A Case Study in Automated Theorem Proving: Otter and EQP

by

Allen Lawrence Mann

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A Case Study in Automated Theorem Proving: Otter and EQP

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A complete proof of the Robbins conjecture is presented, along with a proof that the equation DN_1 axiomatizes Boolean algebra. We also provide a short discussion of how the automated theorem provers Otter and EQP found the proofs.

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Chapter 1

The Robbins Problem

1.1 Boolean algebra

The language of Boolean algebra consists of two binary function symbols $\cup$ and $\cap$, one unary function symbol $\overline{}$, and two constants 0 and 1. The axioms of Boolean algebra, as found in [4, pages 7–8], are:

(B_1)	$x \cup (y \cup z) = (x \cup y) \cup z$	(B'_1)	$x \cap (y \cap z) = (x \cap y) \cap z$
(B_2)	$x \cup y = y \cup x$	(B'_2)	$x \cap y = y \cap x$
(B_3)	$x \cup (x \cap y) = x$	(B'_3)	$x \cap (x \cup y) = x$
(B_4)	$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$	(B'_4)	$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
(B_5)	$x \cup \overline{x} = 1$	(B'_5)	$x \cap \overline{x} = 0$

I will refer to these axioms collectively as B. Axioms B_1 and B'_1 are called the associativity of $\cup$ and $\cap$, respectively. Axioms B_2 and B'_2 are called the commutativity of $\cup$ and $\cap$, respectively. Axioms B_3 and B'_3 are called the absorption axioms. Axioms B_4 and B'_4 are called the distributivity axioms. Axioms B_5 and B'_5 are called the complementation axioms. If one term can be obtained from another simply through applications of B_1, B'_1, B_2, and B'_2, the two terms will be called AC identical. In most proofs, applications of associativity and commutativity will go without mention.

It will aid our discussion to have at our disposal some of the well-known properties of Boolean algebras.

Proposition 1 (Idempotence). $\mathrm{B} \vdash x \cup x = x = x \cap x$.

Proof. By absorption, $x \cup x = x \cup (x \cap (x \cup x)) = x$, and $x \cap x = x \cap (x \cup (x \cap x)) = x$. $\square$

Proposition 2. $\mathrm{B} \vdash x \cup y = y \leftrightarrow x \cap y = x$.

Proof. If $x \cup y = y$, then by absorption $x \cap y = x \cap (x \cup y) = x$. Conversely, if $x \cap y = x$, then by absorption $x \cup y = (x \cap y) \cup y = y$. $\square$

Proposition 3. $\mathrm{B} \vdash x \cup 0 = x = x \cap 1$.

Proof. By B_5' and B_3, $x \cup 0 = x \cup (x \cap \overline{x}) = x$. By B_5 and B_3', $x \cap 1 = x \cap (x \cup \overline{x}) = x$. $\square$

Proposition 4. $\mathrm{B} \vdash x \cap 0 = 0 \wedge x \cup 1 = 1$.

Proof. By B_5' and idempotence, $x \cap 0 = x \cap x \cap \overline{x} = x \cap \overline{x} = 0$. By B_5 and idempotence, $x \cup 1 = x \cup x \cup \overline{x} = x \cup \overline{x} = 1$. $\square$

Definition. Two elements x and y of a Boolean algebra are **complements** if $x \cup y = 1$ and $x \cap y = 0$.

Proposition 5. *$\overline{x}$ is the unique complement of x.*

Proof. The elements x and $\overline{x}$ are complements by B_5 and B_5'. Suppose y and z are both complements of x. Then

$$
\begin{aligned}
y &= y \cap 1 & z &= z \cap 1 && \text{(by 3)} \\
&= y \cap (x \cup z) & &= z \cap (x \cup y) && \text{(by hypothesis)} \\
&= (y \cap x) \cup (y \cap z) & &= (z \cap x) \cup (z \cap y) && \text{(by } \mathrm{B}_4\text{)} \\
&= 0 \cup (y \cap z) & &= 0 \cup (z \cap y) && \text{(by hypothesis)} \\
&= y \cap z. & &= z \cap y. && \text{(by 3)}
\end{aligned}
$$

Hence $y = z$. $\square$

Proposition 6. $\mathrm{B} \vdash \overline{\overline{x}} = x$.

Proof. Both x and $\overline{\overline{x}}$ are complements of $\overline{x}$. □

Proposition 7. $\mathrm{B} \vdash \overline{x} = \overline{y} \to x = y$.

Proof. If $\overline{x} = \overline{y}$, then by the previous proposition, $x = \overline{\overline{x}} = \overline{\overline{y}} = y$. □

Proposition 8. $\mathrm{B} \vdash \overline{0} = 1 \land \overline{1} = 0$.

Proof. By Proposition 3, $0 \cup 1 = 1$ and $0 \cap 1 = 0$, so 0 and 1 are complements. □

Proposition 9 (De Morgan). $\mathrm{B} \vdash \overline{x \cup y} = \overline{x} \cap \overline{y} \land \overline{x \cap y} = \overline{x} \cup \overline{y}$.

Proof. By distributivity, B_5, and Proposition 4 we have:

$$(x \cup y) \cup (\overline{x} \cap \overline{y}) = (x \cup y \cup \overline{x}) \cap (x \cup y \cup \overline{y}) = (y \cup 1) \cap (x \cup 1) = 1 \cap 1 = 1.$$

By distributivity, B'_5, and Proposition 4 we have:

$$(x \cup y) \cap (\overline{x} \cap \overline{y}) = (x \cap \overline{x} \cap \overline{y}) \cup (y \cap \overline{x} \cup \overline{y}) = (0 \cup \overline{y}) \cap (0 \cap \overline{x}) = 0 \cap 0 = 0.$$

Thus $x \cup y$ and $\overline{x} \cap \overline{y}$ are complements. The dual argument shows that $x \cap y$ and $\overline{x} \cup \overline{y}$ are complements. □

It follows immediately from De Morgan's laws and Proposition 6 that $x \cup y = \overline{\overline{x} \cap \overline{y}}$ and $x \cap y = \overline{\overline{x} \cup \overline{y}}$.

1.2 Huntington algebra

In 1933, E.V. Huntington [3, 2] showed that the following three axioms, to which I shall refer as H, form a basis for Boolean algebra. That is, any theorem of Boolean algebra can be derived from the three, and none of the three can be derived from the other two.

(H_1) $x \cup (y \cup z) = (x \cup y) \cup z$ (associativity)

(H_2) $x \cup y = y \cup x$ (commutativity)

(H_3) $\overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y} = x$ (Huntington equation)

The term $\overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y}$ will be called the Huntington expansion of x by y. It is easy to prove that every Boolean algebra satisfies Huntington's axioms.

Theorem 10. B $\vdash$ H.

Proof. Since H_1 and H_2 are identical to B_1 and B_2, all we need to prove is the Huntington equation. We can restate the Huntington equation as:

$$(x \cap y) \cup (x \cap \overline{y}) = x.$$

By distributivity, B_5, and Proposition 3,

$$(x \cap y) \cup (x \cap \overline{y}) = x \cap (y \cup \overline{y}) = x \cap 1 = x.$$

□

Observe that Huntington's axioms use only one binary function symbol $\cup$, and one unary function symbol $\overline{}$. Strictly speaking, to show that H is a basis for Boolean algebra, one must expand the language of Huntington algebra to include $\cap$, 0, and 1 by defining them in terms of $\cup$ and $\overline{}$. On occasion, we will use the abbreviation $nx = \underbrace{x \cup \ldots \cup x}_{n}$.

1.3 Robbins algebra

Shortly after Huntington proved his result, Herbert Robbins conjectured that the following three axioms, to which I shall refer as R, also form a basis for Boolean algebra.

(R_1) $x \cup (y \cup z) = (x \cup y) \cup z$ (associativity)

(R_2) $x \cup y = y \cup x$ (commutativity)

(R_3) $\overline{\overline{x \cup y} \cup \overline{x \cup \overline{y}}} = x$ (Robbins equation)

The term $\overline{\overline{x \cup y} \cup \overline{x \cup \overline{y}}}$ will be called the Robbins expansion of x by y. It is equally easy to prove that every Boolean algebra satisfies Robbins' axioms.

Theorem 11. $\mathrm{B} \vdash \mathrm{R}$.

Proof. Since R_1 and R_2 are identical to B_1 and B_2, all we need to prove is the Robbins equation. We can restate the Robbins equation as:

$$(x \cup y) \cap (x \cup \overline{y}) = x.$$

By distributivity, B_5', and Proposition 3,

$$(x \cup y) \cap (x \cup \overline{y}) = x \cup (y \cap \overline{y}) = x \cup 0 = x.$$

□

The Robbins equation is simpler than the Huntington equation. It has one fewer occurrence of $\overline{}$. Despite the similarity of Huntington's and Robbins' axioms, Robbins and Huntington were unable to find a proof that all Robbins algebras are Boolean. The question "Are all Robbins algebras Boolean?" became known as the Robbins problem.

The problem remained unsolved for many years. According to McCune [5], the first major step toward the solution came in the 1980s when Steve Winker proved several conditions sufficient to make a Robbins algebra Boolean. That is, any Robbins algebra that satisfies one of Winker's conditions is a Boolean algebra. The problem was finally solved in 1997 by EQP, a theorem prover created at Argonne National Laboratory, which under the direction of William McCune proved that all Robbins algebras satisfy what is known as Winker's first condition.

For the rest of the chapter, we will present a complete proof that all Robbins algebras are Boolean.

1.4 $\mathrm{H} \vdash \mathrm{B}$

First, we will prove some basic properties of $\overline{}$.

Proposition 12. $\mathrm{H} \vdash x \cup \overline{x} = \overline{x} \cup \overline{\overline{x}}$.

Proof. Use the Huntington equation to expand x and $\overline{x}$ by $\overline{\overline{x}}$:

$$x \cup \overline{x} = \left(\overline{\overline{x} \cup \overline{\overline{\overline{x}}}} \cup \overline{\overline{x} \cup \overline{\overline{x}}}\right) \cup \left(\overline{\overline{\overline{x}} \cup \overline{\overline{\overline{x}}}} \cup \overline{\overline{\overline{x}} \cup \overline{\overline{x}}}\right).$$

Likewise, use the Huntington equation to expand $\overline{x}$ and $\overline{\overline{x}}$ by $\overline{x}$:

$$\overline{x} \cup \overline{\overline{x}} = \left(\overline{\overline{\overline{x}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{x}} \cup \overline{x}}\right) \cup \left(\overline{\overline{\overline{\overline{x}}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{\overline{x}}} \cup \overline{x}}\right).$$

The right-hand sides of these two equations are AC identical. Therefore $x \cup \overline{x} = \overline{x} \cup \overline{\overline{x}}$. ☐

Proposition 13. $\mathrm{H} \vdash \overline{\overline{x}} = x$.

