$ is r-zero-sum when $r \mid k$. When $r \nmid k$, coloring every integer of $\mathbb{Z}^+$ with the color 1 does not admit a k-term r-zero-sum solution to $\mathcal{E}$ and so we write $S_{\mathfrak{z}}(k;r) = S_{\mathfrak{z},2}(k;r) = \infty$ in this situation.

2. Some Calculations

The author wrote the fortran programs `ZSGS.f` and `ZSGS2.f`, available at `www.aaronrobertson.org`, to determine the numbers $S_{\mathfrak{z}}(k;r)$ and $S_{\mathfrak{z},2}(k;r)$, respectively, for small values of k and r. In addition to a standard backtrack algorithm for traversing colorings, we must have a quick subroutine to determine solutions to $\sum_{i=1}^{k-1} x_i = x_k$ since checking $\approx n^k$ possible arrays $(x_1, x_2, \ldots, x_k)$ on $[1, n]$ will quickly become problematic. To this end, in Algorithm 1, below, we give the pseudocode for our recursive subroutine. In the code, we assume $x_1 \leq x_2 \leq \cdots \leq x_{k-1}$.

```
inputs : t, k;
output: Set S of solutions to Σ_{i=1}^{k-1} x_i = t;
Let n = k − 1;
Let R be a partial solution, initialized to the array (−, −, . . . , −, t) of length k;
Let S be the set of solutions, initialized empty;
call solutions(t, n, S, R);
return;

Subroutine solutions(t, n, S, R)
    if t ≤ 0 then
        return;
    end
    if n = 1 then
        We have found a solution (R is complete) so we add R to S and return;
    else
        for i from ⌈t/n⌉ to R_{n+1} do
            R_n = i;
            call solutions(t − i, n − 1, S, R);
        end
    end
return S;
```

Algorithm 1: Solutions to $\sum_{i=1}^{k-1} x_i = t$

Using this algorithm along with standard backtracking, we calculated the following values.

$k \diagdown r$	2	3	4	5
2	1	∞	∞	∞
3	∞	10	∞	∞
4	5	∞	13	∞
5	∞	∞	∞	38
6	9	15	∞	∞
7	∞	∞	∞	∞
8	13	∞	27	∞
9	∞	24	∞	∞
10	17	∞	∞	≥ 45
11	∞	∞	∞	∞
12	21	33	43	∞

Table 1: Values and a lower bound for $S_{\mathfrak{z}}(k;r)$ for small k and r.
The lower bound was the best one achieved after 28 days of computing time.

$k \diagdown r$	2	3	4	5
2	1	∞	∞	∞
3	∞	5	∞	∞
4	5	∞	11	∞
5	∞	∞	∞	19
6	9	13	∞	∞
7	∞	∞	∞	∞
8	13	∞	25	∞
9	∞	22	∞	∞
10	17	∞	∞	41
11	∞	∞	∞	∞
12	21	31	41	∞

Table 2: Values for $S_{\mathfrak{z},2}(k;r)$ for small k and r

3. Some Formulas and Bounds

Proposition 3.1. *Let k be an even positive integer. Then $S_{\mathfrak{z}}(k;2) = S_{\mathfrak{z},2}(k;2) = 2k-3$.*

Proof. The fact that $S_{\mathfrak{z}}(k;2) = S_{\mathfrak{z},2}(k;2)$ is by definition so we need only show that $S_{\mathfrak{z}}(k;2) = 2k-3$. The formula obviously holds for $k=2$ so we may assume that $k \geq 4$. To see that $S_{\mathfrak{z}}(k;2) \geq 2k-3$ consider the 2-coloring of $[1, 2k-4]$ defined by coloring every integer in $[1, k-2]$ with color 0 and every integer in $[k-1, 2k-4]$ with color 1. (In the sequel, we will describe this coloring by $0^{k-2}1^{k-2}$.) If $x_1, x_2, \ldots, x_{k-1}$ are all of color 0, then x_k must be of color 1 since $\sum_{i=1}^{k-1} x_i \geq k-1$. Assuming that $x_1 \leq x_2 \leq \cdots \leq x_{k-1}$, we can assume that x_{k-1} has color 1. But then $\sum_{i=1}^{k-1} x_i \geq 2k-3$, a contradiction.

