Lab 2: Numerical Integration and Special Functions

In this lab, we will investigate functions whose anti-derivatives cannot be expressed in terms of elementary functions. We will also see how to approximate the value of these "special functions" using numerical integration.

1. Use **Wolfram Alpha** to find the anti-derivative of each of the following functions. Write down the anti-derivative as well as the definition of each special function you encounter.

   - e^{-x^2}
   - $\dfrac{\sin x}{x}$
   - $\dfrac{\cos x}{x}$
   - $\sin x^2$
   - $\cos\left(\frac{\pi}{2}x^2\right)$
   - $\tan x^3$
   - $\dfrac{e^x}{x}$

2. Choose a combination of elementary functions (similar to those above) and use Wolfram Alpha to find its anti-derivative.

3. Now use Wolfram Alpha to approximate $\text{erf}(1)$ by entering to following commands. Find an upper bound for the error in each approximation. How does it compare to the actual error?

   - `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
     `using left endpoint method with 5 intervals`
   - `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
     `using right endpoint method with 5 intervals`
   - `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
     `using midpoint method with 5 intervals`
   - `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
     `using trapezoidal rule with 5 intervals`

4. Use the midpoint method to estimate $\text{erf}(1)$ so that the absolute error is less than 10^{-6}. How many intervals did you need to use (in theory and in practice)?

5. Now use Simpson's rule to estimate $\text{erf}(1)$ so that the absolute error is less than 10^{-6}. Start by entering the following commands:

   - `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
     `using Simpson's rule with 1 interval`

- `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
  `using Simpson's rule with 2 intervals`
- `integrate {2/sqrt(pi)}*e^{-x^2} on [0,1]`
  `using Simpson's rule with 3 intervals`

How many intervals did you need to use (in theory and in practice)? [Note: The number n in the error bound formula is twice the number of intervals according to Wolfram Alpha.]

6. Use Wolfram Alpha to evaluate $\mathrm{erf}(1)$, $\mathrm{erf}(2)$, $\mathrm{erf}(3)$, and $\mathrm{erf}(4)$ to eight decimal places each.

7. What is $\lim_{x\to\infty} \mathrm{erf}(x)$?

8. Finally, enter the following command:

   - `integrate x^{1/3}*e^{-x} on [0,infty]`

   What special function do you encounter? How is it defined?

9. Calculate $\Gamma(n)$ for $n = 1, \ldots, 6$.

10. Predict the value of $\Gamma(7)$ and $\Gamma(8)$ before calculating the values using Wolfram Alpha. Were you predictions correct?