Proof. Use the Huntington equation to expand $\overline{\overline{x}}$ by $\overline{x}$, then simplify with Proposition 12 and the Huntington equation applied to x and $\overline{\overline{x}}$:

$$\overline{\overline{x}} = \overline{\overline{\overline{\overline{x}}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{\overline{x}}} \cup \overline{x}} = \overline{\overline{x} \cup \overline{\overline{\overline{x}}}} \cup \overline{\overline{x} \cup \overline{\overline{x}}} = x.$$

☐

Proposition 14. $\mathrm{H} \vdash \overline{x} = \overline{y} \rightarrow x = y$.

Proof. If $\overline{x} = \overline{y}$, then by the previous proposition, $x = \overline{\overline{x}} = \overline{\overline{y}} = y$. ☐

Next, we define $\cap$ and prove some useful propositions such as De Morgan's laws.

Definition. $x \cap y = \overline{\overline{x} \cup \overline{y}}$.

Proposition 15. $\mathrm{H} \vdash \overline{x \cup y} = \overline{x} \cap \overline{y} \wedge \overline{x \cap y} = \overline{x} \cup \overline{y}$.

Proof. By the definition of $\cap$ and Proposition 13, $\overline{x} \cap \overline{y} = \overline{\overline{\overline{x}} \cup \overline{\overline{y}}} = \overline{x \cup y}$. Similarly, $\overline{x \cap y} = \overline{\overline{\overline{x} \cup \overline{y}}} = \overline{x} \cup \overline{y}$. ☐

Proposition 16. $\mathrm{H} \vdash x \cup y = \overline{\overline{x} \cap \overline{y}}$.

Proof. By Propositions 13 and 15, $x \cup y = \overline{\overline{x \cup y}} = \overline{\overline{x} \cap \overline{y}}$. ☐

Proposition 16 shows that we could formulate the axioms of Huntington algebra in terms of $\overline{}$ and $\cap$ instead of $\overline{}$ and $\cup$. We can also give a more intuitive formulation of the Huntington equation in terms of all three symbols.

Proposition 17. $\mathrm{H} \vdash (x \cap y) \cup (x \cap \overline{y}) = x$.

Proof. $(x \cap y) \cup (x \cap \overline{y}) = \overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup \overline{\overline{y}}} = \overline{\overline{x} \cup \overline{\overline{y}}} \cup \overline{\overline{x} \cup \overline{y}} = x$. $\square$

The associativity and commutativity of $\cap$ follow directly from the associativity and commutativity of $\cup$.

Theorem 18. $\mathrm{H} \vdash \mathrm{B}_1'$.

Proof. By Proposition 15, $x \cap (y \cap z) = \overline{\overline{x} \cup \overline{y \cap z}} = \overline{\overline{x} \cup (\overline{y} \cup \overline{z})} = \overline{(\overline{x} \cup \overline{y}) \cup \overline{z}} = \overline{\overline{x \cap y} \cup \overline{z}} = (x \cap y) \cap z$. $\square$

Theorem 19. $\mathrm{H} \vdash \mathrm{B}_2'$.

Proof. $x \cap y = \overline{\overline{x} \cup \overline{y}} = \overline{\overline{y} \cup \overline{x}} = y \cap x$. $\square$

In any Huntington algebra, the function defined by $f(x) = x \cup \overline{x}$ is constant.

Proposition 20. $\mathrm{H} \vdash x \cup \overline{x} = y \cup \overline{y}$.

Proof. Use the Huntington equation to expand x and $\overline{x}$ by $\overline{y}$:

$$x \cup \overline{x} = \left(\overline{\overline{x} \cup \overline{\overline{y}}} \cup \overline{\overline{x} \cup \overline{y}}\right) \cup \left(\overline{\overline{\overline{x}} \cup \overline{\overline{y}}} \cup \overline{\overline{\overline{x}} \cup \overline{y}}\right).$$

Likewise, use the Huntington equation to expand y and $\overline{y}$ by $\overline{x}$:

$$y \cup \overline{y} = \left(\overline{\overline{y} \cup \overline{\overline{x}}} \cup \overline{\overline{y} \cup \overline{x}}\right) \cup \left(\overline{\overline{\overline{y}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{y}} \cup \overline{x}}\right).$$

The right-hand sides of these two equations are AC identical. Therefore, $x \cup \overline{x} = y \cup \overline{y}$. $\square$

We can now extend the language of Huntington algebra to include the constants 0 and 1.

Definition. $1 = x \cup \overline{x}$.

Definition. $0 = \overline{1} = \overline{x \cup \overline{x}}$.

Observe that by the definition of 1 we have $\mathrm{H} \vdash \mathrm{B}_5$.

Theorem 21. $\mathrm{H} \vdash \mathrm{B}'_5$.

Proof. By Proposition 13, $x \cap \overline{x} = \overline{\overline{x} \cup \overline{\overline{x}}} = \overline{\overline{x} \cup x} = 0$. □

Proposition 22. $\mathrm{H} \vdash x \cup 0 = x = x \cap 1$.

Proof. First, apply the Huntington equation to 0 and 0 to obtain:

$$\overline{1} = 0 = \overline{\overline{0} \cup \overline{0}} \cup \overline{\overline{0} \cup 0} = \overline{\overline{0} \cup \overline{0}} \cup 0 = \overline{1 \cup 1} \cup \overline{1}. \tag{22.1}$$

Second, use equation (22.1) to obtain:

$$1 = 1 \cup \overline{1} = 1 \cup \left(\overline{1 \cup 1} \cup \overline{1}\right) = \left(1 \cup \overline{1}\right) \cup \left(\overline{1 \cup 1}\right) = 1 \cup \overline{1 \cup 1}. \tag{22.2}$$

Third, use equation (22.2) to obtain:

$$1 = (1 \cup 1) \cup \overline{1 \cup 1} = 1 \cup \left(1 \cup \overline{1 \cup 1}\right) = 1 \cup 1. \tag{22.3}$$

Fourth, use equations 22.1 and 22.3 to obtain:

$$0 = \overline{1} = \overline{1 \cup 1} \cup \overline{1} = \overline{1} \cup \overline{1} = 0 \cup 0. \tag{22.4}$$

Fifth, use the Huntington equation applied to x and x and equation (22.4) to obtain,

$$x \cup 0 = \left(\overline{\overline{x} \cup \overline{x}} \cup \overline{\overline{x} \cup x}\right) \cup 0 = \overline{\overline{x} \cup \overline{x}} \cup 0 \cup 0 = \overline{\overline{x} \cup \overline{x}} \cup 0 = \overline{\overline{x} \cup \overline{x}} \cup \overline{\overline{x} \cup x} = x. \tag{22.5}$$

Finally, by equation (22.5) and 13, $x \cap 1 = \overline{\overline{x} \cup \overline{1}} = \overline{\overline{x} \cup 0} = \overline{\overline{x}} = x$. □

Proposition 23. $\mathrm{H} \vdash x \cup x = x = x \cap x$.

Proof. By the Huntington equation applied to x and x, and Proposition 22,

$$x = \overline{\overline{x} \cup \overline{x}} \cup \overline{\overline{x} \cup x} = \overline{\overline{x} \cup \overline{x}} \cup 0 = \overline{\overline{x} \cup \overline{x}} = x \cap x.$$

Therefore, by Propositions 16 and 13 we have $x \cup x = \overline{\overline{x} \cap \overline{x}} = \overline{\overline{x}} = x$. □

Proposition 24. $\mathrm{H} \vdash x \cap 0 = 0 \wedge x \cup 1 = 1$.

Proof. By Proposition 23, $x \cup 1 = x \cup x \cup \overline{x} = x \cup \overline{x} = 1$. Thus $x \cap 0 = \overline{\overline{x} \cup \overline{0}} = \overline{\overline{x} \cup 1} = \overline{1} = 0$. □

Theorem 25. $\mathrm{H} \vdash \mathrm{B}_3$.

Proof. Use the Huntington equation to expand x by y, then simplify with Proposition 23 and the Huntington equation applied to x and y:

$$x \cup (x \cap y) = \overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}} = \overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y} = x.$$

□

Theorem 26. $\mathrm{H} \vdash \mathrm{B}_3'$.

Proof. By Proposition 15, B_3, and Proposition 13,

$$x \cap (x \cup y) = \overline{\overline{x} \cup \overline{x \cup y}} = \overline{\overline{x} \cup (\overline{x} \cap \overline{y})} = \overline{\overline{x}} = x.$$

□

Theorem 27. $\mathrm{H} \vdash \mathrm{B}_4$.

Proof. First, use the Proposition 17 to expand $x \cap (y \cup z)$ by y:

$$x \cap (y \cup z) = [x \cap (y \cup z) \cap y] \cup [x \cap (y \cup z) \cap \overline{y}],$$

which by B_3' applied to y and z simplifies to:

$$x \cap (y \cup z) = [x \cap y] \cup [x \cap (y \cup z) \cap \overline{y}].$$

Now use Proposition 17 to expand each square-bracketed term by z,

$$\begin{aligned}
x \cap (y \cup z) &= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap (y \cup z) \cap \overline{y} \cap z) \cup (x \cap (y \cup z) \cap \overline{y} \cap \overline{z})] \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup (x \cap (y \cup z) \cap \overline{y} \cap \overline{z})] && \text{(by B}_3'\text{)} \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup (x \cap (y \cup z) \cap \overline{y \cup z})] && \text{(by 15)} \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup (x \cap 0)] && \text{(by B}_5'\text{)} \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup 0] && \text{(by 24)} \\
&= (x \cap y \cap z) \cup (x \cap y \cap \overline{z}) \cup (x \cap \overline{y} \cap z) && \text{(by 22)} \\
&= (x \cap y \cap z) \cup (x \cap y \cap \overline{z}) \cup (x \cap y \cap z) \cup (x \cap \overline{y} \cap z) && \text{(by 23)} \\
&= (x \cap y) \cup (x \cap z). && \text{(by 17)}
\end{aligned}$$

□

Theorem 28. $\mathrm{H} \vdash \mathrm{B}_4'$.

Proof.

$$\begin{aligned}
x \cup (y \cap z) &= \overline{\overline{\overline{\overline{x}} \cup \overline{\overline{y} \cup \overline{z}}}} && \text{(by 13)} \\
&= \overline{\overline{x} \cap (\overline{y} \cup \overline{z})} \\
&= \overline{(\overline{x} \cap \overline{y}) \cup (\overline{x} \cap \overline{z})} && \text{(by B}_4\text{)} \\
&= \overline{\overline{\overline{x \cup y} \cup \overline{x \cup z}}} && \text{(by 15)} \\
&= (x \cup y) \cap (x \cup z).
\end{aligned}$$

□

Theorem 29. $\mathrm{H} \vdash \mathrm{B}$.