Moving on to the upper bound, consider an arbitrary coloring $\chi : [1, 2k-3] \to \{0,1\}$. Assume for a contradiction that χ does not admit a 2-zero-sum solution to $\mathcal{E}$. We may assume $\chi(1) = 0$ since χ admits a 2-zero-sum solution if and only if $\widehat{\chi}$ defined by $\widehat{\chi}(i) = 1 - \chi(i)$ does. Using $\chi(1) = 0$ we deduce that $\chi(k-1) = 1$. Considering the solution $(x_1, x_2, \ldots, x_k) = (1, 1, \ldots, 1, k-1, 2k-3)$ we conclude that $\chi(2k-3) = 0$.

We now find that the solution $(x_1, x_2, \ldots, x_k) = (1, 2, 2, \ldots, 2, 2k-3)$ has $\sum_{i=1}^{k} \chi(x_i) \equiv 0 \pmod 2$ regardless of the color of 2 (since k is even so that we have an even number of occurrences of 2), a contradiction. $\square$

Theorem 3.1. *Let $k \in \mathbb{Z}^+$ with $3 \mid k$. Then $S_{\mathfrak{z}}(k;3) \geq 3k-3$.*

Proof. We prove this by giving a coloring $\chi : [1, 3k-4] \to \{0,1,2\}$ that avoids 3-zero-sum solutions to $\mathcal{E}$. To this end, define χ by

$$(012)^{\frac{k}{3}-1}(011)^{\frac{k}{3}}(021)^{\frac{k}{3}-1}02.$$

We will show that no solution to $\mathcal{E}$ is 3-zero-sum under χ. We assume that $x_1 \leq x_2 \leq \cdots \leq x_{k-1}$ and will use the notation

$$T(m) = \sum_{i=1}^{m} x_i \qquad \text{and} \qquad C(m) = \sum_{i=1}^{m} \chi(x_i).$$

For an arbitrary solution to $\mathcal{E}$ given by $(x_1, x_2, \ldots, x_k)$, we let A_j be the set of x_i of color j, for $j = 0, 1, 2$ restricted to $i \leq k - 1$.

`Case I.` $x_{k-1} \leq k - 1$. Since $x_{k-1} \leq k - 1$, we see that, for $1 \leq i \leq k - 1$, in this case we have:

(1) $\chi(x_i) = 0$ if and only if $x_i \equiv 1 \pmod 3$;

(2) $\chi(x_i) = 1$ if and only if $x_i \equiv 2 \pmod 3$; and

(3) $\chi(x_i) = 2$ if and only if $x_i \equiv 0 \pmod 3$.

`Subcase i.` $T(k-1) \equiv 0 \pmod 3$. We must have $\chi(x_k) = 1$ since $x_k \equiv 0 \pmod 3$ and $x_k \geq k - 1$. We will show that $C(k-1) \not\equiv 2 \pmod 3$ thereby showing that there is no 3-zero-sum solution to $\mathcal{E}$ in this subcase. Assume, for a contradiction, that there exists a solution with $C(k-1) \equiv 2 \pmod 3$.

We know that $T(k-1) \equiv |A_0| + 2|A_1| \equiv (k - 1 - |A_1| - |A_2|) + 2|A_1| \equiv |A_1| - |A_2| + k - 1 \equiv |A_1| - |A_2| + 2 \pmod 3$. Hence, $|A_2| \equiv |A_1| + 2 \pmod 3$. Using this, we have $C(k-1) \equiv |A_1| + 2|A_2| \equiv 3|A_1| + 4 \equiv 1 \pmod 3$, contradicting our assumption that $C(k-1) \equiv 2 \pmod 3$. †

`Subcase ii.` $T(k-1) \equiv 1 \pmod 3$. We must have $\chi(x_k) = 0$ since $x_k \equiv 1 \pmod 3$. We will show that we do not have $C(k-1) \equiv 0 \pmod 3$. Following the argument in Subcase i, we have $T(k-1) \equiv |A_1| - |A_2| + 2 \pmod 3$ so that $|A_2| \equiv |A_1| + 1 \pmod 3$. Then we have $C(k-1) \equiv |A_1| + 2|A_2| \equiv 3|A_1| + 2$ and we conclude that $C(k-1) \equiv 2 \pmod 3$ so that our solution is not 3-zero-sum. †

