Independence-Friendly Cylindric Set Algebras

by

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Independence-friendly logic is a conservative extension of first-order logic that has the same expressive power as existential second-order logic. In her Ph.D. thesis, Dechesne introduces a variant of independence-friendly logic called IFG logic. We attempt to algebraize IFG logic in the same way that Boolean algebra is the algebra of propositional logic and cylindric algebra is the algebra of first-order logic.

We define independence-friendly cylindric set algebras and prove three main results. First, the concept of "iff" is not expressible in IFG logic. Second, every independence-friendly cylindric set algebra over a structure has an underlying Kleene algebra. Moreover, the class of such underlying Kleene algebras generates the variety of all Kleene algebras. Hence the equational theory of the class of Kleene algebras that underly an independence-friendly cylindric set algebra is finitely axiomatizable. Third, every one-dimensional independence-friendly cylindric set algebra over a structure has an underlying monadic Kleene algebra. However, the class of such underlying monadic Kleene algebras does not generate the variety of all monadic Kleene algebras. Finally, we offer a conjecture about which subvariety of monadic Kleene algebras the class of such monadic Kleene algebras does generate.

Dedication

To Paul and Lorraine

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IFG Logic

1.1   Introduction

Imagine a sentence of first-order logic. Most likely, it has an initial block of quantifiers. If there are multiple quantifiers in the block, then some of the quantifiers are dependent on other quantifiers. In fact, there is only one possible dependence relation among the quantifiers: later quantifiers depend on prior quantifiers. The first attempt to allow dependence relations other than the usual one on the quantifiers of a first-order sentence was made by Henkin [10]. He allowed the dependence relation to be a partial order, rather than a linear order. For example, in the Henkin sentence

$$\left(\begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array} \right) \phi(x, y, z, w)$$

the variable y depends only on x, while w depends only on z. It is a result due to Ehrenfeucht [10] that the above quantifier is not definable in ordinary first-order logic. Later it was shown independently by Enderton [8] and Walkoe [25] that first-order logic with these "branching quantifiers" has the same expressive power as existential second-order logic.

Independence-friendly logic (IF logic) was introduced by Hintikka and Sandu [14] as a way to allow arbitrary dependence relations between the quantifiers of a first-order sentence. Independence-friendly logic is a conservative extension of ordinary first-order logic in that every ordinary first-order sentence has an independence-friendly counterpart that is true in exactly the same models. However, the ability to specify arbitrary dependence relations between the

quantifiers (and even the connectives) means that there are far more sentences than before. It turns out that a given independence-friendly sentence has the same expressive power as a pair of existential second-order sentences. In The Principles of Mathematics Revisited [13], Hintikka argues that IF logic is the correct first-order logic and advocates for its adoption as the foundation of mathematics.

In IF logic the truth of a sentence (or formula) is defined in terms of games. Let ϕ be a sentence, and let $\mathfrak{A}$ be a suitable structure. Informally, the semantic game $G(\mathfrak{A}, \phi)$ is played between two players, player 0 (Abélard) and player 1 (Eloïse). Eloïse's goal is to verify the sentence ϕ in the structure $\mathfrak{A}$, while Abélard's goal is to falsify it. A familiar example from calculus is the definition of continuity:

$$\forall x \forall \varepsilon (\varepsilon > 0 \rightarrow \exists \delta (\delta > 0 \wedge \forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon))).$$

Given a function f, Abélard picks an x and an ε. If $\varepsilon \leq 0$, Eloïse wins. If $\varepsilon > 0$, Eloïse chooses a δ. If $\delta \leq 0$, Abélard wins. If $\delta > 0$, Abélard chooses a y. If $|x - y| \geq \delta$, Eloïse wins. Otherwise, Eloïse wins if $|f(x) - f(y)| < \varepsilon$, and Abélard wins if $|f(x) - f(y)| \geq \varepsilon$.