Proof. By the definition of 1 and Theorems 18, 19, 25, 26, 21, 27, and 28. □

1.5 $\mathrm{R} + \mathrm{W}_1 \vdash \mathrm{H}$

In 1992, Winker [8] proved that each of the following axioms is a sufficient condition for a Robbins algebra to satisfy the Huntington equation, and therefore to be a Boolean algebra.

(W_{-2})	$\overline{\overline{x}} = x$	(double negation)
(W_{-1})	$x \cup 0 = x$	(zero)
(W_0)	$a \cup a = a$	(idempotent)
(W_1)	$a \cup b = b$	(absorption)
(W_2)	$\overline{a \cup b} = \overline{b}$	(absorption within negation)

Note that in the above equations, x is a variable, while 0, a, and b are constants.

Theorem 30. $\mathrm{R} + \mathrm{W}_{-2} \vdash \mathrm{H}$.

Proof. Since H_1 and H_2 are identical to R_1 and R_2, all we need to prove is the Huntington equation. Apply the Robbins equation to $\overline{x}$ and y to get $\overline{\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}}} = \overline{x}$. Then by W_{-2},

$$\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}} = \overline{\overline{\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}}}} = \overline{\overline{x}} = x.$$

Commuting terms on the left yields the Huntington equation. $\square$

Theorem 31. $\mathrm{R} + \mathrm{W}_{-1} \vdash \mathrm{W}_{-2}$.

Proof. Use the Robbins equation to expand 0 by x, then simplify with W_{-1} to obtain:

$$0 = \overline{\overline{0 \cup x} \cup \overline{0 \cup \overline{x}}} = \overline{\overline{x} \cup \overline{\overline{x}}}. \tag{31.1}$$

Use the Robbins equation to expand $\overline{x}$ by $\overline{\overline{x}}$, then simplify with equation (31.1) and W_{-1} to obtain:

$$\overline{x} = \overline{\overline{\overline{x} \cup \overline{\overline{x}}} \cup \overline{\overline{x} \cup \overline{\overline{\overline{x}}}}} = \overline{0 \cup \overline{\overline{x} \cup \overline{\overline{\overline{x}}}}} = \overline{\overline{\overline{x} \cup \overline{\overline{\overline{x}}}}}.$$

Use the Robbins equation to expand $\bar{\bar{\bar{x}}}$ by $\bar{x}$, then simplify with equation (31.1) applied to $\bar{x}$ to obtain:

$$\bar{\bar{\bar{x}}} = \overline{\overline{\bar{\bar{\bar{x}}} \cup \bar{x}} \cup \overline{\bar{\bar{\bar{x}}} \cup \bar{\bar{x}}}} = \overline{\overline{\bar{\bar{\bar{x}}} \cup \bar{x}} \cup 0} = \overline{\overline{\bar{\bar{\bar{x}}} \cup \bar{x}}}.$$

Therefore $\bar{x} = \bar{\bar{\bar{x}}}$. Substitute $\overline{x \cup y} \cup \overline{x \cup \bar{y}}$ in this equation, then apply the Robbins equation to both sides to get:

$$x = \overline{\overline{x \cup y} \cup \overline{x \cup \bar{y}}} = \overline{\overline{\overline{\overline{\overline{x \cup y} \cup \overline{x \cup \bar{y}}}}}} = \bar{\bar{x}}.$$

$\square$

Theorem 32. $\mathrm{R} + \mathrm{W}_0 \vdash \mathrm{W}_{-1}$.

Proof. Suppose $a \cup a = a$, and define $0 = \overline{a \cup \bar{a}}$. We will show that $x \cup 0 = x$.

Use the Robbins equation to expand a by a, then simplify with W_0 to get:

$$a = \overline{\overline{a \cup a} \cup \overline{a \cup \bar{a}}} = \overline{\bar{a} \cup 0}. \tag{32.1}$$

Use the Robbins equation to expand $a \cup x$ by a, then simplify with W_0 to get:

$$a \cup x = \overline{\overline{a \cup x \cup a} \cup \overline{a \cup x \cup \bar{a}}} = \overline{\overline{a \cup x} \cup \overline{a \cup x \cup \bar{a}}}. \tag{32.2}$$

Use the Robbins equation to expand x by $\bar{a} \cup 0$, then simplify with equation (32.1) to get:

$$x = \overline{\overline{x \cup \bar{a} \cup 0} \cup \overline{x \cup \overline{\bar{a} \cup 0}}} = \overline{\overline{x \cup \bar{a} \cup 0} \cup \overline{x \cup a}}. \tag{32.3}$$

Use the Robbins equation to expand $\bar{a}$ by $a \cup \bar{a}$, then simplify with equation (32.1) to get:

$$\bar{a} = \overline{\overline{\bar{a} \cup a \cup \bar{a}} \cup \overline{\bar{a} \cup \overline{a \cup \bar{a}}}} = \overline{\overline{a \cup \bar{a} \cup \bar{a}} \cup a}. \tag{32.4}$$

Apply equation (32.3) to a, then simplify with W_0 to get:

$$a = \overline{\overline{a \cup \bar{a} \cup 0} \cup \overline{\bar{a} \cup a}} = \overline{\overline{a \cup \bar{a} \cup 0} \cup \bar{a}}. \tag{32.5}$$

Use the Robbins equation to expand a by $a \cup \overline{a} \cup \overline{a}$, then simplify with equation (32.4) and W_0 to get:

$$a = \overline{\overline{a \cup a \cup \overline{a} \cup \overline{a}} \cup \overline{a \cup \overline{a \cup \overline{a} \cup \overline{a}}}} = \overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup \overline{a}}. \tag{32.6}$$

Use the Robbins equation to expand $\overline{a \cup \overline{a} \cup \overline{a}}$ by a, then simplify with equations (32.4) and (32.6) to get:

$$\overline{a \cup \overline{a} \cup \overline{a}} = \overline{\overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup a} \cup \overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup \overline{a}}} = \overline{\overline{a} \cup a} = 0. \tag{32.7}$$

By equations (32.4) and (32.7):

$$\overline{a} = \overline{a \cup \overline{a \cup \overline{a} \cup \overline{a}}} = \overline{a \cup 0}. \tag{32.8}$$

Apply equation (32.2) to 0, then simplify with equations (32.8) and (32.5) to get:

$$a \cup 0 = \overline{\overline{a \cup 0} \cup \overline{a \cup 0 \cup \overline{a}}} = \overline{\overline{a} \cup \overline{a \cup 0 \cup \overline{a}}} = a. \tag{32.9}$$

Use the Robbins equation to expand $x \cup 0$ by a, then simplify with equations (32.9) and (32.3) to get:

$$x \cup 0 = \overline{\overline{x \cup 0 \cup a} \cup \overline{x \cup 0 \cup \overline{a}}} = \overline{\overline{x \cup a} \cup \overline{x \cup 0 \cup \overline{a}}} = x.$$

□

The proof of Theorem 32 was found by EQP. It is shorter than Winker's original proof.

Lemma 33. $\mathrm{R} \vdash \overline{a \cup \overline{b \cup c}} = \overline{a \cup b \cup \overline{c}} \rightarrow a \cup b = a.$

Proof. Use the Robbins equation to expand $a \cup b$ by c, then apply the hypothesis and simplify with the Robbins equation applied to a and $b \cup c$:

$$a \cup b = \overline{\overline{a \cup b \cup c} \cup \overline{a \cup b \cup \overline{c}}} = \overline{\overline{a \cup b \cup c} \cup \overline{a \cup \overline{b \cup c}}} = a.$$

□

Lemma 34. $\mathrm{R} \vdash \overline{a \cup \overline{b \cup c}} = \overline{b \cup \overline{a \cup c}} \to a = b.$

Proof. Use the Robbins equation to expand a by $b \cup c$, then apply the hypothesis and simplify with the Robbins equation applied to b and $a \cup c$:

$$a = \overline{\overline{a \cup b \cup c} \cup \overline{a \cup \overline{b \cup c}}} = \overline{\overline{b \cup a \cup c} \cup \overline{b \cup \overline{a \cup c}}} = b.$$

□

Lemma 35. $\mathrm{R} \vdash \overline{a \cup \overline{b}} = c \to \overline{\overline{a \cup b} \cup c} = a.$

Proof. Use the Robbins equation to expand a by b, then apply the hypothesis:

$$a = \overline{\overline{a \cup b} \cup \overline{a \cup \overline{b}}} = \overline{\overline{a \cup b} \cup c}.$$

□

Lemma 36. *For every positive integer k,* $\mathrm{R} \vdash \overline{a \cup \overline{b}} = c \to \overline{a \cup \overline{b \cup k(a \cup c)}} = c.$

Proof. By induction on k. Let $b_0 = b$ and $b_k = b \cup k(a \cup c)$. By hypothesis, $\overline{a \cup \overline{b_0}} = c$. Now assume $\overline{a \cup \overline{b_k}} = c$. Then by Lemma 35, $a = \overline{\overline{a \cup b_k} \cup c}$, so we have

$$\overline{a \cup \overline{b_{k+1}}} = \overline{\overline{\overline{a \cup b_k} \cup c} \cup \overline{b_k \cup a \cup c}} = c,$$

by the Robbins equation applied to c and $a \cup b_k$. □

Lemma 37. *For every positive integer k,* $\mathrm{R} \vdash \overline{\overline{a \cup \overline{b}} \cup \overline{b}} = a \to \overline{b \cup k(a \cup \overline{a \cup \overline{b}})} = \overline{b}.$

Proof. Let $\overline{a \cup \overline{b}} = c$ and $b_k = b \cup k(a \cup c)$. Then by Lemma 36,

$$\overline{a \cup \overline{b_k}} = \overline{a \cup \overline{b \cup k(a \cup c)}} = c.$$

By hypothesis, $\overline{c \cup \overline{b}} = a$, so by Lemma 36,

$$\overline{c \cup \overline{b_k}} = \overline{c \cup \overline{b \cup k(c \cup a)}} = a.$$

Therefore,

$$\overline{\overline{b_k} \cup \overline{\overline{b} \cup c}} = \overline{\overline{b_k} \cup a} = c = \overline{\overline{b} \cup a} = \overline{\overline{b} \cup \overline{\overline{b_k} \cup c}},$$

so by Lemma 34 applied to $\overline{b_k}$, $\overline{b}$, and c, we have $\overline{b_k} = \overline{b}$. □

This proof of Lemma 37 was discovered by an automated theorem prover [9].

Lemma 38. *For every positive integer k,* $\mathrm{R} \vdash \overline{a \cup b} = \bar{b} \to \overline{b \cup k(a \cup \overline{\overline{a \cup \bar{b}}})} = \bar{b}$.