`Subcase iii.` $T(k-1) \equiv 2 \pmod 3$. We must have $\chi(x_k) \neq 0$ since $x_k \equiv 2 \pmod 3$. Following the argument in Subcase i, we have $T(k-1) \equiv |A_1| - |A_2| + 2 \pmod 3$ so that $|A_2| \equiv |A_1| \pmod 3$. Then we have $C(k-1) \equiv |A_1| + 2|A_2| \equiv 3|A_1|$ and we conclude that $C(k-1) \equiv 0 \pmod 3$. Since $\chi(x_k) \not\equiv 0 \pmod 3$, our solution is not 3-zero-sum. †

This completes Case I. ◇

`Case II.` $x_{k-1} \geq k$. In order to have $x_k \leq 3k - 4$ we must have $x_{k-2} \leq k - 1$. Since $x_{k-2} \leq k-1$, we can use the arguments in Case I by considering A_j restricted to $i \leq k-2$. To this end, let $B_j = A_j \setminus \{x_{k-1}\}$ for $j = 0, 1, 2$.

`Subcase i.` $T(k-2) \equiv 0 \pmod 3$. We must have $x_{k-1} \equiv x_k \pmod 3$ in this subcase. Hence, we know that $\chi(x_{k-1}) + \chi(x_k) \not\equiv 1 \pmod 3$ (we cannot have $\chi(x_{k-1}) = 2$ since this gives $x_k > 3k-4$, which is out of bounds). We will show that we must have $C(k-2) \equiv 2 \pmod 3$ so that we cannot have $C(k) \equiv 0 \pmod 3$. Using the arguments in Case I we can conclude that $|B_2| \equiv |B_1| + 1 \pmod 3$. From here we deduce that $C(k-2) \equiv 2 \pmod 3$, so that $C(k) \not\equiv 0 \pmod 3$, and we are done with this subcase. †

`Subcase ii.` $T(k-2) \equiv 1 \pmod 3$. In this situation we must have $1 + x_{k-1} \equiv x_k \pmod 3$. Looking at χ we see that $\chi(x_{k-1}) + \chi(x_k) \not\equiv 0 \pmod 3$. We will show that $C(k-2) \equiv 0 \pmod 3$ so that χ does not contain a 3-zero-sum solution to $\mathcal{E}$ in this subcase. Using the arguments in Case I we conclude that $T(k-2) \equiv |B_1| - |B_2| + 1 \pmod 3$ so that $|B_1| \equiv |B_2| \pmod 3$. This gives us $C(k-2) \equiv 3|B_1| \equiv 0 \pmod 3$, finishing this subcase. †

`Subcase iii.` $T(k-2) \equiv 2 \pmod 3$. In this situation we must have $\chi(x_{k-1}) + \chi(x_k) \not\equiv 2 \pmod 3$. We will show that $C(k-2) \equiv 1 \pmod 3$ so that χ does not contain a 3-zero-sum solution to $\mathcal{E}$ in this subcase. Using the arguments in Case I we conclude that $T(k-2) \equiv |B_1| - |B_2| + 1 \pmod 3$ so that $|B_2| \equiv |B_1| + 2 \pmod 3$. This gives us $C(k-2) \equiv 3|B_1| + 4 \equiv 1 \pmod 3$, finishing this subcase. †

This concludes the proof of Case II. $\diamond$

Having exhausted all possibilities, the proof is complete. $\square$

When we restrict the number of colors to just two, we can provide a formula for the associated number.

Theorem 3.2. *Let* $k \in \mathbb{Z}^+$ *with* $3 \mid k$. *Then* $S_{\mathfrak{z},2}(k;3) = 3k-5$.

Proof. To see that $S_{\mathfrak{z},2}(k;3) > 3k-6$ consider the coloring $0^{k-2}1^{2k-4}$. In any solution to $\mathcal{E}$ we must have at least one integer of color 1. In order to be 3-zero-sum we must then have at least 3 integers of color 1. But then $\sum_{i=1}^{k-1} x_i \geq 1(k-3) + 2(k-1) = 3k-5 > 3k-6$ so we cannot have a solution with more than 2 integers of color 1.

To show that $S_{\mathfrak{z},2}(k;3) \leq 3k-5$, assume, for a contradiction, that $\chi : [1, 3k-5] \to \{0,1\}$ does not admit a 3-zero-sum solution to $\mathcal{E}$. We may assume that $\chi(1) = 0$ by considering $\widehat{\chi}(i) = 1 - \chi(i)$ and noticing that a solution is 3-zero-sum under χ if and only if the solution is 3-zero-sum under $\widehat{\chi}$ (by the divisibility property of k). Considering the solution $(1,1,\ldots,1,k-1)$ we must have $\chi(k-1) = 1$.

