MATH 483 SYLLABUS AND THESIS PROBLEMS

Fall 2020

Prof. Robertson

NOTES: Grading for your thesis is based on a departmentally decided standard. To receive an A, the thesis must be of sufficient quality to stand for high honors. To receive an A−, the thesis must be of sufficient quality to stand for honors. Grades for the course are based equally on the thesis grade and an "engagement" grade, which is determined through the weekly reports and meetings. All theses must be written in LaTeX. Visit `www.aaronrobertson.org` and click on Teaching to find a LaTeX skeleton file. The department has software on all computers that will compile your LaTeX file; however the online LaTeX editor at `overleaf.com` is very good and easy to use.

DUE DATES: Due <u>every Thursday by noon</u> is weekly report of your progress over the week. Go to my website to download a template for this report (to be written in LaTeX). Please email your report to me as a PDF with file name `NameX.pdf` where `Name` is your last name and `X` is the week number. So, next Thursday you will send me a PDF named `Name1.pdf`.

As you progress, your weekly reports will turn into drafts of your thesis. These will also be due every Thursday, just as the weekly reports.

Your thesis is due by the last day of classes, Wednesday, December 9, at 5:00pm.

MEETINGS: I will make comments on your weekly reports and email these back to you on Friday. We will set up a Zoom meeting for the next week as needed. Also, please email me any time you want to set up a Zoom meeting.

WHAT YOU CAN USE: You can use any resource available to you except for people. If you need help with computer programs, contact me. The department has licenses for Matlab and languages based on C (e.g., python).

HONORS: Honors and high honors are based, in part, on the production of a thesis of distinction. Such theses are put forward by me and I will inform you of my intention to do so near the end of the process (based on a first draft). However, just because your thesis is put forward to stand for honors does not mean honors will be granted as your thesis and its defense is graded by two other independent faculty members and the decision must be unanimous.

FOR FRIDAY, AUGUST 28: Please email me an **ordered** list of 5 problems you would like to work on. I will attempt to give everyone one of their first 3 choice if at all possible.

Problem 1. Consider the quintic polynomial $f_a(x) = x^5 - x - a$, where a is a (perhaps complex) constant. First explain how we know that there is no general algebraic (i.e., with radicals) formula for the roots of an arbitrary quintic polynomial. Why must there be at least one real root if $a \in \mathbb{R}$ but not necessarily if $a \in \mathbb{C}$? Next, show that any quintic polynomial can be transformed into $f_a(x)$ for some a. Explain how Lagrange Inversion can be used to find a root r of $f_a(x)$ in terms of an infinite series, so that $f_a(x)$ can be written as $(x - r)g(x)$ where $g(x)$ is a quartic polynomial (which *can* be solved in radicals). Write a short computer (Maple/Matlab/Mathematica are fine) program that follows this method to produce the five roots for a given value of a (to arbitrary floating point degree of accuracy) of a general quintic polynomial (Maple's standard response can be unsatisfying). You can use the known algorithm(s) for polynomials of degree less than 4 provided you code the algorithm yourself. You can also use Lagrange Inversion multiple times, if possible.

Problem 2. Let $D = \{d_1, d_2, \ldots, d_n\}$ be a set of n distances (so that the d_i are all positive). Define $M(n)$ to be the maximum number of points M in $\mathbb{R}^2$ such that all distances between every pair of points is from D, i.e., for points $x_1, x_2, \ldots, x_M \in \mathbb{R}^2$ we have $\|x_i - x_j\| \in D$ for all $i \neq j$.

Show that $M(1) = 3$ and $M(2) \geq 5$ (bonus if you can show $M(2) = 5$). For general n, show that

$$2n + 1 \leq M(n) \leq (n + 1)(2n + 1).$$

Problem 3. Attempt to find the exact value of the Hales-Jewett number $HJ(4, 2)$. The only known Hales-Jewett number $HJ(3, 2) = 4$ was determined in 2014. (Don't let the small value lead you to believe this is easy.) First, give a description of what the Hales-Jewett theorem is. For this problem, it asserts that there exists a minimum integer n such that any 2-coloring of all length n words over $[4]^n = \{a_1a_2 \cdots a_n : a_i \in \{1, 2, 3, 4\}\}$ will contain a monochromatic variable word (aka, combinatorial line). The best-known bounds are $6 \leq HJ(4, 2) \leq 10^{11}$. I suspect $HJ(4, 2) = 6$ or 7 is the correct value. Programming will prove helpful here in order to deduce some forced word colors, but brute force will take too long.