Proof. Observe that by hypothesis, $\overline{\overline{a \cup \bar{b}} \cup \bar{b}} = \overline{\overline{a \cup b} \cup \overline{a \cup \bar{b}}} = a$, so the conclusion follows from Lemma 37. □

Lemma 39. $\mathrm{R} \vdash \overline{2a \cup b} = \bar{b} = \overline{3a \cup b} \to 2a \cup b = 3a \cup b$.

Proof. Applying Lemma 38 to $2a$ and b with $k = 1$ yields:

$$\overline{2a \cup b \cup \overline{\overline{2a \cup \bar{b}}}} = \overline{b \cup 2a \cup \overline{\overline{2a \cup \bar{b}}}} = \bar{b}.$$

Applying Lemma 38 with $k = 1$ to a and $2a \cup b$ yields:

$$\overline{2a \cup b \cup a \cup \overline{\overline{a \cup \bar{b}}}} = \overline{2a \cup b \cup a \cup \overline{\overline{a \cup \overline{2a \cup b}}}} = \overline{2a \cup b} = \bar{b}.$$

Hence

$$\overline{2a \cup b \cup \overline{\overline{2a \cup \bar{b}}}} = \overline{2a \cup b \cup a \cup \overline{\overline{a \cup \bar{b}}}},$$

so by Lemma 33 we have $2a \cup b = 2a \cup b \cup a$. □

Lemma 40. $\mathrm{R} \vdash (\overline{a \cup b} = \bar{b} \vee \overline{\overline{a \cup \bar{b}} \cup \bar{b}} = a) \to b \cup 2(a \cup \overline{a \cup \bar{b}}) = b \cup 3(a \cup \overline{a \cup \bar{b}})$.

Proof. Apply either Lemma 37 or 38 to obtain

$$\overline{b \cup 2(a \cup \overline{a \cup \bar{b}})} = \bar{b} = \overline{b \cup 3(a \cup \overline{a \cup \bar{b}})}.$$

Then use Lemma 39 applied to $a \cup \overline{a \cup \bar{b}}$ and b to conclude $b \cup 2(a \cup \overline{a \cup \bar{b}}) = b \cup 3(a \cup \overline{a \cup \bar{b}})$. □

Theorem 41. $\mathrm{R} + \mathrm{W}_1 \vdash \mathrm{W}_0$.

Proof. Let $a \cup b = b$. Define $c = b \cup 2(\overline{a \cup \bar{b}})$ and $d = c \cup \overline{c \cup \bar{c}}$. We will show that $3d \cup 3d = 3d$.

First observe that by W_1,

$$a \cup c = a \cup b \cup 2(\overline{a \cup \overline{b}}) = b \cup 2(\overline{a \cup \overline{b}}) = c. \tag{41.1}$$

By W_1 we have $\overline{a \cup b} = \overline{b}$, so by Lemma 38 and W_1,

$$\overline{b} = \overline{b \cup 2(a \cup \overline{a \cup \overline{b}})} = \overline{b \cup 2(\overline{a \cup \overline{b}})} = \overline{c}. \tag{41.2}$$

By equation (41.2), W_1, and Lemma 40,

$$\begin{aligned} c \cup \overline{a \cup \overline{c}} &= b \cup 2(\overline{a \cup \overline{b}}) \cup \overline{a \cup \overline{b}} \\ &= b \cup 3(\overline{a \cup \overline{b}}) \\ &= b \cup 3(a \cup \overline{a \cup \overline{b}}) \\ &= b \cup 2(a \cup \overline{a \cup \overline{b}}) \\ &= b \cup 2(\overline{a \cup \overline{b}}) \\ &= c \end{aligned} \tag{41.3}$$

By equations (41.1), (41.3), and the Robbins equation applied to c and $a \cup \overline{c}$ we have

$$\overline{\overline{c \cup \overline{c}} \cup \overline{c}} = \overline{\overline{c \cup a \cup \overline{c}} \cup \overline{c \cup \overline{a \cup \overline{c}}}} = c, \tag{41.4}$$

which satisfies the hypothesis of Lemma 40 applied to c and c. Hence,

$$c \cup 2d = c \cup 2(c \cup \overline{c \cup \overline{c}}) = c \cup 3(c \cup \overline{c \cup \overline{c}}) = c \cup 3d. \tag{41.5}$$

Therefore, $4d = 3d \cup c \cup \overline{c \cup \overline{c}} = 2d \cup c \cup \overline{c \cup \overline{c}} = 3d$. Repeat twice more to obtain $6d = 3d$. □

Theorem 42. $\mathrm{R} + \mathrm{W}_1 \vdash \mathrm{H}$

Proof. By Theorems 30, 32, and 41. □

1.6 $\mathrm{R} \vdash \mathrm{W}_1$

Lemma 43. $\mathrm{R} \vdash \overline{\overline{\overline{x} \cup y} \cup \overline{x \cup y}} = y$.

Proof. This is just a restatement of the Robbins equation. □

Lemma 44. $\mathrm{R} \vdash \overline{\overline{\overline{x \cup y} \cup \overline{x} \cup y} \cup y} = \overline{x \cup y}$.

Proof. Applying Lemma 43 to $\overline{x} \cup y$ and $\overline{x \cup y}$ yields:

$$\overline{\overline{\overline{\overline{x} \cup y} \cup \overline{x \cup y}} \cup \overline{\overline{x} \cup y \cup \overline{x \cup y}}} = \overline{x \cup y}.$$

Use Lemma 43 applied to x and y to simplify the left-hand side of the equation:

$$\overline{y \cup \overline{\overline{x} \cup y \cup \overline{x \cup y}}} = \overline{x \cup y}.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 45. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} = \overline{\overline{x} \cup y}$.

Proof. Applying Lemma 43 to $x \cup y$ and $\overline{\overline{x} \cup y}$ yields:

$$\overline{\overline{\overline{x \cup y} \cup \overline{\overline{x} \cup y}} \cup \overline{x \cup y \cup \overline{\overline{x} \cup y}}} = \overline{\overline{x} \cup y}.$$

Use Lemma 43 applied to x and y to simplify the left-hand side of the equation:

$$\overline{y \cup \overline{x \cup y \cup \overline{\overline{x} \cup y}}} = \overline{\overline{x} \cup y}.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 46. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y \cup \overline{\overline{x} \cup y}} = y$.

Proof. Applying Lemma 43 to $\overline{\overline{x} \cup y} \cup x \cup y$ and y yields:

$$\overline{\overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{\overline{x} \cup y} \cup x \cup y \cup y}} = y.$$

Use Lemma 45 applied to x and y to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{x} \cup y} \cup \overline{\overline{\overline{x} \cup y} \cup x \cup y \cup y}} = y.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 47. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z}\cup\overline{y\cup z}} = z.$

Proof. Applying Lemma 43 to w and z yields:

$$\overline{\overline{\overline{w}\cup z}\cup\overline{w\cup z}} = z.$$

Let $w = \overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}$. Then by Lemma 46, $\overline{w} = y$, so we have:

$$\overline{\overline{y\cup z}\cup\overline{w\cup z}} = z.$$

which expands to:

$$\overline{\overline{y\cup z}\cup\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z}} = z.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 48. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}\cup z} = \overline{y\cup z}.$

Proof. Applying Lemma 43 to $\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z$ and $\overline{y\cup z}$ yields:

$$\overline{\overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z}\cup\overline{y\cup z}}\cup\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z\cup\overline{y\cup z}}} = \overline{y\cup z}.$$

Use Lemma 47 applied to x, y, and z to simplify the left-hand side of the equation:

$$\overline{z\cup\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}} = \overline{y\cup z}.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 49.

$$\mathrm{R} \vdash \overline{\overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}\cup z\cup u}\cup\overline{\overline{y\cup z}\cup u}} = u.$$

Proof. Applying Lemma 43 to w and u yields:

$$\overline{\overline{\overline{w}\cup u}\cup\overline{w\cup u}} = z.$$

Let $w = \overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}\cup z$. Then by Lemma 48, $\overline{w} = \overline{y\cup z}$, so we have:

$$\overline{\overline{\overline{y\cup z}\cup u}\cup\overline{w\cup u}} = u,$$

which expands to:

$$\overline{\overline{\overline{y\cup z}\cup u}\cup\overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{x\cup y}\cup\overline{y\cup z}\cup z}\cup z\cup u}}=u.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 50. $\mathrm{R}\vdash\overline{\overline{\overline{\overline{3x}\cup x}\cup\overline{3x}}\cup\overline{\overline{\overline{3x}\cup x}\cup 5x}}=\overline{\overline{3x}\cup x}$.

Proof. Applying Lemma 44 to $3x$ and $\overline{\overline{3x}\cup x\cup 2x}$ yields:

$$\overline{\overline{\overline{3x\cup\overline{\overline{3x}\cup x}\cup 2x}\cup\overline{3x}\cup\overline{\overline{3x}\cup x}\cup 2x}\cup\overline{\overline{3x}\cup x}\cup 2x}=\overline{3x\cup\overline{\overline{3x}\cup x}\cup 2x},$$

which simplifies to:

$$\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup 5x}\cup\overline{3x}\cup\overline{\overline{3x}\cup x}\cup 2x}\cup\overline{\overline{3x}\cup x}\cup 2x}=\overline{\overline{\overline{3x}\cup x}\cup 5x}. \tag{50.1}$$

Applying Lemma 49 to $3x$, x, $2x$, and $\overline{\overline{3x}\cup x}$ yields:

$$\overline{\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup 3x\cup x\cup x}\cup\overline{\overline{3x}\cup x}\cup\overline{x\cup 2x}\cup 2x}\cup 2x\cup\overline{\overline{3x}\cup x}}\cup x\cup\overline{\overline{x\cup 2x}\cup\overline{\overline{3x}\cup x}}}=\overline{\overline{3x}\cup x},$$

which simplifies to:

$$\overline{\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup 5x}\cup\overline{\overline{3x}\cup x}\cup\overline{3x}\cup 2x}\cup 2x\cup\overline{\overline{3x}\cup x}}\cup\overline{\overline{3x}\cup\overline{\overline{3x}\cup x}}}=\overline{\overline{3x}\cup x}.$$

Now use equation (50.1) to further simplify, yielding:

$$\overline{\overline{\overline{\overline{3x}\cup x}\cup 5x}\cup\overline{\overline{3x}\cup\overline{\overline{3x}\cup x}}}=\overline{\overline{3x}\cup x}.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 51. $\mathrm{R}\vdash\overline{\overline{\overline{3x}\cup x}\cup 5x}=\overline{3x}$.

