ON THE DEGREE OF REGULARITY OF GENERALIZED VAN DER WAERDEN TRIPLES

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Abstract:

Let $1 \leq a \leq b$ be integers. A triple of the form $(x, ax + d, bx + 2d)$, where x, d are positive integers is called an *(a,b)-triple*. The *degree of regularity* of the family of all (a, b)-triples, denoted $\text{dor}(a, b)$, is the maximum integer r such that every r-coloring of $\mathbb{N}$ admits a monochromatic (a, b)-triple. We settle, in the affirmative, the conjecture that $\text{dor}(a, b) < \infty$ for all $(a, b) \neq (1, 1)$. We also disprove the conjecture that $\text{dor}(a, b) \in \{1, 2, \infty\}$ for all (a, b).

1. Introduction

B.L. van der Waerden [5] proved that for any positive integers k and r, there is a positive integer $w(k, r)$ such that any r-coloring of $\{1, 2, ..., w(k, r)\}$ must admit a monochromatic k-term arithmetic progression. In [3], a generalization of van der Waerden's theorem for 3-term arithmetic progressions was investigated. Namely, for integers $1 \leq a \leq b$, define an (a, b)*-triple* to be any 3-term sequence of the form $(x, ax+d, bx+2d)$, where x, d are positive integers. Taking $a = b = 1$ gives a 3-term arithmetic progression, and by van der Waerden's theorem the associated van der Waerden number $w(3, r)$ is finite for all r.

Throughout this note, we assume that a and b are integers and that $1 \leq a \leq b$. For $r \geq 1$, denote by $n = n(a, b; r)$ the least positive integer, if it exists, such that every r-coloring of $[1, n]$ admits a monochromatic (a, b)-triple. If no such n exists, we write $n(a, b; r) = \infty$. We say that (a, b) is *regular* if $n(a, b; r) < \infty$ for each $r \in \mathbb{N}$. By van der Waerden's theorem $(1, 1)$ is regular. If (a, b) is not regular, the *degree of regularity* of (a, b), denoted $\text{dor}(a, b)$, is the largest integer r such that (a, b) is r-regular.

In [3], it is shown that for a wide class of pairs $(a, b) \neq (1, 1)$, (a, b) is not regular, i.e., $\text{dor}(a, b) < \infty$, and its authors conjectured that, in fact, $(1, 1)$ is the *only* regular pair. In

Section 2 we confirm this conjecture.

Also in [3], it was shown that

$$\mathrm{dor}(a,b) = 1 \text{ if and only if } b = 2a, \tag{1}$$

and upper bounds on $\mathrm{dor}(a,b)$ are given for those pairs which are shown not to be regular. Further, those authors speculated that $\mathrm{dor}(a,b) \in \{1,2,\infty\}$ for all pairs (a,b). In Section 3 we show this conjecture to be false. We also obtain upper bounds on $\mathrm{dor}(a,b)$ for all $(a,b) \neq (1,1)$, which improve upon the results of [3], and provide an alternate proof that (1,1) is the only regular triple.

2. The Only Regular Triples are Arithmetic Progressions

In this section we give a short proof which shows that (1,1)-triples are the only regular (a,b)-triples. The proof makes use of Rado's regularity theorem (see [4]) which states, in particular, that the linear equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$ has a monochromatic solution in $\mathbb{N}$ under any finite coloring of $\mathbb{N}$ if and only if some nonempty subset of the nonzero coefficients sums to zero. It also uses the following lemma.

Lemma 1 *For all $1 \leq a \leq b$, and all $i \geq 1$,*

$$n(a,b;r) \leq n(a+i, b+2i; r),$$

and hence $dor(a,b) \geq dor(a+i, b+2i)$.