Notice that Eloïse winning a single play of the game is not sufficient to guarantee that f is continuous. Likewise, Abélard winning a single play is not sufficient to show that f is not continuous. What is required for f to be continuous is for Eloïse to have a way to win every play of the game, given correct play by both players—that is, Eloïse must have a winning strategy. Dually, f is not continuous if and only if Abélard has a winning strategy. By extension, a first-order sentence is true in a model $\mathfrak{A}$ if and only if Eloïse has a winning strategy for the game $G(\mathfrak{A}, \phi)$, and ϕ is false in $\mathfrak{A}$ if and only if Abélard has a winning strategy for $G(\mathfrak{A}, \phi)$.

In IF logic a sentence is defined to be true in a model if Eloïse has a winning strategy for the corresponding semantic game. Dually, a sentence is false in a model if Abélard has a winning strategy. For any ordinary first-order sentence ϕ the semantic game $G(\mathfrak{A}, \phi)$ is a two-player, win-loss game of perfect information. Hence Zermelo's theorem tells us that one of the two players must have a winning strategy. Thus every ordinary first-order sentence is either true

of false. That is, one can prove the principle of bivalence for ordinary first-order sentences from the game-theoretical definition of truth.

The extension beyond ordinary first-order logic comes from the fact that in IF logic one can write down a sentence whose corresponding semantic game is not a game of perfect information by restricting the information available to the existential player. For such a sentence it is no longer the case that one of the two players must have a winning strategy. Thus it is possible for a sentence in IF logic to be neither true nor false. For example, consider the sentence

$$\forall x \exists y_{/x} (x = y).$$

First Abélard chooses an element of the universe and assigns it as the value of the variable x. Then Eloïse chooses an element of the universe and assigns it to y, but Eloïse must make her choice without knowing the value of x. If Eloïse correctly guesses which element Abélard chose, she wins. Otherwise Abélard wins. It should be clear that in any structure with at least two elements, Eloïse does not have a winning strategy. Therefore the sentence is not true. But neither does Abélard have a winning strategy because there is always the possibility that Eloïse will guess correctly. Therefore the sentence is not false. It is worth noting that whether or not a sentence is undetermined depends on the structure in which the semantic game is played. For example, the sentence

$$\forall x \exists y_{/x} (y \leq x)$$

is neither true nor false in $\mathbb{R}$, but it is true in $\mathbb{N}$ because Eloïse can always choose 0.

In her Ph.D. thesis [7] Dechesne provides a rigorous mathematical foundation for an extension of IF logic in which one allows the information available to Eloïse and Abélard to be restricted. She calls her extension IFG logic (for "generalized independence-friendly logic"). Her thesis is the basis for our work. With the possible exception of Proposition 1.5, none of the results in the present chapter are original.

Instead of focusing on which quantifiers are independent of one another, it might seem more natural to focus on which quantifiers depend on which other quantifiers. Väänänen does

exactly that in his forthcoming book <u>Dependence Logic</u> [24].

Cylindric algebra is the algebra of ordinary first-order logic in the same way that Boolean algebra is the algebra of ordinary propositional logic. Cylindric algebra was first studied by Henkin, Monk, and Tarski [11, 12]. The goal of this thesis is to algebraize IFG logic in the same spirit as cylindric algebra. Our algebraization will depend heavily on the compositional semantics for independence-friendly logic put forth by Hodges [15, 16].

1.2 Syntax

In regular first-order logic, a formula ϕ is a string of symbols that satisfies certain conditions, and a variable x is said to occur in ϕ if the symbol "x" appears in the string. In our version of IFG logic, each formula will be a pair $\langle \phi, X \rangle$ where ϕ is a formula in the standard sense (a string of symbols satisfying certain conditions) and X is a finite set of variables. A variable is said to occur in $\langle \phi, X \rangle$ if and only if it belongs to X. We require that any variable that appears in ϕ must belong to X. However, we will allow variables that do not appear in ϕ to belong to X. Thus every variable that appears in ϕ must occur in $\langle \phi, X \rangle$, but a variable may occur in $\langle \phi, X \rangle$ without appearing in ϕ.

Definition. Given a first-order signature σ, an **atomic IFG-formula** is a pair $\langle \phi, X \rangle$ where ϕ is an atomic first-order formula and X is a finite set of variables that includes every variable that appears in ϕ.