Proof. Applying Lemma 43 to $\overline{\overline{3x}\cup x}\cup\overline{3x}$ and $\overline{\overline{\overline{3x}\cup x}\cup 5x}$ yields:

$$\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup\overline{3x}}\cup\overline{\overline{\overline{3x}\cup x}\cup 5x}}\cup\overline{\overline{\overline{3x}\cup x}\cup\overline{3x}\cup\overline{\overline{\overline{3x}\cup x}\cup 5x}}}=\overline{\overline{\overline{3x}\cup x}\cup 5x}.$$

Use Lemma 50 applied to x to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{3x}\cup x}\cup\overline{\overline{\overline{3x}\cup x}\cup\overline{3x}\cup\overline{\overline{\overline{3x}\cup x}\cup 5x}}}=\overline{\overline{\overline{3x}\cup x}\cup 5x}. \tag{51.1}$$

Applying Lemma 47 to $3x$, x, and $\overline{3x}$ yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 3x \cup x \cup x} \cup \overline{\overline{3x} \cup x} \cup \overline{3x}} \cup \overline{x \cup \overline{3x}}} = \overline{3x}. \tag{51.2}$$

The left-hand sides of equations (51.1) and (51.2) are AC identical. Therefore, $\overline{\overline{\overline{3x} \cup x} \cup 5x} = \overline{3x}$. ∎

Lemma 52. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{3x}} = \overline{\overline{3x} \cup x} \cup 2x$.

Proof. Applying Lemma 43 to $3x$ and $\overline{\overline{3x} \cup x} \cup 2x$ yields:

$$\overline{\overline{\overline{3x} \cup \overline{\overline{3x} \cup x} \cup 2x} \cup \overline{3x \cup \overline{\overline{3x} \cup x} \cup 2x}} = \overline{\overline{3x} \cup x} \cup 2x,$$

which simplifies to:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{\overline{3x} \cup x} \cup 5x} = \overline{\overline{3x} \cup x} \cup 2x.$$

Use Lemma 51 to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{3x}} = \overline{\overline{3x} \cup x} \cup 2x.$$

∎

Lemma 53. $\mathrm{R} \vdash \overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} = x$.

Proof. Applying Lemma 43 to $\overline{\overline{3x} \cup x} \cup 4x$ and x yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} \cup \overline{\overline{\overline{3x} \cup x} \cup 4x \cup x}} = x,$$

which simplifies to:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} \cup \overline{\overline{\overline{3x} \cup x} \cup 5x}} = x.$$

Use Lemma 51 applied to x to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} \cup \overline{3x}} = x. \tag{53.1}$$

Applying Lemma 45 to $3x$ and x yields:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup 3x \cup x} \cup x} = \overline{\overline{3x} \cup x},$$

which simplifies to:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} = \overline{\overline{3x} \cup x}. \tag{53.2}$$

Use equation (53.2) to simplify the left-hand side of equation (53.1), yielding:

$$\overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} = x.$$

□

Lemma 54. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup y} \cup \overline{x \cup y}} = y.$

Proof. Applying Lemma 43 to $\overline{\overline{3x} \cup x} \cup \overline{3x}$ and y yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} \cup y} \cup \overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup y}} = y.$$

Use Lemma 53 applied to x to simplify the left-hand side of the equation:

$$\overline{\overline{x \cup y} \cup \overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup y}} = y.$$

Commuting terms on the left-hand side yields the desired result. □

Theorem 55. $\mathrm{R} \vdash \mathrm{W}_1$

Proof. Applying Theorem 54 to x and $2x$ yields:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{x \cup 2x}} = 2x,$$

which simplifies to:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{3x}} = 2x.$$

Use this equation to simplify the left-hand side of Theorem 52, yielding:

$$2x = \overline{\overline{3x} \cup x} \cup 2x.$$

□

Theorem 56. $\mathrm{R} \vdash \mathrm{B}.$

Proof. By Theorems 29, 42, and 55. □

Chapter 2

A Single Axiom for Boolean Algebra

The simplest possible axiomatization of Boolean algebra would be a single equation. For some time, such axiomatizations have been known to exist. McCune *et al.* [6] write:

> In 1973, Padmanabhan and Quackenbush [7] presented a method of constructing a single axiom for any finitely based theory that has particular distributive and permutable congruences. Boolean algebra has these properties. However, straightforward application of the method usually yields a single axiom of enormous length (sometimes with tens of millions of symbols).

According to McCune *et al.* [6], the following short single axiom for Boolean algebra was discovered by "automatically generating and semantically filtering a great number of equations, then sending the surviving candidates to the theorem prover Otter to search for a proof of a known basis."

$$(\mathrm{DN}_1) \quad \overline{\overline{\overline{x \cup y} \cup z} \cup \overline{x \cup \overline{\overline{z} \cup \overline{z \cup u}}}} = z.$$

A simple computation shows that DN_1 holds in any Boolean algebra. Otter discovered the following proof that DN_1 proves R, presented with slight modifications.

2.1 $\mathrm{B} \vdash \mathrm{DN}_1$

Theorem 57. $\mathrm{B} \vdash \mathrm{DN}_1$.

Proof. First, rewrite DN_1 as $(\overline{x \cup y} \cup z) \cap (x \cup (z \cap (z \cup u)) = z$. Then

$$
\begin{aligned}
(\overline{x \cup y} \cup z) \cap (x \cup (z \cap (z \cup u)) &= (\overline{x \cup y} \cup z) \cap (x \cup z) && \text{(by } \mathrm{B}_3'\text{)} \\
&= (\overline{x \cup y} \cap x) \cup z && \text{(by } \mathrm{B}_4'\text{)} \\
&= (\overline{x} \cap \overline{y} \cap x) \cup z && \text{(by De Morgan)} \\
&= (0 \cap \overline{y}) \cup z && \text{(by } \mathrm{B}_5'\text{)} \\
&= 0 \cup z && \text{(by 4)} \\
&= z. && \text{(by 3)}
\end{aligned}
$$

□

2.2 $\mathrm{DN}_1 \vdash \mathrm{R}$

Lemma 58. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{\overline{\overline{z \cup u} \cup x} \cup \overline{\overline{y} \cup \overline{y \cup v}}}}} = y$.

Proof. Apply DN_1 to $\overline{\overline{z \cup u} \cup x}$, $\overline{z \cup \overline{\overline{x} \cup \overline{x \cup w}}}$, y, and v to get:

$$
\overline{\overline{\overline{\overline{\overline{z \cup u} \cup x} \cup \overline{z \cup \overline{\overline{x} \cup \overline{x \cup w}}}} \cup y} \cup \overline{\overline{\overline{z \cup u} \cup x} \cup \overline{\overline{y} \cup \overline{y \cup v}}}} = y.
$$

Then use DN_1 applied to z, u, x, and w to simplify the left-hand side. □

Lemma 59. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{z \cup x} \cup \overline{\overline{y} \cup \overline{y \cup u}}}} = y$.

Proof. Let $s = \overline{x \cup z}$ and $t = \overline{\overline{\overline{y \cup v} \cup x} \cup \overline{\overline{z} \cup \overline{z \cup w}}}$. Apply Lemma 58 to x, y, s, t and u to get:

$$
\overline{\overline{x \cup y} \cup \overline{\overline{\overline{s \cup t} \cup x} \cup \overline{\overline{y} \cup \overline{y \cup u}}}} = y.
$$

By Lemma 58, $\overline{s \cup t} = z$. □

Lemma 60. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup \overline{x}} \cup x} = \overline{x}$.

Proof. Apply Lemma 58 to x, $\overline{x}$, $\overline{\overline{x}}$, y, and $x \cup v$ to get:

$$
\overline{\overline{x \cup \overline{x}} \cup \overline{\overline{\overline{\overline{\overline{\overline{x}} \cup y} \cup x} \cup \overline{\overline{\overline{x}} \cup \overline{\overline{x} \cup x \cup v}}}}} = \overline{x}. \tag{60.1}
$$

Apply DN_1 to $\overline{\overline{x}}$, y, x, and v to get:

$$\overline{\overline{\overline{\overline{\overline{x}} \cup y} \cup x} \cup \overline{\overline{\overline{x}} \cup \overline{\overline{x} \cup \overline{x \cup v}}}} = x. \tag{60.2}$$

Use equation (60.2) to simplify equation (60.1). □

Lemma 61. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{z \cup x} \cup \overline{\overline{\overline{y \cup \overline{y}} \cup y} \cup \overline{y \cup u}}}} = y$.

Proof. Use Lemma 60 applied to y to expand $\overline{y}$ in Lemma 59. □

Lemma 62. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{z \cup x} \cup y}} = y$.

Proof. Apply Lemma 61 to x, y, z, and $\overline{\overline{y} \cup \overline{y \cup u}}$ to get:

$$\overline{\overline{x \cup y} \cup \overline{\overline{z \cup x} \cup \overline{\overline{\overline{y \cup \overline{y}} \cup y} \cup \overline{y \cup \overline{\overline{y} \cup \overline{y \cup u}}}}}} = y. \tag{62.1}$$

Apply DN_1 to y, $\overline{y}$, y, and u to get:

$$\overline{\overline{\overline{y \cup \overline{y}} \cup y} \cup \overline{y \cup \overline{\overline{y} \cup \overline{y \cup u}}}} = y. \tag{62.2}$$

Use equation (62.2) to simplify equation (62.1). □

Theorem 63. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{x} \cup y}} = y$.

Proof. Apply Lemma 62 to x, y, and $\overline{x \cup \overline{x}}$ to get:

$$\overline{\overline{x \cup y} \cup \overline{\overline{\overline{x \cup \overline{x}} \cup x} \cup y}} = y.$$

Then use Lemma 60 applied to x to simplify the left-hand side. □

Lemma 64. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{x \cup y} \cup x} \cup \overline{x \cup y}} = x$.

Proof. Apply Lemma 61 to $\overline{x \cup y}$, x, x, and y to get:

$$\overline{\overline{\overline{x \cup y} \cup x} \cup \overline{\overline{x \cup \overline{x \cup y}} \cup \overline{\overline{\overline{x \cup \overline{x}} \cup x} \cup \overline{x \cup y}}}} = x. \tag{64.1}$$

Apply Lemma 62 to x, $\overline{x \cup y}$, and $\overline{x \cup \overline{x}}$ to get:

$$\overline{\overline{x \cup \overline{x \cup y}} \cup \overline{\overline{\overline{x \cup \overline{x}} \cup x} \cup \overline{x \cup y}}} = \overline{x \cup y}. \tag{64.2}$$

Use equation (64.2) to simplify equation (64.1). □

Lemma 65. $\mathrm{DN}_1 \vdash \overline{x \cup \overline{\overline{x \cup y} \cup x}} = \overline{x \cup y}$.

