MATH 483 THESIS PROBLEMS

Fall 2018

Prof. Robertson

NOTES: Grading for your thesis is based on a departmentally decided standard. To receive an A, the thesis must be of sufficient quality to stand for high honors. To receive an A−, the thesis must be of sufficient quality to stand for honors. After each problem heading is the maximum grade possible for the chosen problem (as not all topics are of equal difficulty). Grades for the course are based equally on the thesis grade and an "engagement" grade, which is determined through the weekly meetings. All theses must be written in LaTeX. Visit `www.aaronrobertson.org` and click on Teaching to find a LaTeX skeleton file. The department has software on all computers that will compile your LaTeX file. Alternatively, there are many free programs available online that you can download to your own personal computer. *After having met for at least 4 times weekly without significant progress, you will be reassinged to one of the expository problems.*

EXPOSITORY THESIS TOPIC

Problem 1. (`max grade: B+`) Write about Google's word search algorithm and image search algorithm in mathematical detail (e.g., **how** does it do fast operations on matrices; what is the eigenvalue really calculating (hint: it's in probability theory, etc).

`Starting reference:` *Mathematics at Google*, by Javier Tordable

SEMI-EXPOSITORY THESIS TOPICS

Problem 2. (`max grade: B+`) Write about Issai Schur's attempt to prove Fermat's Last Theorem. In particular, explain his proof that for any $n \in \mathbb{Z}^+$, there exists an integer $F(n)$ such that for any prime $p \geq F(n)$ there exists a solution to $a^n + b^n \equiv c^n \pmod{p}$ with $a, b, c \in \mathbb{Z}^+$ and $abc \not\equiv 0 \pmod{p}$. Determine good lower and upper bounds for $F(n)$.

Problem 3. (`max grade: A-`) Write about the WZ-Method for evaluating hypergeometric sums (WZ stands for Wilf-Zeilberger). Use the method along with the associated Maple package, to demonstrate – in detail – the WZ-Method on some involved hypergeometric sums.

`Required Books:` (1) $A = B$ by Petkovsek, Wilf, and Zeilberger;

(2) Maple Intro: `https://math.la.asu.edu/~kuang/class/Mapletut.pdf` (free)

RESEARCH-TYPE THESIS TOPICS

Problem 4. (`max grade: B+`) Find

$$\lim_{n\to\infty} 2^{n-1} \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}_{n-1 \text{ square roots}} \, .$$

What can be said if we replace the 2 by $a > 1$? Make sure you use the real definition of a limit (from real analysis) and not just a calculus-type limit.

Problem 5. (`max grade: B+`) Consider

$$\mathcal{M} = \left\{ A = \begin{bmatrix} \frac{1}{a} & \frac{1}{a+b} \\ 1 & \frac{1}{b} \end{bmatrix} : a, b \in \mathbb{R}^+, \det(A) = 0 \right\}.$$

(a) Show that if A and B are both members of $\mathcal{M}$, then $AB = kC$ with $C \in \mathcal{M}$, for some constant $k > 0$. Does this fact show that $\mathcal{M}$ is a setoid (a.k.a., an E-set) (where we use k to define equivalence classes)?

(b) For any positive integer n, show that there exists $\mathcal{S} \subseteq \mathcal{M}$ where $\mathcal{S} = \{A_1, A_2, \ldots, A_j\}$ such that

$$\text{trace}\left(\sum_{i=1}^{j} A_i \right) = n,$$

where the sum is component-wise addition.

`Required Books:` (1) Most any linear algebra book

(2) Most any abstract algebra book

Problem 6. (`max grade: A-`) Consider the quintic polynomial $f_a(x) = x^5 - x - a$, where a is a (perhaps complex) constant. First explain how we know that there is no general algebraic (i.e., with radicals) formula for the roots of an arbitrary quintic polynomial. Next, show that any quintic polynomial can be transformed into $f_a(x)$ for some a. Explain how Langrange Inversion can be use to find a root r of $f_a(x)$ in terms of an infinite series, so that $f_a(x)$ can be written as $(x - r)g(x)$ where $g(x)$ is a quartic polynomial (which *can* be solved in radicals). Write a short Maple program that follows this method to produce the five roots (to arbitrary floating point degree of accuracy) of a general quintic polynomial (Maple's standard response can be unsatisfying).

`Required Books:` (1) Most any abstract algebra book

(2) Maple Intro `https://math.la.asu.edu/∼kuang/class/Mapletut.pdf` (free)

Problem 7. (`max grade: A`) Define $B(k)$ to be the minimal integer such that every 2-coloring of the integers $1, 2, \ldots, B(k)$ admits a monochromatic sequence $x_1, x_2, \ldots, x_k$ such that there exist positive integers $d_1 \neq d_2$ with $x_i - x_{i-1} \in \{d_1, d_2\}$ for $2 \leq i \leq k$. Write a computer program to calculate $B(k)$ for small values of k and determine upper and lower bounds on $B(k)$ for arbitrary k (use of the probabilistic method may prove fruitful).

`Required Books:` (1) *Ramsey Theory on the Integers, Second Edition* by Landman and Robertson;

(2) Maple Intro: `https://math.la.asu.edu/∼kuang/class/Mapletut.pdf` (free)

Problem 8. (`max grade: A`) Define $v(k)$ to be the minimal integer such that every 2-coloring of $\{1, 2, \ldots, v(k)\}$ that uses an equal number of each color admits a k-term arithmetic progression with exactly half of its term of one color (and the other half the other color). Does $v(k)$ exist for all even k (clearly it does not exist if k is odd)? Use the Lovasz Local Lemma to provide a lower bound for $v(k)$.

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(2) Maple Intro: `https://math.la.asu.edu/∼kuang/class/Mapletut.pdf` (free)