Definition. Given a first-order signature σ, the language $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$ is the smallest set of formulas such that:

(1) Every atomic IFG-formula is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$.

(2) If $\langle \phi, Y \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$ and $Y \subseteq X$, then $\langle \phi, X \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$.

(3) If $\langle \phi, X \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$, then $\langle \sim\phi, X \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$.

(4) If $\langle \phi, X \rangle$ and $\langle \psi, X \rangle$ are in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$, and $Y \subseteq X$, then $\langle \phi \vee_{/Y} \psi, X \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$.

(5) If $\langle \phi, X \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$, $x \in X$, and $Y \subseteq X$, then $\langle \exists x_{/Y}\phi, X \rangle$ is in $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$.

Above X and Y are finite sets of variables.

From now on we will make certain assumptions about IFG-formulas that will allow us to simplify our notation. First, we will assume that the set of variables of $\mathscr{L}_{\mathrm{IFG}}^{\sigma}$ is $\{\, v_n \mid n \in \omega \,\}$. Second, since it does not matter much which particular variables appear in a formula, we will assume that variables with smaller indices are used before variables with larger indices. More precisely, if $\langle \phi, X \rangle$ is a formula, $v_j \in X$, and $i \leq j$, then $v_i \in X$. By abuse of notation, if $\langle \phi, X \rangle$ is a formula and $|X| = N$, then we will say that ϕ has N variables and write ϕ for $\langle \phi, X \rangle$. As a shorthand, we will call ϕ an IFG_N-formula. Let $\mathscr{L}_{\mathrm{IFG}_N}^{\sigma} = \{\, \phi \in \mathscr{L}_{\mathrm{IFG}}^{\sigma} \mid \phi \text{ has } N \text{ variables} \,\}$. Third, sometimes we will write $\phi \vee_{/J} \psi$ instead of $\phi \vee_{/Y} \psi$ and $\exists v_{n/J}\phi$ instead of $\exists v_{n/Y}\phi$, where $J = \{\, j \mid v_j \in Y \,\}$. Finally, we will use $\phi \wedge_{/J} \psi$ to abbreviate $\sim(\sim\phi \vee_{/J} \sim\psi)$ and $\forall v_{n/J}\phi$ to abbreviate $\sim\exists v_{n/J} \sim\phi$.

Definition. Let ϕ be an IFG-formula. The **subformula tree** of ϕ, denoted $\mathrm{Sub}(\phi)$, is the smallest tree satisfying the following conditions.

(1) $\langle \emptyset, \phi \rangle \in \mathrm{Sub}(\phi)$.

(2) If $\langle s, \sim\psi \rangle \in \mathrm{Sub}(\phi)$, then $\langle s^\frown 0, \psi \rangle \in \mathrm{Sub}(\phi)$.

(3) If $\langle s, \psi_1 \vee_{/J} \psi_2 \rangle \in \mathrm{Sub}(\phi)$, then $\langle s^\frown 1, \psi_1 \rangle \in \mathrm{Sub}(\phi)$ and $\langle s^\frown 2, \psi_2 \rangle \in \mathrm{Sub}(\phi)$.

(4) If $\langle s, \exists v_{n/J}\psi \rangle \in \mathrm{Sub}(\phi)$, then $\langle s^\frown 3, \psi \rangle \in \mathrm{Sub}(\phi)$.

For every $\langle s, \psi \rangle \in \mathrm{Sub}(\phi)$, $\langle s, \psi \rangle \in \mathrm{Sub}^+(\phi)$ if s contains an even number of 0s, and $\langle s, \psi \rangle \in \mathrm{Sub}^-(\phi)$ if s contains an odd number of 0s.

From now on, we will assume that all subformulas are indexed by their position in the subformula tree. This will allow us to distinguish between multiple instances of the same formula that may occur as subformulas of ϕ. For example, if ϕ is $v_0 = v_1 \vee_{/v_0} v_0 = v_1$ we will distinguish between the left and right disjuncts.