Proof. Apply Lemma 64 to $\overline{x \cup y}$, and x to get:

$$\overline{\overline{\overline{\overline{x \cup y} \cup x} \cup \overline{x \cup y}} \cup \overline{\overline{x \cup y} \cup x}} = \overline{x \cup y}.$$

Use Lemma 64 to simplify the left-hand side. □

Lemma 66. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{x \cup y} \cup z} \cup \overline{x \cup z}} = z$.

Proof. Apply Lemma 62 to $\overline{x \cup y}$, z, and $\overline{\overline{x \cup y} \cup x}$ to get:

$$\overline{\overline{\overline{x \cup y} \cup z} \cup \overline{\overline{\overline{\overline{x \cup y} \cup x} \cup \overline{x \cup y}} \cup z}} = z.$$

Use Lemma 64 to simplify the left-hand side. □

Lemma 67. $\mathrm{DN}_1 \vdash \overline{x \cup \overline{\overline{y \cup z} \cup \overline{y \cup x}}} = \overline{y \cup x}$.

Proof. Apply Lemma 66 to $\overline{y \cup z}$, x, and $\overline{y \cup x}$ to get:

$$\overline{\overline{\overline{\overline{y \cup z} \cup x} \cup \overline{y \cup x}} \cup \overline{\overline{y \cup z} \cup \overline{y \cup x}}} = \overline{y \cup x}. \tag{67.1}$$

Apply Lemma 66 to y, z, and x to get:

$$\overline{\overline{\overline{y \cup z} \cup x} \cup \overline{y \cup x}} = x. \tag{67.2}$$

Use equation (67.2) to simplify equation (67.1). □

Lemma 68. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{\overline{x \cup y} \cup z} \cup \overline{\overline{x} \cup y}} \cup y} = \overline{\overline{x} \cup y}$.

Proof. Apply Lemma 66 to $\overline{x \cup y}$, z, and $\overline{\overline{x} \cup y}$ to get:

$$\overline{\overline{\overline{\overline{x \cup y} \cup z} \cup \overline{\overline{x} \cup y}} \cup \overline{\overline{x \cup y} \cup \overline{\overline{x} \cup y}}} = \overline{\overline{x} \cup y}.$$

Use Theorem 63 to simplify the left-hand side. □

Lemma 69. $\mathrm{DN}_1 \vdash \overline{x \cup \overline{\overline{y \cup z} \cup \overline{z \cup x}}} = \overline{z \cup x}$.

Proof. Apply Lemma 67 to x, z, and $\overline{\overline{\overline{y \cup u} \cup \overline{y \cup z}}}$ to get:

$$\overline{x \cup \overline{\overline{z \cup \overline{\overline{y \cup u} \cup \overline{y \cup z}}} \cup \overline{z \cup x}}} = \overline{z \cup x}. \tag{69.1}$$

Apply Theorem 67 to z, y and u to get:

$$\overline{z \cup \overline{\overline{y \cup u} \cup \overline{y \cup z}}} = \overline{y \cup z}. \tag{69.2}$$

Use equation (69.2) to simplify equation (69.1). □

Lemma 70. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{z \cup x} \cup \overline{\overline{y} \cup \overline{u \cup y}}}} = y$.

Proof. Apply Lemma 59 to x, y, z and $\overline{\overline{u \cup v} \cup \overline{u \cup y}}$ to get:

$$\overline{\overline{x \cup y} \cup \overline{\overline{z \cup x} \cup \overline{\overline{y} \cup \overline{y \cup \overline{\overline{u \cup v} \cup \overline{u \cup y}}}}}} = y. \tag{70.1}$$

Apply Theorem 67 to y, u and v to get:

$$\overline{y \cup \overline{\overline{u \cup v} \cup \overline{u \cup y}}} = \overline{u \cup y}. \tag{70.2}$$

Use equation (70.2) to simplify equation (70.1). □

Proposition 71. $\mathrm{DN}_1 \vdash \overline{x \cup y} = \overline{y \cup x}$.

Proof. Apply Lemma 69 to x, $\overline{y \cup x}$, and y to get:

$$\overline{x \cup \overline{\overline{\overline{y \cup x} \cup y} \cup \overline{y \cup x}}} = \overline{y \cup x}. \tag{71.1}$$

Apply Theorem 64 to y and x to get:

$$\overline{\overline{\overline{y \cup x} \cup y} \cup \overline{y \cup x}} = y. \tag{71.2}$$

Use equation (71.2) to simplify equation (71.1). □

Lemma 72. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{x \cup y} \cup \overline{y \cup z}} \cup z} = \overline{y \cup z}$.

Proof. By Proposition 71 and Lemma 69 applied to z, x, and y,

$$\overline{\overline{\overline{x \cup y} \cup \overline{y \cup z}} \cup z} = \overline{z \cup \overline{\overline{x \cup y} \cup \overline{y \cup z}}} = \overline{y \cup z}.$$

□

Lemma 73. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup \overline{\overline{x \cup y} \cup z}} \cup z} = \overline{\overline{x \cup y} \cup z}$.

Proof. Apply Lemma 72 to $\overline{\overline{x \cup y} \cup x}$, $\overline{x \cup y}$, and z to get:

$$\overline{\overline{\overline{\overline{\overline{x \cup y} \cup x} \cup \overline{x \cup y}} \cup \overline{\overline{x \cup y} \cup z}} \cup z} = \overline{\overline{x \cup y} \cup z}.$$

Use Lemma 62 to simplify the left-hand side. □

Lemma 74. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{x \cup y} \cup x} \cup y} = \overline{y \cup y}$.

Proof. Apply Lemma 73 to $\overline{x \cup y}$, x, and y to get:

$$\overline{\overline{\overline{x \cup y} \cup \overline{\overline{\overline{x \cup y} \cup x} \cup y}} \cup y} = \overline{\overline{\overline{x \cup y} \cup x} \cup y}. \tag{74.1}$$

Apply Lemma 62 to x, y, and $\overline{x \cup y}$ to get:

$$\overline{\overline{x \cup y} \cup \overline{\overline{\overline{x \cup y} \cup x} \cup y}} = y. \tag{74.2}$$

Use equation (74.2) to simplify equation (74.1). □

Proposition 75. $\mathrm{DN}_1 \vdash \overline{\overline{x} \cup \overline{x \cup y}} = x$.

Proof. Apply Lemma 70 to y, x, $\overline{x}$, and y to get:

$$\overline{\overline{y \cup x} \cup \overline{\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y \cup x}}}} = x.$$

Apply Lemma 67 to $\overline{y \cup x}$, $\overline{x}$, and y, then use Proposition 71 to get:

$$\overline{\overline{y \cup x} \cup \overline{\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y \cup x}}}} = \overline{\overline{x} \cup \overline{y \cup x}} = \overline{\overline{x} \cup \overline{x \cup y}}.$$

□

Lemma 76. $\mathrm{DN}_1 \vdash \overline{x \cup \overline{y \cup \overline{x}}} = \overline{x}$.

Proof. Apply Lemma 66 to y, x, and $\overline{x}$ to get:

$$\overline{\overline{\overline{y \cup x} \cup \overline{x}} \cup \overline{y \cup \overline{x}}} = \overline{x}.$$

Use Propositions 71 and 75 to simplify the left-hand side. □

Proposition 77. $\mathrm{DN}_1 \vdash \overline{x \cup x} = \overline{x}$.

Proof. Apply Lemma 76 to x and $\overline{y \cup x}$ to get:

$$\overline{x \cup \overline{\overline{y \cup x} \cup \overline{x}}} = \overline{x}.$$

Use Propositions 71 and 75 to simplify the left-hand side. □

Lemma 78. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{x \cup y} \cup x} \cup y} = \overline{y}$.

Proof. Apply Lemma 77 to y to get $\overline{y \cup y} = \overline{y}$. Then use Lemma 74 to expand the left-hand side. □

Proposition 79. $\mathrm{DN}_1 \vdash \overline{\overline{x}} = x$.

Proof. Apply Lemma 78 to x and $\overline{x}$ to get:

$$\overline{\overline{\overline{x \cup \overline{x}} \cup x} \cup \overline{x}} = \overline{\overline{x}}.$$

Apply Proposition 75 to x and $\overline{x \cup \overline{x}}$ to get:

$$\overline{\overline{x} \cup \overline{x \cup \overline{x \cup \overline{x}}}} = x.$$

By Proposition 71 the left-hand sides of these two equations are equal. □

Corollary 80. $\mathrm{DN}_1 \vdash \overline{x} = \overline{y} \to x = y$.

Proof. If $\overline{x} = \overline{y}$, then by Proposition 79, $x = \overline{\overline{x}} = \overline{\overline{y}} = y$. □

Theorem 81. $\mathrm{DN}_1 \vdash x \cup y = y \cup x$.

Proof. By Proposition 71 and Corollary 80. ☐

Lemma 82. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup x} \cup y = y$.

Proof. By Lemma 78 and Corollary 80. ☐

Lemma 83. $\mathrm{DN}_1 \vdash \overline{x \cup \overline{y}} \cup \overline{\overline{y} \cup y} = \overline{x \cup \overline{y}}$.

Proof. Apply Lemma 82 to $\overline{x \cup \overline{y}}$ and y to get:

$$\overline{\overline{y \cup \overline{x \cup \overline{y}}} \cup y} \cup \overline{x \cup \overline{y}} = \overline{x \cup \overline{y}}. \tag{83.1}$$

Apply Lemma 76 to y and x to get:

$$\overline{y \cup \overline{\overline{x \cup \overline{y}}}} = \overline{y}. \tag{83.2}$$

Use equation (83.2) to simplify equation (83.1), then use Theorem 81 to commute terms on the left-hand side. ☐

Lemma 84. $\mathrm{DN}_1 \vdash \overline{x \cup y} \cup \overline{\overline{y} \cup y} = \overline{x \cup y}$.

Proof. Apply Lemma 82 to $\overline{y}$ and $\overline{x \cup y}$ to get:

$$\overline{\overline{\overline{y} \cup \overline{x \cup y}} \cup \overline{y}} \cup \overline{x \cup y} = \overline{x \cup y}.$$

Use Proposition 75 and Theorem 81 to simplify the left-hand side. ☐

Lemma 85. $\mathrm{DN}_1 \vdash \overline{(x \cup \overline{y}) \cup y} = \overline{\overline{y} \cup y}$.

Proof. Apply Lemma 72 to x, $\overline{y}$, and y to get:

$$\overline{\overline{\overline{x \cup \overline{y}} \cup \overline{\overline{y} \cup y}} \cup y} = \overline{\overline{y} \cup y}.$$

Use Propositions 83 and 79 to simplify the left-hand side. ☐

Lemma 86. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{(x \cup \overline{y}) \cup z} \cup y} \cup \overline{\overline{y} \cup y}} = y$.