Problem 4. Consider three lines $\overleftrightarrow{u}$, $\overleftrightarrow{v}$, and $\overleftrightarrow{w}$. We call them *equi-angular* if the angles between $\overleftrightarrow{u}$ and $\overleftrightarrow{v}$, between $\overleftrightarrow{u}$ and $\overleftrightarrow{w}$, and between $\overleftrightarrow{v}$ and $\overleftrightarrow{w}$ are all the same. The extension to more that three lines should be obvious. What is the maximum number of equi-angular lines in $\mathbb{R}^3$? Find an upper bound for the maximum number in $\mathbb{R}^n$ for $n > 3$. Next, call a set of lines *almost equi-angular* if there are only two distinct values for all of the angles between pairs of lines. Determine the maximum number of almost equi-angular lines in $\mathbb{R}^3$.

Problem 5. Let $m < n$ be positive integers. Determine necessary and sufficient conditions for a sequence $\{x_j\}_{j=1}^n$ of real numbers to satisfy

$$\left| \sum_{j \in S} x_j \right| = \left| \sum_{\substack{1 \leq j \leq n \\ j \notin S}} x_j \right|$$

for every m-element subset S of $\{1, 2, \ldots, n\}$. Next, determine necessary and sufficient conditions for a convergent sequence $\{x_j\}_{j=1}^\infty$ of real numbers to satisfy $\left| \sum_{j \in T} x_j \right| = \left| \sum_{j \in \mathbb{Z}^+ \setminus T} x_j \right|$ for every $T \subset \mathbb{Z}^+$ with $|T| = |\mathbb{Z}^+ \setminus T|$.

Problem 6. A k-term polynomial progression of degree n is $\{p(0), p(1), \ldots, p(k-1)\}$ for some polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ where the coefficients a_i are nonnegative integers with a_n positive. For any $r \in \mathbb{Z}^+$, it is known that there is a minimal integer, denoted $pp(k,n;r)$, where every r-coloring of the integers $1, 2, \ldots, pp(k,n;r)$ contains a monochromatic k-term polynomial progression of degree n. Use the probabilistic (or other) method to find a non-trivial lower bound for $pp(k,n;r)$.

Problem 7. In this problem you'll be modifying the finite difference method to be used on PDEs over one space variable and one time variable. First, give a detailed mathematical summary of how the finite difference method would be used with a mesh of $\Delta x \times \Delta t$ rectangles. You should find that the approximation $f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$ plays a role in the finite difference method. Modify this approach by trying $f'(x) \approx \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$. Calling this approximation $A(x)$, approximate $f''(x)$ by $\frac{f'(x+\Delta x)-f'(x)}{\Delta x} \approx \frac{A(x+\Delta x)-A(x)}{\Delta x}$.

Implement your new method in Matlab (or some other computer language) to produce 3-d plots of your approximate solutions. Compare with equations with known analytic solutions (e.g., heat equation in one dimension). Test your new method with other PDEs without known analytic solutions and compare to other people's results.

Problem 8. Consider a simple graph G on n vertices. We call G a *semi-small network* if the average shortest path length between two vertices is less than $\log n$. To help determine the proportion of all simple graphs on n vertices that are semi-small networks, we will find the distribution of the random variable X = average shortest path over all such graphs. Write an algorithm to randomly choose a simple graph for given values of n (will be fairly small) and calculate the value of X for it. By implementing this algorithm many times (e.g., 100,000) we can find an approximate histogram for X. Can you conjecture the related density function for X?

Problem 9. Let $A \subseteq [1,n]$. Describe how to use Fourier analysis to enumerate solutions to $x^2 + y^2 \equiv z^2 \pmod{n}$ with $x, y, z \in A$. Generalize this to prove that any r-coloring of $\{1, 2, \ldots, n\}$ admits a monochromatic solutions to $x^4 + y^4 \equiv z^2 \pmod{n}$ for sufficiently large n, while proving that $x^4 + y^4 = z^2$ has no solution (monochromatic or otherwise) over the integers.