1.3   Game semantics

Dechesne defines her semantic games in extensive form [7], which is standard practice in game theory. It turns out that games in extensive form are more general than is necessary for our purposes. We modify her definition in order to focus on those aspects of the games that are relevant to the present discussion.

Definition. Given a first-order signature σ and a formula $\langle \phi, X \rangle \in \mathscr{L}_{\mathrm{IFG}}^\sigma$, a structure $\mathfrak{A}$ is called **suitable** for ϕ if $\mathfrak{A}$ has an interpretation for every non-logical symbol in σ.

Definition. If $\langle \phi, X \rangle$ is a formula and $\mathfrak{A}$ is a suitable structure, then a **valuation** for $\langle \phi, X \rangle$ over $\mathfrak{A}$ is a function from X to A. Since we assuming that X has the form $\{v_0, \ldots, v_{N-1}\}$, we will identify valuations with sequences of individuals in A, denoted $\vec{a} \in {}^N A$. A set of valuations $V \subseteq {}^N A$ is called a **team**.

Definition. Let $\vec{a}, \vec{b} \in {}^N A$ be two valuations, and let $J \subseteq N$. We say that $\vec{a}$ and $\vec{b}$ **agree outside of** J, denoted $\vec{a} \approx_J \vec{b}$, if $\vec{a} \restriction (N \setminus J) = \vec{b} \restriction (N \setminus J)$.

Note that $\approx_J$ is an equivalence relation on ${}^N A$. Also note that $\approx_\emptyset$ is the identity relation and $\approx_N$ is the total relation on ${}^N A$.

Lemma 1.1. *Let $\vec{a}, \vec{b} \in {}^N A$, and let $J \subseteq K \subseteq N$. Then $\vec{a} \approx_J \vec{b}$ implies $\vec{a} \approx_K \vec{b}$.*

Proof. If $\vec{a} \approx_J \vec{b}$, then $\vec{a} \restriction (N \setminus J) = \vec{b} \restriction (N \setminus J)$, which implies $\vec{a} \restriction (N \setminus K) = \vec{b} \restriction (N \setminus K)$. Hence $\vec{a} \approx_K \vec{b}$. $\qquad \square$

Definition. If $\vec{a} \in {}^N A$, $b \in A$, and $n < N$, define $\vec{a}(n : b)$ to be the valuation that is like $\vec{a}$ except that v_n is assigned the value b instead of a_n. In other words,

$$\vec{a}(n : b) = \vec{a} \restriction (N \setminus \{n\}) \cup \{\langle n, b \rangle\}.$$

We call $\vec{a}(n : b)$ an n-**variant of** $\vec{a}$.

Definition. If $V \subseteq {}^N A$ is a team and $b \in A$, define

$$V(n : b) = \{\, \vec{a}(n : b) \mid \vec{a} \in V \,\}.$$

Furthermore, if $B \subseteq A$ define

$$V(n : B) = \{\, \vec{a}(n : b) \mid \vec{a} \in V, \ b \in B \,\}.$$

A set $V' \subseteq V(n : A)$ is called an n-**variation of** V if for every $\vec{a} \in V$ there is at least one n-variant of $\vec{a}$ in V'. Finally if $f : V \to A$, and $V' \subseteq V$, define the n-**variation of** V' **by** f to be

$$V'(n : f) = \{\, \vec{a}(n : f(\vec{a})) \mid \vec{a} \in V' \,\}.$$

Definition. Let ϕ be a formula with N variables, let $\mathfrak{A}$ be a suitable structure, and let $V \subseteq {}^{N}A$ be a team. The **semantic game** $G(\mathfrak{A}, \phi, V)$ is defined as follows. A **position** of the game is a triple $\langle \psi, \vec{b}, \varepsilon \rangle$, where ψ is a subformula of ϕ, $\vec{b} \in {}^{N}A$, and $\varepsilon \in \{0, 1\}$. A **terminal position** is a position in which ψ is an atomic formula. A **play** of the game is a sequence of positions $\langle p_0, \ldots, p_q \rangle$ that satisfies the following conditions.