`Case I.` $\chi(2) = 0$. Since $\chi(k-1) = 1$, from $(1,1,\ldots,1,k-1,k-1,3k-5)$ we see that $\chi(3k-5) = 0$. In turn, since $1+3+3+\cdots+3 = 3k-5$ we see that $\chi(3) = 1$. Finally, consider $(2,2,3,3,\ldots,3,3k-5)$. The sum of the colors for this solution is $(k-3)$, which is congruent to 0 modulo 3 since $3 \mid k$, a contradiction. †

`Case II.` $\chi(2) = 1$. Since $2+2+\cdots+2 = 2k-2$, we have $\chi(2k-2) = 0$. We also have $\chi(3k-5) = 0$ by considering the solution $(2,2,\ldots,2,k-1,3k-5)$. In turn, since $1+3+3+\cdots+3 = 3k-5$ we have $\chi(3) = 1$. Now, for $0 \leq i \leq k-3$, by considering the solution $(2,2,\ldots,2,3,3,2k)$ we have $\chi(2k) = 0$. Next, consider $(1,1,\ldots,1,k,2k-2)$ to see that $\chi(k) = 1$. But now $(1,1,\ldots,1,2,2,k,2k)$ is a 3-zero-sum solution to $\mathcal{E}$, a contradiction. †

As the two cases cover all situations, the proof is complete. $\square$

Theorem 3.3. *Let* $k \in \mathbb{Z}^+$ *with* $4 \mid k$. *Then* $S_{\mathfrak{z}}(k;4) \geq 4k-5$.

Proof. We prove this by giving a coloring $\chi : [1, 4k-6] \to \{0,1,2,3\}$ that avoids 4-zero-sum solutions to $\mathcal{E}$. To this end, define χ by

$$(0123)^{\frac{k}{4}-1}(0120)(0220)^{\frac{k}{2}-1}(3210)^{\frac{k}{4}-1}32.$$

We will show that no solution to $\mathcal{E}$ is 4-zero-sum under χ. We assume that $x_1 \leq x_2 \leq \cdots \leq x_{k-1}$ and will again use the notation

$$T(m) = \sum_{i=1}^{m} x_i \qquad \text{and} \qquad C(m) = \sum_{i=1}^{m} \chi(x_i).$$

For an arbitrary solution to $\mathcal{E}$ given by $(x_1, x_2, \ldots, x_k)$, we let A_j be the set of x_i of color j, for $j = 0, 1, 2, 3$ restricted to $i \leq k-1$.

`Case I.` $x_{k-1} \leq k-1$. Since $x_{k-1} \leq k-1$, we see that, for $1 \leq i \leq k-1$, in this case we have:

(1) $\chi(x_i) = 0$ if and only if $x_i \equiv 1 \pmod 4$;

(2) $\chi(x_i) = 1$ if and only if $x_i \equiv 2 \pmod 4$;

(3) $\chi(x_i) = 2$ if and only if $x_i \equiv 3 \pmod 4$; and

(4) $\chi(x_i) = 3$ if and only if $x_i \equiv 0 \pmod 4$.

`Subcase i.` $T(k-1) \equiv 0 \pmod 4$. We must have $\chi(x_k) = 0$ since $x_k \equiv 0 \pmod 4$. We will show that $C(k-1) \not\equiv 0 \pmod 4$ thereby showing that there is no 4-zero-sum solution to $\mathcal{E}$ in this subcase.

We have $T(k-1) \equiv |A_0|+2|A_1|+3|A_2| \equiv (k-1-|A_1|-|A_2|-|A_3|)+2|A_1|+3|A_2| \equiv |A_1| + 2|A_2| - |A_3| + 3 \pmod 4$. Since we have $T(k-1) \equiv 0 \pmod 4$ in this subcase, we conclude that $|A_3| \equiv |A_1| + 2|A_2| + 3 \pmod 4$. Using this, we have $C(k-1) \equiv |A_1| + 2|A_2| + 3|A_3| \equiv 4|A_1| + 8|A_2| + 9 \equiv 1 \pmod 4$. Hence, $C(k) \equiv 1 \pmod 4$ so that there is no 4-zero-sum solution in this subcase. †