Problem 10. Describe, in mathematical detail, what the Stone-Čech compactification of $\mathbb{Z}^+$ is in terms of ultrafilters and ultrafilter addition. Prove that for any ultrafilter p and any partition of $\mathbb{Z}^+$ into r sets, one of the sets is in p. Prove that the Stone-Čech compactification contains an idempotent ultrafilter. Use this to prove that every r-coloring of $\mathbb{Z}^+$ contains a monochromatic set of the form $\{x, y, z, x+y, x+z, y+z, x+y+z\}$.

Problem 11. Let $a \in \mathbb{R}$. Consider the set $S = \{ia : i \in \mathbb{Z}^+\}$. Let $\epsilon > 0$ be arbitrary. Prove that there exists $s \in S$ and $j \in \mathbb{Z}^+$ such that $|s - j| < \epsilon$. Let $\{ia\}$ be the fractional part of ia (i.e., the part after the decimal point). Let $T = \{\{ia\} : i \in \mathbb{Z}^+\}$. Determine conditions on a for T to be dense in $(0,1)$. Use this to show that for any $k \in \mathbb{Z}^+$ with $k \notin \{10^{i-1} : i \in \mathbb{Z}^+\}$, if $s_1 s_2 \cdots s_t$ is a sequence of t (with $t \in \mathbb{Z}^+$ arbitrarily) integers such that $s_i \in \{0, 1, \ldots, 9\}$ for all i (but $s_1 \neq 0$), then there exists $n \in \mathbb{Z}^+$ such that the first t digits of k^n are $s_1 s_2 \ldots s_t$ (in order).

Problem 12. Let $f_m(x) = x^m$. Determine necessary and sufficient conditions on n and m for

$$S_m(n) = \sum_{i=1}^{n} f_m(i)$$

to be divisible by $m+1$. Next, show that for any $k \in \mathbb{Z}^+$,

$$\sum_{j=0}^{k-1} \binom{k}{j+1} n^{j+1} = \sum_{j=0}^{k-1} \binom{k}{j} S_j(n).$$

Use this to give a method of evaluating $S_m(n)$ (e.g., $S_1(n) = \binom{n+1}{2}$).

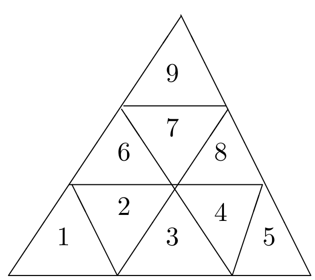
Problem 13. Let $\mathcal{F}$ be a family of m distinct subsets of $\{1, 2, \ldots, n\}$. Let $S_1, S_2, \ldots, S_m$ be these subsets. Consider the following properties:

- P1) All $|S_i|$ are even;
- P2) All $|S_i|$ are odd;
- P3) $|S_i \cap S_j|$ is even for all $i \neq j$;
- P4) $|S_i \cap S_j|$ is odd for all $i \neq j$.

If P1 and P3 hold, show that $m \leq 2^{\lfloor \frac{n}{2} \rfloor}$. If P2 and P3 hold, show $m \leq n$. If P1 and P4 hold, what can be said? If P2 and P4 hold, what can be said?

Problem 14. Describe, in detail, what the Circle Method is. Use the circle method to show that there are infinitly many 3-term arithmetic progressions of primes. This is a heavy-duty analysis problem, but is also an expository project as you will not be solving anything.

Problem 15. Give a description of 3×3 Magic squares and proof of their existence, along with any relevant results about them. Now consider the following definition of Magic triangles (these are not the same as what you will find online and in the literature). Consider the triangle split into 9 triangles as in the figure below.



Place the integers 1 through 9 into the 9 sub-triangles. If the sum of the integers in each of the regions $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 6, 7, 9\}$ and $\{4, 5, 7, 8, 9\}$ (i.e., along the sides) are the same, we call it a Magic triangle.

Do Magic triangles exists? If so, explain how to find them. If not, offer a proof.

Can you generalize to larger triangles as is done with magic squares?