(1) The initial position $p_0 = \langle \phi, \vec{a}, 1 \rangle$, where $\vec{a} \in V$.

(2) If $p_k = \langle \sim\!\psi, \vec{b}, \varepsilon \rangle$, then $p_{k+1} = \langle \psi, \vec{b}, 1 - \varepsilon \rangle$.

(3) If $p_k = \langle \psi_1 \vee_{/J} \psi_2, \vec{b}, \varepsilon \rangle$, then $p_{k+1} = \langle \psi_1, \vec{b}, \varepsilon \rangle$ or $p_{k+1} = \langle \psi_2, \vec{b}, \varepsilon \rangle$.

(4) If $p_k = \langle \exists v_{n/J} \psi, \vec{b}, \varepsilon \rangle$, then $p_{k+1} = \langle \psi, \vec{b}(n : c), \varepsilon \rangle$ for some $c \in A$.

(5) The final position p_q is a terminal position, and p_q is the only terminal position in the play.

A **partial play** $\langle p_0, \ldots, p_\ell \rangle$ is an initial segment of a play. A partial play that is not a play is called a **proper partial play**.

For a given play of the game with final position $p_q = \langle \psi, \vec{b}, \varepsilon \rangle$ where ψ is an atomic formula, player ε **wins** if $\mathfrak{A} \models \psi[\vec{b}]$, and player $1 - \varepsilon$ **wins** if $\mathfrak{A} \not\models \psi[\vec{b}]$. In a given position $\langle \psi, \vec{b}, \varepsilon \rangle$, player ε is called the **verifier** and player $1 - \varepsilon$ is called the **falsifier**. The game $G(\mathfrak{A}, \phi, V)$ is the set of all possible plays. We will use $G(\mathfrak{A}, \phi)$ to abbreviate $G(\mathfrak{A}, \phi, {}^{N}A)$.

Definition. A **strategy** for player ε for the game $G(\mathfrak{A}, \phi, V)$ is a function S from the set of all non-terminal positions of the game in which player ε is the verifier to the set of all positions. A strategy is **legal** if for every proper partial play $\langle p_0, \ldots, p_\ell \rangle$ where player ε is the verifier in p_ℓ, the sequence $\langle p_0, \ldots, p_\ell, p_{\ell+1} \rangle$ is a partial play, where $p_{\ell+1} = S(p_\ell)$, and

(1) if $p_\ell = \langle \psi_1 \vee_{/J} \psi_2, \vec{a}, \varepsilon \rangle$ and $p'_\ell = \langle \psi_1 \vee_{/J} \psi_2, \vec{b}, \varepsilon \rangle$, where $\vec{a} \approx_J \vec{b}$, then $S(p_\ell) = S(p'_\ell)$;

(2) if $p_\ell = \langle \exists v_{n/J} \psi, \vec{a}, \varepsilon \rangle$ and $p'_\ell = \langle \exists v_{n/J} \psi, \vec{b}, \varepsilon \rangle$, where $\vec{a} \approx_J \vec{b}$, then $S(p_\ell) = S(p'_\ell)$.

Given a play $p = \langle p_0, \ldots, p_q \rangle$ and a strategy S for player ε, player ε is said to **follow S in p** if for every non-terminal position p_k in which player ε is the verifier, $p_{k+l} = S(p_k)$. A strategy for player ε is **winning** if it is legal and player ε wins every play in which he or she follows S.

Observe that if V is empty the game $G(\mathfrak{A}, \phi, V)$ has no positions nor plays. Hence the empty strategy $\emptyset \colon \emptyset \to \emptyset$ is a winning strategy for both players. As we will see later when we define a Tarski-style satisfaction relation for IFG-formulas, this apparent defect is actually a feature.

Definition. We say that ϕ is **true in $\mathfrak{A}$ relative to V** if player 1 has a winning strategy for the semantic game $G(\mathfrak{A}, \phi, V)$, and that ϕ is **false in $\mathfrak{A}$ relative to V** if player 0 has a winning strategy for $G(\mathfrak{A}, \phi, V)$. In the first case, we call V a **winning team** (or **trump**) for ϕ in $\mathfrak{A}$. In the second case, we call V a **losing team** (or **cotrump**) for ϕ in $\mathfrak{A}$. We say that ϕ is **true in $\mathfrak{A}$** if it is true in $\mathfrak{A}$ relative to ${}^N A$, and that ϕ is **false in $\mathfrak{A}$** if it is false relative to ${}^N A$.