Proof. Let a,b,i be given. To prove the lemma, it suffices to show that every $(a+i, b+2i)$-triple is also an (a,b)-triple. Let $X = (x,y,z)$ be an $(a+i,b+2i)$-triple. So $y = (a+i)x + d$ and $z = (b+2i)x + 2d$ for some $d > 0$. But then X is also an (a,b) triple, since $y = ax + (ix + d)$ and $z = bx + 2(ix + d)$. □

Theorem 1 *Let $1 \leq a \leq b$. If $(a,b) \neq (1,1)$, then (a,b) is not regular.*

Proof. Since the triple $\{x, ax+d, bx+2d\}$ satisfies the equation $(2a-b)x - 2y + z = 0$, by Rado's regularity theorem an (a,b)-triple is regular only if $b - 2a \in \{-2,-1,1\}$. Hence, this leaves three cases to consider: (i) $b = 2a+1$, (ii) $b = 2a-1$, and (iii) $b = 2a-2$. In [3] it was shown that $\mathrm{dor}(1,3) \leq 3$, $\mathrm{dor}(2,3) = 2$, and $\mathrm{dor}(2,2) \leq 5$. By Lemma 1, these three facts cover Cases (i), (ii), and (iii), respectively. □

Remark 1 In Section 3 we will show that $\mathrm{dor}(2,2) \leq 4$. We see from this fact, the proof of Theorem 1, and (1), that $2 \leq \mathrm{dor}(a, 2a-2) \leq 4$ for all $a \geq 2$; that $\mathrm{dor}(a, 2a-1) = 2$ for all $a \geq 2$; and that $2 \leq \mathrm{dor}(a, 2a+1) \leq 3$ for all $a \geq 1$.

3. More on the Degree of Regularity

Using the Fortran program `AB.f`, available from the third author's website[1], we have found that $n(2,2;3) = 88$. This implies

$$\mathrm{dor}(2,2) \geq 3, \tag{2}$$

which is a counterexample to the suggestion made in [3] that $\mathrm{dor}(a,b) \in \{1,2,\infty\}$ for all (a,b). The program uses a well-known backtracking algorithm (see [4], Algorithm 2, page 31) which checks that all 3-colorings of $[1,88]$ contain a monochromatic $(2,2)$-triple.

Although (2) shows the existence of a pair besides (1,1) whose degree of regularity is greater than two, we wonder if $\mathrm{dor}(a,b) = 2$ for "almost all" (a,b). In particular, we pose the following questions.

Question 1 Is it true that $\mathrm{dor}(a,b) \leq 2$ whenever $b \neq 2a-2$ and $a \geq 2$?

Question 2 For $b \neq 2a$, are there only a finite number of pairs (a,b) such that $\mathrm{dor}(a,b) \neq 2$?

While we do not yet have the answers to these questions, we have been able to improve the upper bounds for $\mathrm{dor}(a,b)$, as established in [3], for many (a,b)-triples. These new bounds are supplied by the next two theorems. The proofs of both theorems use the following coloring.

Notation Let $c \geq 3$ be an integer and let $p = 2 - \frac{2}{c}$. Denote by γ_c the c-coloring of $\mathbb{N}$ defined by coloring, for each $k \geq 0$, the interval $[p^k, p^{k+1})$ with color $k \pmod c$.

Theorem 2 Let $a,i,c \in \mathbb{Z}$ such that $a \geq 2$ and $c \geq 5$. Define $p = 2 - \frac{2}{c}$ and let $0 \leq i \leq p^c(p^{c-1}-2)$. If $a \leq \frac{p^c}{c-1}$, then $\mathrm{dor}(a, a+i) \leq c-1$.

Proof. We use the c-coloring γ_c. Assume, for a contradiction, that $\{x, ax+d, (a+i)x+2d\}$ is a monochromatic $(a,a+i)$-triple under γ_c. Let $x \in [p^k, p^{k+1})$. Since $p < 2$ and $a \geq 2$, we have that $ax + d \in [p^{k+cj}, p^{k+cj+1})$ for some $j \in \mathbb{N}$. This gives us that $d > p^{k+cj} - ap^{k+1}$, which, in turn, gives us $(a+i)x + 2d > 2p^{k+cj} - ap^{k+1} + ip^k$. We now show that this lower bound is more that p^{k+cj+1}: By choice of a we have $a \leq p^{c-1}(2-p)$ so that $2 - \frac{a}{p^{cj}} \geq p$ for all $j \in \mathbb{N}$. This gives us $2p^{k+cj} - ap^{k+1} > p^{k+cj+1}$ which is sufficient for all $i \geq 0$.

Next, we will show that $(a+i)x + 2d < p^{k+c(j+1)}$. Since $d < ax + d < p^{k+cj+1}$ and $ix < ip^{k+1}$ it suffices to show that $2p^{k+cj+1} + ip^{k+1} < p^{k+cj+c}$. We have $i \leq p^c(p^{c-1}-2)$, which implies that $2 + \frac{i}{p^c j} < p^{c-1}$ for all $j \in \mathbb{N}$, which, in turn, implies the desired bound.