Proof. Apply Lemma 66 to $x \cup \overline{y}$, z, and y to get:

$$\overline{\overline{\overline{(x \cup \overline{y}) \cup z} \cup y} \cup \overline{(x \cup \overline{y}) \cup y}} = y.$$

Use Lemma 85 to simplify the left-hand side. □

Lemma 87. $\mathrm{DN}_1 \vdash x \cup \overline{\overline{y \cup z} \cup \overline{y \cup x}} = y \cup x$.

Proof. By Proposition 67 and Corollary 80. □

Lemma 88. $\mathrm{DN}_1 \vdash x \cup \overline{y \cup \overline{\overline{\overline{z \cup y} \cup x}}} = \overline{z \cup y} \cup x$.

Proof. Apply Lemma 87 to x, $\overline{z \cup y}$, and $\overline{y}$ to get:

$$x \cup \overline{\overline{\overline{z \cup y} \cup \overline{y}} \cup \overline{\overline{z \cup y} \cup x}} = \overline{z \cup y} \cup x.$$

Use Theorem 81 and Proposition 75 to simplify the left-hand side. □

Lemma 89. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup \overline{\overline{\overline{x \cup y} \cup \overline{x \cup z}}}} \cup y = y$.

Proof. Apply Lemma 82 to $\overline{\overline{x \cup y} \cup \overline{x \cup z}}$ and y to get:

$$\overline{\overline{\overline{\overline{x \cup y} \cup \overline{x \cup z}} \cup y} \cup \overline{\overline{x \cup y} \cup \overline{x \cup z}}} \cup y = y. \tag{89.1}$$

Apply Lemma 87 to y, x, and z to get:

$$y \cup \overline{\overline{x \cup z} \cup \overline{x \cup y}} = x \cup y. \tag{89.2}$$

Use Theorem 81 and equation (89.2) to simplify equation (89.1). □

Lemma 90. $\mathrm{DN}_1 \vdash x \cup \overline{(y \cup \overline{x}) \cup z} = x$.

Proof. Apply Lemma 86 to y, x, and z to get:

$$\overline{\overline{\overline{(y \cup \overline{x}) \cup z} \cup x} \cup \overline{\overline{x} \cup x}} = x.$$

Apply Lemma 84 to $\overline{(y \cup \overline{x}) \cup z}$ and x to get:

$$\overline{\overline{(y \cup \overline{x}) \cup z} \cup x} \cup \overline{\overline{x} \cup x} = \overline{\overline{(y \cup \overline{x}) \cup z} \cup x}.$$

Therefore, by Theorem 81 and Proposition 79,

$$x \cup \overline{(y \cup \overline{x}) \cup z} = \overline{(y \cup \overline{x}) \cup z} \cup x = \overline{\overline{\overline{(y \cup \overline{x}) \cup z} \cup x}} = x.$$

$\square$

Lemma 91. $\mathrm{DN}_1 \vdash \overline{x} \cup \overline{(y \cup x) \cup z} = \overline{x}$.

Proof. Apply Lemma 90 to $\overline{x}$, y, and z to get:

$$\overline{x} \cup \overline{(y \cup \overline{\overline{x}}) \cup z} = \overline{x}$$

then simplify with Proposition 79. $\square$

Lemma 92. $\mathrm{DN}_1 \vdash \overline{x \cup y} \cup x = x \cup \overline{y}$.

Proof. Use Theorem 81 and Lemma 88 applied to x, y, and x to get:

$$x \cup \overline{\overline{\overline{x \cup y} \cup x} \cup y} = x \cup \overline{y \cup \overline{\overline{x \cup y} \cup x}} = \overline{x \cup y} \cup x.$$

Use Lemma 82 to simplify the left-hand side. $\square$

Lemma 93. $\mathrm{DN}_1 \vdash \overline{x \cup \overline{x \cup \overline{y}}} = \overline{x \cup y}$.

Proof. Use Lemma 92 to simplify Lemma 65. $\square$

Lemma 94. $\mathrm{DN}_1 \vdash \overline{\overline{x \cup y} \cup (x \cup z)} \cup y = y$.

Proof. Apply Lemma 93 to $\overline{x \cup y}$ and $x \cup z$ to get:

$$\overline{\overline{x \cup y} \cup \overline{\overline{x \cup y} \cup \overline{(x \cup z)}}} = \overline{\overline{x \cup y} \cup (x \cup z)}.$$

Use this equation to simplify Lemma 89. $\square$

Lemma 95. $\mathrm{DN}_1 \vdash \overline{\overline{\overline{x \cup y} \cup z} \cup \overline{\overline{x} \cup y}} \cup y = \overline{x} \cup y$.

Proof. By Lemma 68 and Proposition 79. $\square$

Lemma 96. $\mathrm{DN}_1 \vdash \overline{\overline{x} \cup \overline{(y \cup x) \cup (y \cup z)}} \cup (y \cup z) = (y \cup x) \cup (y \cup z)$.

Proof. Apply Lemma 95 to $\overline{y \cup x}$, $y \cup z$, and x to get:

$$\overline{\overline{\overline{\overline{y \cup x} \cup (y \cup z)} \cup x} \cup \overline{\overline{\overline{y \cup x}} \cup (y \cup z)}} \cup (y \cup z) = \overline{\overline{y \cup x}} \cup (y \cup z). \tag{96.1}$$

Apply Lemma 94 to y, x, and z to get:

$$\overline{\overline{y \cup x} \cup (y \cup z)} \cup x = x. \tag{96.2}$$

Use equation (96.2) and Proposition 79 to simplify equation (96.1). □

Lemma 97. $\mathrm{DN}_1 \vdash x \cup (y \cup z) = (y \cup x) \cup (y \cup z)$.

Proof. Apply Lemma 91 to x, y, and $y \cup z$ to get:

$$\overline{x} \cup \overline{(y \cup x) \cup (y \cup z)} = \overline{x}.$$

Use this equation and Proposition 79 to simplify equation Lemma 96. □

Lemma 98. $\mathrm{DN}_1 \vdash z \cup (y \cup x) = (y \cup z) \cup (y \cup x)$.

Proof. Apply Lemma 97 to z, y, and x to get:

$$z \cup (y \cup x) = (y \cup z) \cup (y \cup x).$$

□

Theorem 99. $\mathrm{DN}_1 \vdash x \cup (y \cup z) = (x \cup y) \cup z$.

Proof. By Lemma 97, Theorem 81, and Lemma 98 we have:

$$x \cup (y \cup z) = (y \cup x) \cup (y \cup z) = (y \cup z) \cup (y \cup x) = z \cup (y \cup x) = (x \cup y) \cup z.$$

□

Theorem 100. $\mathrm{DN}_1 \vdash \mathrm{R}$.

Proof. By Theorems 63, 81, and 99. □

Chapter 3

How Otter and EQP work

OTTER (Organized Techniques for Theorem-proving and Effective Reasoning) and EQP (Equational Theorem Prover) are automated theorem proving programs for first-order logic with equality developed by the Mathematics and Computer Science Division of Argonne National Laboratory. They are coded in the C programming language and run mostly on UNIX systems, but there are versions for both Macintosh and Windows.

Our presentation follows Wos *et al.* [10].

3.1 A Language for Automated Theorem Provers

A **literal** is an n-ary predicate symbol together with its arguments, possibly negated. An unnegated literal is called a **positive literal**. A negated literal is called a **negative literal**. A **clause** is a disjunction of literals. A clause containing only positive literals is called a **positive clause**. A clause containing only negative literals is called a **negative clause**. A clause containing both positive and negative literals is called a **mixed clause**. A clause containing a single literal is called a **unit clause**. A clause containing more than one literal is called a **nonunit clause**. A set of clauses is treated as the conjunction of the clauses [10, pages 191–192].

Duplicate literals are automatically removed from clauses since the proposition $\mathrm{P} \vee \mathrm{P}$ is logically equivalent to P [10, page 193].

All variables are universally quantified. Existentially quantified variables are replaced by functions (or constants). We will use the lowercase letters u, v, w, x, y, and z to denote variables [10, page 193].

Equality is treated as a special binary predicate that obeys the laws of equality. Thus an equation is a positive literal [10, page 192].

3.2 Inference Rules

An inference rule is an algorithm that, when successfully applied to a set of clauses (the premises), yields a new clause (the conclusion) that follows logically from the premises. OTTER and EQP have several inference rules at their disposal. We will discuss only those that were used to prove the results presented in this thesis.

3.2.1 Unification

Two literals can be **unified** if there exists a substitution that makes them identical, except possibly for sign. For example, consider the set of clauses:

```
MALE(x) | FEMALE(x).
-MALE(Alice).
```

where `|` is the symbol for disjunction and `-` is the symbol for negation. The literals `MALE(x)` and `-MALE(Alice)` can be unified by substituting the constant `Alice` for the variable `x`, ultimately yielding the conclusion `FEMALE(Alice)` [10, pages 197–198].

Unification is recursive. There is an algorithm due to Robinson that, when given two terms as inputs, will produce a substitution that makes them identical if possible, and will fail otherwise. Moreover, the substitution produced is unique up to renaming of variables and is the simplest substitution that will work. Our presentation of the Unification algorithm follows [1, pages 139–145].

Definition. The **length** of a term t is the number of nodes in its tree representation.

For example, the term $f(x, g(x, y))$ has a length of 5.

Definition. A **substitution** is a function $\sigma : Variable \to Term$, where $Variable$ is the set of variables and $Term$ is the set of terms of our language. We will use postfix notation for substitutions, i.e., if σ is a substitution that maps the variable x to the term t, we will write $x\sigma = t$. We will use the symbol ϵ for the identity substitution. Substitutions can be composed as functions, so that $x(\sigma\tau) = (x\sigma)\tau$. We can extend σ to all terms by setting

$$
\begin{aligned}
c\sigma &= c && \text{if } c \text{ is a constant,} \\
[f(t_1, \ldots, t_n)]\sigma &= f(t_1\sigma, \ldots, t_n\sigma) && \text{if } f \text{ is an } n\text{-ary function symbol.}
\end{aligned}
$$

Definition. The **set of support** of a substitution σ is the set of variables $\{x : x\sigma \neq x\}$. A substitution has **finite support** if its set of support is finite. If a substitution σ has finite support and maps the variable x_1 to the term t_1, the variable x_2 to to the term t_2, and so on, we will use the notation $\sigma = (x_1 \mapsto t_1, \ldots, x_n \mapsto t_n)$.

Definition. The **range of variables** of a substitution σ is the set of variables occuring as subterms of terms in the range of σ.

Definition. A substitution σ is called a **renaming of variables** if σ is injective and for every variable x the term $x\sigma$ is a variable.

Definition. Two terms t_1 and t_2 are **isomorphic** if there exists a renaming of variables σ such that $t_1\sigma = t_2$.