`Subcase ii.` $T(k-1) \equiv 1 \pmod 4$. We must have $\chi(x_k) = 0$ or 3 since $x_k \equiv 1 \pmod 4$. We will show that $C(k-1) \equiv 2 \pmod 4$ so that we know $C(k) \not\equiv 0 \pmod 4$, and hence we do not have a 4-zero-sum solution. Following the argument in Subcase i, we have $|A_3| \equiv |A_1| + 2|A_2| + 2 \pmod 4$. Then we have $C(k-1) \equiv |A_1| + 2|A_2| + 3|A_3| \equiv 4|A_1| + 8|A_2| + 6 \equiv 2 \pmod 4$ and we conclude that $C(k-1) \equiv 2 \pmod 4$ so that $C(k) \equiv 1$ or $2 \pmod 4$ and our solution is not 4-zero-sum. †

`Subcase iii.` $T(k-1) \equiv 2 \pmod 4$. We must have $\chi(x_k) = 2$ since $x_k \equiv 2 \pmod 4$ and we cannot have $\chi(x_k) = 1$ since this means $x_k = k-2$, which is not possible. Following the argument in Subcase i, we have $|A_3| \equiv |A_1| + 2|A_2| + 1 \pmod 4$. Hence, $C(k-1) \equiv |A_1| + 2|A_2| + 3|A_3| \equiv 4|A_1| + 8|A_2| + 3 \equiv 3 \pmod 4$, so that $C(k) \equiv 1 \pmod 4$ and our solution is not 4-zero-sum. †

`Subcase iv.` $T(k-1) \equiv 3 \pmod 4$. We must have $\chi(x_k) = 1$ or 2 since $x_k \equiv 3 \pmod 4$. Following the argument in Subcase i, we have $|A_3| \equiv |A_1| + 2|A_2| \pmod 4$. Hence, $C(k-1) \equiv |A_1| + 2|A_2| + 3|A_3| \equiv 4|A_1| + 8|A_2| \equiv 0 \pmod 4$. This gives us that $C(k) \equiv 1$ or $2 \pmod 4$ so that our solution is not 4-zero-sum. †

This completes Case I. ◇

`Case II.` $x_{k-2} \leq k-1$ and $x_{k-1} \geq k$. Since $x_{k-2} \leq k-1$, we can use the arguments in Case I by considering A_j restricted to $i \leq k-2$. Thus, we let $B_j = A_j \setminus \{x_{k-1}\}$ for $j = 0, 1, 2, 3$.

`Subcase i.` $T(k-2) \equiv 0 \pmod 4$. We must have $x_{k-1} \equiv x_k \pmod 4$. Looking at χ, we see that $\chi(x_{k-1}) + \chi(x_k) \in \{0, 3\}$. We will show that $C(k-2) \not\equiv 0, 1 \pmod 4$ thereby showing that there is no 4-zero-sum solution to $\mathcal{E}$ in this subcase.

We have $T(k-2) \equiv |B_0|+2|B_1|+3|B_2| \equiv (k-2-|B_1|-|B_2|-|B_3|)+2|B_1|+3|B_2| \equiv |B_1| + 2|B_2| - |B_3| + 2 \pmod 4$. Since we have $T(k-2) \equiv 0 \pmod 4$ in this subcase, we conclude that $|B_3| \equiv |B_1| + 2|B_2| + 2 \pmod 4$. Using this, we have $C(k-2) \equiv |B_1| + 2|B_2| + 3|B_3| \equiv 4|B_1| + 8|B_2| + 6 \equiv 2 \pmod 4$, hence $C(k-2) \not\equiv 0, 1 \pmod 4$ and this subcase is complete. †

`Subcase ii.` $T(k-2) \equiv 1 \pmod 4$. We must have $x_k \equiv x_{k-1} + 1 \pmod 4$. From this we conclude that $\chi(x_{k-1}) + \chi(x_k) \in \{0, 2, 3\}$. We will show that $C(k-2) \equiv 3 \pmod 4$ so that we know $C(k) \not\equiv 0 \pmod 4$, and hence we do not have a 4-zero-sum solution. Following the argument in Subcase i, we have $|B_3| \equiv |B_1| + 2|B_2| + 1 \pmod 4$. Then we have $C(k-2) \equiv |B_1| + 2|B_2| + 3|B_3| \equiv 4|B_1| + 8|B_2| + 3 \equiv 3 \pmod 4$ and we conclude that $C(k-2) \equiv 3 \pmod 4$ so that $C(k) \equiv 1$ or $2 \pmod 4$ and our solution is not 4-zero-sum. †