[1] `http://math.colgate.edu/~aaron/programs.html`

Hence, we have $p^{k+cj+1} < (a+i)x + 2d < p^{k+c(j+1)}$. By the definition of γ_c, we see that if x and $ax+d$ are the same color, then $(a+i)x+2d$ must be a different color under γ_c, a contradiction. □

Example By Theorem 2 and (2), $\mathrm{dor}(2,2) \in \{3,4\}$.

Theorem 3 Let $b, c \in \mathbb{N}$ such that $b \geq 2$ and $c \geq 5$. Let $p = 2 - \frac{2}{c}$. If $b < \frac{2+p^c}{p}$, then $\mathrm{dor}(1,b) \leq c-1$.

Proof. The proof is quite similar to that of Theorem 2. Assume, for a contradiction, that $\{x, x+d, bx+2d\}$ is monochromatic under γ_c. Let $x \in [p^k, p^{k+1})$ so that $bx + 2d \in [p^{k+cj}, p^{k+cj+1})$ (since $b \geq 2 > c$) for some $j \in \mathbb{N}$. This gives $d \geq \frac{1}{2}p^{k+cj} - \frac{b}{2}p^{k+1}$ so that $x + d > p^k + \frac{1}{2}p^{k+cj} - \frac{b}{2}p^{k+1}$. The condition on b implies that this last bound is larger than p^{k+1}.

We next show that $x + d < p^{k+cj}$. We have $d < \frac{1}{2}p^{k+cj+1}$ so that $x + d < p^{k+1} + \frac{1}{2}p^{k+cj+1}$. Since $2 < p^{c-1}(2-p)$ for all $c \geq 5$, we have $p^{k+1} + \frac{1}{2}p^{k+cj+1} < p^{k+cj}$ for all $j \in \mathbb{N}$. Hence, $p^{k+1} < x + d < p^{k+cj}$ so that $x + d$ is not the same color, under γ_c, as x and $bx + 2d$, a contradiction. □

Corollary 1 For $a \geq 1$ and $1 \leq j \leq 5$, $\mathrm{dor}(a, 2a+j) \leq 4$.

Proof. This follows from Theorem 3 and Lemma 1. □

Remark 2 Theorems 2 and 3, along with the following result from [3], provide an alternate proof of Theorem 1 without the use of Rado's regularity theorem.

Lemma 2 *Assume* $b \geq (2^{3/2}-1)a - 2^{3/2} + 2$. *Then* $dor(a,b) \leq \lceil 2\log_2 c \rceil$, *where* $c = \lceil b/a \rceil$.

Below we give a table showing the known bounds on the degrees of regularity for some small values of a and b. The entries in the table that improve the previously known bounds are marked with *; all others are from [3]. The improved bounds for dor(1,5), dor(1,6), dor(1,7), dor(1,8), and dor(1,9) follow from Theorem 3; the upper bound on dor(2,10) follows from Theorem 2; and the upper bounds on dor(3,4) and dor(3,7) follow from Lemma 1.

(a,b)	$\mathrm{dor}(a,b)$	(a,b)	$\mathrm{dor}(a,b)$	(a,b)	$\mathrm{dor}(a,b)$
$(1,1)$	∞	$(2,2)$	3^*-4^*	$(3,3)$	$2-5$
$(1,2)$	1	$(2,3)$	2	$(3,4)$	$2-3^*$
$(1,3)$	$2-3$	$(2,4)$	1	$(3,5)$	2
$(1,4)$	$2-4$	$(2,5)$	$2-3$	$(3,6)$	1
$(1,5)$	$2-4^*$	$(2,6)$	$2-3$	$(3,7)$	$2-3^*$
$(1,6)$	$2-4^*$	$(2,7)$	$2-4$	$(3,8)$	$2-3$
$(1,7)$	$2-4^*$	$(2,8)$	$2-4$	$(3,9)$	$2-3$
$(1,8)$	$2-5^*$	$(2,9)$	$2-4$	$(3,10)$	$2-4$
$(1,9)$	$2-5^*$	$(2,10)$	$2-4^*$	$(3,11)$	$2-4$

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