Proposition 101. *If t is a term and σ is a substitution, then $length(t) \leq length(t\sigma)$.*

Lemma 102. *Let σ be a substitution. If there exists another substitution σ' such that $\sigma\sigma' = \epsilon$, then σ is a renaming of variables.*

Proof. Let x and y be variables. If $x\sigma = y\sigma$, then $x = x\sigma\sigma' = y\sigma\sigma' = y$. Therefore σ is injective. The term $x\sigma$ cannot be a constant, because then $x = x\sigma\sigma'$ would also be a constant. Furthermore, the term $x\sigma$ cannot be of the form $f(\cdots)$ for some function symbol f, because then we would have

$$length(x) < length(x\sigma) \leq length(x\sigma\sigma') = length(x).$$

□

Definition. Let σ_1 and σ_2 be substitutions. Define $\sigma_1 \lesssim \sigma_2$ if there exists a substitution τ such that $\sigma_1\tau = \sigma_2$. If $\sigma_1 \lesssim \sigma_2$ we will say that σ_1 is more general than σ_2. If $\sigma_1 \lesssim \sigma_2$ and $\sigma_2 \lesssim \sigma_1$ we will write $\sigma_1 \sim \sigma_2$ and say that σ_1 and σ_2 are equally general.

Proposition 103. *The relation $\lesssim$ is a quasi-order.*

Proof. The relation $\lesssim$ is reflexive because for every substitution σ we have $\sigma\epsilon = \sigma$. Let σ_1, σ_2, and σ_3 be substitutions. If $\sigma_1 \lesssim \sigma_2$ and $\sigma_2 \lesssim \sigma_3$ then there exist substitutions τ and τ' such that $\sigma_1\tau = \sigma_2$ and $\sigma_2\tau' = \sigma_3$. Hence $\sigma_1\tau\tau' = \sigma_3$. Thus $\sigma_1 \lesssim \sigma_3$. Therefore $\lesssim$ is transitive. □

Proposition 104. *If $\sigma_1 \sim \sigma_2$ then there exists a renaming of variables τ on the range of variables of σ_1 such that $\sigma_1\tau = \sigma_2$.*

Proof. If $\sigma_1 \sim \sigma_2$ then there exist substitutions τ and τ' such that $\sigma_1\tau = \sigma_2$ and $\sigma_2\tau' = \sigma_1$. Hence $\sigma_1\tau\tau' = \sigma_1$. Thus $\tau\tau'$ is the identity substitution on the range of variables of σ_1. Therefore by Lemma 102, τ is a renaming of variables on the range of variables of σ_1. □

Definition. A substitution σ **unifies** two terms t_1 and t_2 if $t_1\sigma = t_2\sigma$. Such a substitution is called a **unifier** of t_1 and t_2. Two terms are **unifiable** if they have a unifier. A substitution σ is a **most general unifier** of t_1 and t_2 if σ unifies t_1 and t_2 and is more general than every other unifier.

Definition. Given two distinct terms t_1 and t_2, a **disagreement pair** d_1, d_2 is a pair of terms such that d_1 is a subterm of t_1, d_2 is a subterm of t_2, the symbols at the root of their respective tree representations are distinct, but the paths leading from the root of t_1 to the root of d_1 and from the root of t_2 to the root of d_2 are the same.

Unification Algorithm.

```
let σ := ε;
while t₁σ ≠ t₂σ do
    begin
    choose a disagreement pair, d₁, d₂ for t₁σ, t₂σ;
    if neither d₁ nor d₂ is a variable then FAIL;
    let x be whichever of d₁, d₂ is a variable
        (if both are, choose one),
        and let t be the other one of d₁, d₂;
    if x occurs in t then FAIL;
    let σ := σ(x ↦ t)
    end.
```

Lemma 105. *Suppose a substitution τ unifies two terms u and v that have a disagreement pair d_1, d_2. Then* (1) *at least one of d_1 and d_2 is a variable that does not occur in the other. Call the variable x and the other term t. Then* (2) $(x \mapsto t)\tau = \tau$.

Proof. Since u and v are unifiable we must have $d_1\tau = d_2\tau$. If d_1 and d_2 were distinct constants we would have $d_1\tau = d_1 \neq d_2 = d_2\tau$. If d_1 were a constant and d_2 had the form $f(\cdots)$ for some function symbol f we would have $\mathit{length}(d_1\tau) = \mathit{length}(d_1) < \mathit{length}(d_2) \leq \mathit{length}(d_2\tau)$. If d_1 had the form $f(\cdots)$ and d_2 had the form $g(\cdots)$ for distinct function symbols f and g we would have $d_1\tau = f(\cdots\tau) \neq g(\cdots\tau) = d_2$. Therefore at least one of d_1 and d_2 must be a variable. Let x be the variable and t the other term. If t is a constant or a variable other than x, then obviously x does not occur in t. Suppose t has the form $f(\cdots)$. If x were a proper part of t, then $x\tau$ would be a proper part of $t\tau$. Therefore x does not occur in t.

To show that $(x \mapsto t)\tau = \tau$ it suffices to show they have the same effect on every variable. If y is a variable other than x, then $y(x \mapsto t)\tau = y\tau$, and $x(x \mapsto t)\tau = t\tau = x\tau$. □

Theorem 106 (Unification Theorem). *Let t_1 and t_2 be terms. If t_1 and t_2 are not unifiable, the unification algorithm will FAIL. If t_1 and t_2 are unifiable, the unification algorithm will terminate without FAILure, and the final value of σ will be a most general unifier for t_1 and t_2.*

Proof. First, we will show that the algorithm always terminates. Let $S(\sigma)$ be the set of variables that occur in either $t_1\sigma$ or $t_2\sigma$. Each pass through the while loop that does not FAIL decreases the size of $S(\sigma)$ by at least 1 (because x is replaced by a term t that cannot contain occurrences of x). Since t_1 and t_2 have only a finite number of variables, termination is ensured.

If the algorithm does not FAIL, then it must terminate because of the while loop condition, that is, because $t_1\sigma = t_2\sigma$. Now suppose t_1 and t_2 are unifiable, and let τ be any unifier. We must show that the algorithm does not FAIL, and that the unifier σ produced is more general than τ.

Consider the statement: $\sigma\tau = \tau$. When the while loop is first encountered this statement is true, because σ is the identity substitution. If we show the statement is loop invariant, then it will be true when the loop terminates, which will show that σ is more general than τ. If we also show the loop can not terminate because of FAILure, we are done.

Suppose we are the beginning of the loop body, and for the current value of σ, $t_1\sigma \neq t_2\sigma$, and $\sigma\tau = \tau$. Then $t_1\sigma$ and $t_2\sigma$ are unifiable because $t_1\sigma\tau = t_1\tau = t_2\tau = t_2\sigma\tau$. Let d_1, d_2 be a disagreement pair for t_1, t_2. By Lemma 105, at least one of d_1 and d_2 must be a variable that does not occur in the other. Therefore we do not exit the while loop because of the first FAIL condition. Let x be the variable and t the other term. Since x does not occur in t, we do not exit the while loop because of the second FAIL condition either. Then we must execute the assignment statement at the bottom of the loop. Let us denote the new value of σ by σ', and continue to use σ for the old value;

thus $\sigma' = \sigma(x \mapsto t)$. By Lemma 105, $\sigma'\tau = \sigma(x \mapsto t)\tau = \sigma\tau = \tau$. □

3.2.2 Paramodulation

Paramodulation is a type of equality substitution that procedes as follows. First, select two clauses, one of which must contain a positive equality literal (i.e. an equation). The clause containing the positive equality literal is called the **from clause**. The other clause is called the **into clause**. Select one of the positive equality literals contained in the from clause, and choose one of its arguments (i.e. one side of the equation). Next, select a term of the into clause and attempt to unify it with the selected argument. If unification is successful, replace the unified term with the unselected argument of the positive equality literal (i.e. the other side of the equation). For example, suppose we are given the equations:

```
n(n(x) + n(y)) + n(n(x) + y) = x.
x + x = x.
```

Select the second equation as the from clause and the first equation as the into clause. Choose the left-hand side of the from clause (`x + x`) and the term `n(x) + n(y)` of the into clause. We can unify the chosen argument and term by substituting `x` for the variable `y` in the into clause, and `n(x)` for `x` in the from clause, yielding:

```
n(n(x) + n(x)) + n(n(x) + x) = x.
n(x) + n(x) = n(x).
```

Now substitute `n(x)` for `n(x) + n(x)` in the into clause, yielding:

```
n(n(x)) + n(n(x) + x) = x.
```

The final step of paramodulation is to form the disjunction of all literals in the from and into clauses, except the selected positive equality literal [10, pages 201–203].

3.2.3 Demodulation

Demodulation is the process of rewriting expressions using unit equality clauses (equations) designated for this purpose, called **demodulators**. Typically, all terms of all newly generated clauses are examined for possible demodulation. A term is demodulated if it can be unified with one of the arguments of a demodulator, in which case the term is replaced by the other argument of the demodulator. The original clause is then discarded, being replaced by the demodulated clause [10, page 206].

The set of demodulators can change over time, depending on the instructions given to the progam. When a new unit equality clause (equation) is added to the set of demodulators, all previously retained clauses are examined for possible demodulation by the new demodulator. Such a process is called **back demodulation**. Typically, a successful demodulation is followed immediately by further attempts to apply other demodulators [10, pages 206–207]. For example, the equation:

```
n(n(x + y)) = n(n(y + x)).
```

can be simplified by two applications of the demodulator:

```
n(n(x)) = x.
```

First, substitute `x + y` for the variable `x` in the demodulator to get:

```
n(n(x + y)) = x + y.
```

then unify with the left-hand side of the original equation, yielding:

```
x + y = n(n(y + x)).
```

Second, substitute `y + x` for the variable `x` in the demodulator to get:

```
n(n(y + x)) = y + x.
```

then unify with the right-hand side of the previously demodulated equation to obtain:

```
x + y = y + x.
```

Demodulation is closely related to paramodulation. Both are forms of equality substitution. Let us highlight the differences. First, demodulation requires the positive equality literal to belong to a unit clause, while paramodulation does not. Second, demodulation allows substitution only in the demodulator (the "from" clause), whereas paramodulation allows substitution in both the from and the into clause. Third, demodulation discards the original clause; paramodulation retains both the original clause and the new clause. Fourth, successful demodulation immediately triggers further demodulation attempts, while paramodulation stops after one equality substitution [10, page 207–208].

3.3 AC Unification

EQP is a variant of OTTER. The two major differences between them are that EQP is restricted to first-order equational logic, and that EQP has associative-commutative (AC) unification built in to the inference process. That is, all binary operations are assumed to be associative and commutative, so that (in the case of +) the equations:

```
x + (y + z) = (x + y) + z.
x + y = y + x.
```

need not be present as explicit axioms [5].

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