Thus ϕ is true in $\mathfrak{A}$ if and only if player 1 has a winning strategy for the game $G(\mathfrak{A}, \phi)$, and ϕ is false in $\mathfrak{A}$ if and only if player 0 has a winning strategy for $G(\mathfrak{A}, \phi)$. It is important to realize that restricting the information available to the players at different moves does not change the set of possible plays of the game $G(\mathfrak{A}, \phi, V)$. It only restricts the strategies the players are allowed to use.

Definition. Let $p_k = \langle \psi, \vec{b}, \varepsilon \rangle$ be a position of the game $G(\mathfrak{A}, \phi, V)$. The **dual position** of p_k is $\widetilde{p}_k = \langle \psi, \vec{b}, 1 - \varepsilon \rangle$. If S_1 is a strategy for player 1 for $G(\mathfrak{A}, \phi, V)$, the **dual strategy** $\widetilde{S}_1$ for

player 0 for $G(\mathfrak{A}, \sim\phi, V)$ is defined by $\widetilde{S}_1(\widetilde{p}_k) = S_1(p_k)$ for all $p_k \in \mathrm{dom}(S_1)$. If S_0 is a strategy for player 0 for $G(\mathfrak{A}, \phi, V)$, the **dual strategy** $\widetilde{S}_0$ for player 1 for $G(\mathfrak{A}, \sim\phi, V)$ is defined by $\widetilde{S}_0(\langle \sim\phi, \vec{a}, 1\rangle) = S_0(\langle \phi, \vec{a}, 0\rangle)$ for all $\vec{a} \in V$, and $\widetilde{S}_0(\widetilde{p}_k) = S_0(p_k)$ for all $p_k \in \mathrm{dom}(S_0)$.

Lemma 1.2. *Let ϕ be an IFG_N-formula, let $\mathfrak{A}$ be a suitable structure, and let $V \subseteq {}^N A$. Then S is a winning strategy for player ε for the game $G(\mathfrak{A}, \phi, V)$ if and only if $\widetilde{S}$ is a winning strategy for player $1 - \varepsilon$ for $G(\mathfrak{A}, \sim\phi, V)$.*

Proof. Suppose S_1 is a winning strategy for player 1 for $G(\mathfrak{A}, \phi, V)$. Then $\widetilde{S}_1$ is a legal strategy for player 0 for $G(\mathfrak{A}, \sim\phi, V)$. To show that $\widetilde{S}_1$ is a winning strategy, let $\widetilde{p} = \langle \widetilde{p}_0, \widetilde{p}_1, \ldots, \widetilde{p}_q\rangle$ be a play of $G(\mathfrak{A}, \sim\phi, V)$ in which player 0 follows $\widetilde{S}_1$, and let $\widetilde{p}_q = \langle \psi, \vec{b}, 1 - \varepsilon\rangle$. Then the corresponding play $p = \langle p_1, \ldots, p_q\rangle$ of $G(\mathfrak{A}, \phi, V)$ is a play in which player 1 follows S_1 and $p_q = \langle \psi, \vec{b}, \varepsilon\rangle$. By hypothesis, player 1 wins p. Hence $\mathfrak{A} \models \psi[\vec{b}]$ if $\varepsilon = 1$ and $\mathfrak{A} \not\models \psi[\vec{b}]$ if $\varepsilon = 0$. Thus $\mathfrak{A} \models \psi[\vec{b}]$ if $1 - \varepsilon = 0$ and $\mathfrak{A} \not\models \psi[\vec{b}]$ if $1 - \varepsilon = 1$. In either case, player 0 wins $\widetilde{p}$.