`Subcase iii.` $T(k-2) \equiv 2 \pmod 4$. Looking at our coloring, we see that we can only have $\chi(x_{k-1}) + \chi(x_k) = 0$ if $x_{k-1} \geq 3k-3$. But then $T(k-1) > 4k-6$, which is out of bound. Thus, we must have $x_k \equiv x_{k-1} + 2 \pmod 4$. From this we conclude that $\chi(x_{k-1}) + \chi(x_k) \in \{1, 2\}$. Following the argument in Subcase i, we have $|B_3| \equiv |B_1| + 2|B_2| \pmod 4$. Then we have $C(k-2) \equiv |B_1| + 2|B_2| + 3|B_3| \equiv 4|B_1| + 8|B_2| \equiv 0 \pmod 4$ and we conclude that $C(k) \equiv 1$ or $2 \pmod 4$ so that our solution is not 4-zero-sum. †

`Subcase iv.` $T(k-1) \equiv 3 \pmod 4$. We must have $x_k \equiv x_{k-1} + 3 \pmod 4$. As in Subcase iii directly above, we cannot have $\chi(x_{k-1}) + \chi(x_k) = 3$ since that would imply that $T(k-1) > 4k-6$. Hence, we see that $\chi(x_{k-1}) + \chi(x_k) \in \{0, 1, 2\}$ Following the argument in Subcase i, we have $|B_3| \equiv |B_1| + 2|B_2| + 3 \pmod 4$. Then we have $C(k-2) \equiv |B_1| + 2|B_2| + 3|B_3| \equiv 4|B_1| + 8|B_2| + 9 \equiv 1 \pmod 4$. But then $C(k) \equiv 1, 2,$ or $3 \pmod 4$ so that $C(k) \equiv 0 \pmod 4$ is not possible.

This completes Case II. ◇

`Case III.` $x_{k-2} \geq k$. We must have $x_{k-3} \leq k-1$ for otherwise $T(k-1) \geq 4k-4$. Also, we have $x_k \geq 3k-3$ so there is a one-to-one correspondence between x_k and $\chi(x_k)$. Since $x_{k-3} \leq k-1$, we can use the arguments in Case I by considering A_j restricted to $i \leq k-3$. To this end, let $D_j = A_j \setminus \{x_{k-2}, x_{k-1}\}$ for $j = 0, 1, 2, 3$. Note that, under χ, the only possible colors of x_{k-2} and x_{k-1} are 0 and 2.

`Subcase i.` $T(k-3) \equiv 0 \pmod 4$. We have $T(k-3) \equiv |D_0| + 2|D_1| + 3|D_2| \equiv (k-3-|D_1|-|D_2|-|D_3|) + 2|D_1| + 3|D_2| \equiv |D_1| + 2|D_2| - |D_3| + 1 \pmod 4$. Since we have $T(k-3) \equiv 0 \pmod 4$ in this subcase, we conclude that $|D_3| \equiv |D_1| + 2|D_2| + 1 \pmod 4$. Using this, we have $C(k-3) \equiv |D_1| + 2|D_2| + 3|D_3| \equiv 4|D_1| + 8|D_2| + 3 \equiv 3 \pmod 4$.

`Subsubcase a.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 0$. We have $x_{k-2}, x_{k-1} \equiv 0$ or $1 \pmod 4$ so that $T(k-1) \equiv 0, 1,$ or $2 \pmod 4$. We also know that $C(k-1) \equiv 3 \pmod 4$ so we need $\chi(x_k) = 1$ in order to have a 4-zero-sum solution. But the only possible integers of color 1 are congruent to 3 modulo 4, which is not possible since $T(k-1) \not\equiv 3 \pmod 4$.

`Subsubcase b.` $\chi(x_{k-2}) = 0$ and $\chi(x_{k-1}) = 2$ or $\chi(x_{k-2}) = 2$ and $\chi(x_{k-1}) = 0$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 3$, which yields $x_k \equiv 1 \pmod 4$. This means we need $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ since we have $T(k-3) \equiv 0 \pmod 4$ in this subcase. But we know that one of x_{k-2} and x_{k-1} has color 0, and so is congruent to 0 or 1 modulo 4, and the other has color 2, and so is congruent to 2 or 3 modulo 4. Hence, we cannot have $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ and, consequently, no 4-zero-sum solution to $\mathcal{E}$ exists.

`Subsubcase c.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 2$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 1$, which yields $x_k \equiv 3 \pmod 4$. We know that x_{k-2} and x_{k-1} are congruent to either 2 or 3 modulo 4 so that $x_{k-2} + x_{k-1} \equiv 0, 1$, or $2 \pmod 4$. But then $T(k-1) \not\equiv 3 \equiv x_k \pmod 4$. Hence, $C(k) \not\equiv 0 \pmod 4$.

This completes Subcase i. †

`Subcase ii.` $T(k-3) \equiv 1 \pmod 4$. Following the argument in Case III.i we conclude that $|D_3| \equiv |D_1| + 2|D_2| \pmod 4$ so that $C(k-3) \equiv |D_1| + 2|D_2| + 3|D_3| \equiv 4|A_1| + 8|A_2| \equiv 0 \pmod 4$.

`Subsubcase a.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 0$. We have $x_{k-2}, x_{k-1} \equiv 0$ or $1 \pmod 4$ so that $T(k-1) \equiv 1, 2$, or $3 \pmod 4$. We also know that $C(k-1) \equiv 0 \pmod 4$ so we need $\chi(x_k) = 0$ in order to have a 4-zero-sum solution. But the only possible integers of color 0 are congruent to 0 modulo 4, which is not possible since $T(k-1) \not\equiv 0 \pmod 4$.

`Subsubcase b.` $\chi(x_{k-2}) = 0$ and $\chi(x_{k-1}) = 2$ or $\chi(x_{k-2}) = 2$ and $\chi(x_{k-1}) = 0$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 2$, which yields $x_k \equiv 2 \pmod 4$. This means we need $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ since we have $T(k-3) \equiv 1 \pmod 4$ in this subcase. But we know that one of x_{k-2} and x_{k-1} has color 0, and so is congruent to 0 or 1 modulo 4, and the other has color 2, and so is congruent to 2 or 3 modulo 4. Hence, we cannot have $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ and, consequently, no 4-zero-sum solution to $\mathcal{E}$ exists.

`Subsubcase c.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 2$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 0$, which yields $x_k \equiv 0 \pmod 4$. We know that x_{k-2} and x_{k-1} are congruent to either 2 or 3 modulo 4 so that $x_{k-2} + x_{k-1} \equiv 0, 1$, or $2 \pmod 4$. But then $T(k-1) \equiv 1, 2$ or $3 \pmod 4$ so that $T(k-1) \not\equiv x_k \pmod 4$. Hence, $C(k) \not\equiv 0 \pmod 4$.

This completes Subcase ii. †

`Subcase iii.` $T(k-3) \equiv 2 \pmod 4$. Following the argument in Case III.i we conclude that $|D_3| \equiv |D_1| + 2|D_2| + 3 \pmod 4$. Hence, $C(k-3) \equiv |D_1| + 2|D_2| + 3|D_3| \equiv 4|A_1| + 8|A_2| + 9 \equiv 1 \pmod 4$.

`Subsubcase a.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 0$. We have $x_{k-2}, x_{k-1} \equiv 0$ or $1 \pmod 4$ so that $T(k-1) \equiv 0, 2$, or $3 \pmod 4$. We also know that $C(k-1) \equiv 1 \pmod 4$ so we need $\chi(x_k) = 3$ in order to be 4-zero-sum. But the only possible integers of color 3 are congruent to 1 modulo 4, which is not possible since $T(k-1) \not\equiv 1 \pmod 4$.

`Subsubcase b.` $\chi(x_{k-2}) = 0$ and $\chi(x_{k-1}) = 2$ or $\chi(x_{k-2}) = 2$ and $\chi(x_{k-1}) = 0$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 1$, so that $x_k \equiv 3 \pmod 4$. This means we need $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ since we have $T(k-3) \equiv 2 \pmod 4$ in this subcase. As in Case III.ii.b, we cannot have $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ and, consequently, no 4-zero-sum solution to $\mathcal{E}$ exists.

`Subsubcase c.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 2$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 3$, so that $x_k \equiv 1 \pmod 4$. We know that x_{k-2} and x_{k-1} are congruent to either 2 or 3 modulo 4 so that $x_{k-2} + x_{k-1} \equiv 0, 1$, or $2 \pmod 4$. But then $T(k-1) \equiv 0, 2$

or 3 $\pmod 4$ so that $T(k-1) \not\equiv x_k \pmod 4$. Hence, $C(k) \not\equiv 0 \pmod 4$.

This completes Subcase iii. †

`Subcase iv.` $T(k-3) \equiv 3 \pmod 4$. Following the argument in Case III.i we conclude that $|D_3| \equiv |D_1| + 2|D_2| + 2 \pmod 4$ so that $C(k-3) \equiv |D_1| + 2|D_2| + 3|D_3| \equiv 4|A_1| + 8|A_2| + 6 \equiv 2 \pmod 4$.

`Subsubcase a.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 0$. We have $x_{k-2}, x_{k-1} \equiv 0$ or $1 \pmod 4$ so that $T(k-1) \equiv 0, 1,$ or $3 \pmod 4$. We also know that $C(k-1) \equiv 2 \pmod 4$ so we need $\chi(x_k) = 2$ in order to be 4-zero-sum. But the only possible integers of color 2 are congruent to 2 modulo 4, which is not possible since $T(k-1) \not\equiv 2 \pmod 4$.

`Subsubcase b.` $\chi(x_{k-2}) = 0$ and $\chi(x_{k-1}) = 2$, or $\chi(x_{k-2}) = 2$ and $\chi(x_{k-1}) = 0$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 0$, which yields $x_k \equiv 0 \pmod 4$. This means we need $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ since we have $T(k-3) \equiv 3 \pmod 4$ in this case. As in Case III.ii.b, we cannot have $x_{k-2} + x_{k-1} \equiv 1 \pmod 4$ and, consequently, no 4-zero-sum solution to $\mathcal{E}$ exists.

`Subsubcase c.` $\chi(x_{k-2}) = \chi(x_{k-1}) = 2$. In order to have $C(k) \equiv 0 \pmod 4$ we require $\chi(x_k) = 2$, which yields $x_k \equiv 2 \pmod 4$. As in Case III.ii.c, we have $x_{k-2} + x_{k-1} \equiv 0, 1,$ or $2 \pmod 4$. But then $T(k-1) \equiv 0, 1$ or $3 \pmod 4$ so that $T(k-1) \not\equiv x_k \pmod 4$. Hence, $C(k) \not\equiv 0 \pmod 4$.

This is the end of the proof of Subcase iv. †

This completes Case III. ⋄

Having exhausted all cases, the theorem's proof is complete. □

The last instance we investigate are those numbers along the diagonal.

Proposition 3.2. *Let k be an odd positive integer. Then $S_{\mathfrak{z}}(k;k) \geq 2(k^2-k-1)$.*

Proof. We will show that the k-coloring $(01)^{k-2}(0\,(k-1))^{(k-1)(k-2)}(01)^{k-2}0$ avoids k-zero-sum solutions to $\mathcal{E}$. It is easy to check that we cannot have a solution to $\mathcal{E}$ with all integers of color 1 or all integers of color $k-1$. Hence, in order to have a k-zero-sum solution, the number of integers colored 1 must equal the number of integers colored $k-1$. Now, k being odd implies that we have an odd number of integers of color 0. Next, we note that the only integers of color 0 are odd, while the only integers of color 1 or $k-1$ are even. By comparing the parities of $\sum_{i=1}^{k-1} x_i$ and x_k, this cannot occur. Hence, we cannot have a k-zero-sum solution to $\mathcal{E}$ under this coloring. □

If we restrict to just two colors, then the associated number is the same as $S(k;2)$ since any k-zero-sum solution to $\mathcal{E}$ must necessarily be monochromatic. In other word, for $k \in \mathbb{Z}^+$ we have $S_{\mathfrak{z},2}(k;k) = k^2 - k - 1$.

4. Conclusion and Open Questions

The area of inquiry of zero-sum sequences with rigid structure is ripe for future research. What can be said about zero-sum Rado numbers in addition to what is found in [10]? What can we say about zero-sum sequences $x_1 < x_2 < \cdots < x_k$ with $x_{i+1} - x_i$ from a prescribed set (say the (shifted) primes, powers of 2, etc)?

For specific questions related to this article, we ask the following.

Q1. Is it true that $S_{\mathfrak{z}}(k;3) = 3k - 3$ for $k \geq 6$?

Q2. Prove or disprove: $S_{\mathfrak{z}}(k;4) = 4k - 5$ for $k \geq 8$.

Q3. What is the exact value of $S_{\mathfrak{z},2}(k;4)$?

Q4. Is it true that $S_{\mathfrak{z}}(k;k)$ is of order k^2?

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