Conversely, suppose $\widetilde{S}_1$ is a winning strategy for player 0 for $G(\mathfrak{A}, \sim\phi, V)$. Then S_1 is a legal strategy for player 1 for $G(\mathfrak{A}, \psi, V)$. To show S_1 is a winning strategy, let $p = \langle p_1, \ldots, p_q\rangle$ be a play of $G(\mathfrak{A}, \psi, V)$ in which player 1 follows S_1, where $p_1 = \langle \phi, \vec{a}, 1\rangle$ and $p_q = \langle \psi, \vec{b}, \varepsilon\rangle$. Then $\widetilde{p} = \langle \widetilde{p}_0, \widetilde{p}_1, \ldots, \widetilde{p}_q\rangle$ is a play of $G(\mathfrak{A}, \phi, V)$, where $\widetilde{p}_0 = \langle \sim\phi, \vec{a}, 1\rangle$ and $\widetilde{p}_q = \langle \psi, \vec{b}, 1 - \varepsilon\rangle$. By hypothesis, player 0 wins $\widetilde{p}$. Hence $\mathfrak{A} \models \psi[\vec{b}]$ if $1 - \varepsilon = 0$ and $\mathfrak{A} \not\models \psi[\vec{b}]$ if $1 - \varepsilon = 1$. Thus $\mathfrak{A} \models \psi[\vec{b}]$ if $\varepsilon = 1$ and $\mathfrak{A} \not\models \psi[\vec{b}]$ if $\varepsilon = 0$. In either case, player 1 wins.

Similarly, S_0 is a winning strategy for player 0 for $G(\mathfrak{A}, \phi, V)$ if and only if $\widetilde{S}_0$ is a winning strategy for player 1 for $G(\mathfrak{A}, \sim\phi, V)$. $\qquad\square$

Proposition 1.3. *Let ϕ be an IFG_N-formula, let $\mathfrak{A}$ be a suitable structure, and let $V \subseteq {}^N A$. Then ϕ is true in $\mathfrak{A}$ relative to V if and only if $\sim\phi$ is false in $\mathfrak{A}$ relative to V, and vice versa.*

Proof. By the previous lemma, Eloïse has a winning strategy for $G(\mathfrak{A}, \phi, V)$ if and only if Abélard has a winning strategy for $G(\mathfrak{A}, \sim\phi, V)$, and Eloïse has a winning strategy for $G(\mathfrak{A}, \sim\phi, V)$ if and only if Abélard has a winning strategy for $G(\mathfrak{A}, \phi, V)$. $\qquad\square$

Proposition 1.4. *Let ϕ be an IFG$_N$-formula, let $\mathfrak{A}$ be a suitable structure, and let $V \subseteq {}^N A$.*

Then ϕ is true in $\mathfrak{A}$ relative to V if and only if $\sim(\sim\phi)$ is true in $\mathfrak{A}$ relative to V.

Proof. By Proposition 1.3, ϕ is true in $\mathfrak{A}$ relative to V if and only if $\sim\phi$ is false in $\mathfrak{A}$ relative to V if and only if $\sim(\sim\phi)$ is true in $\mathfrak{A}$ relative to V. $\qquad\qquad\square$

Proposition 1.5. *Let ϕ be an IFG$_N$-sentence. If ϕ is true in $\mathfrak{A}$ relative to some nonempty $V \subseteq {}^N A$, then ϕ is true in $\mathfrak{A}$.*

Proof. Suppose S is a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, V)$. We will construct a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, {}^N A)$. To do so, we will need to keep track of which variables the players have had the opportunity to modify during the play of the game. For each subformula ψ of ϕ, define a set of indices J_ψ of those variables of ψ that have been unbound:

(1) $J_\phi = \emptyset$.

(2) If ψ is a subformula of ϕ of the form $\sim\chi$, then $J_\chi = J_\psi$.

(3) If ψ is a subformula of ϕ of the form $\chi_1 \vee_{/K} \chi_2$, then $J_{\chi_1} = J_\psi$ and $J_{\chi_2} = J_\psi$.

(4) If ψ is a subformula of ϕ of the form $\exists v_{n/K}\chi$, then $J_\chi = J_\psi \cup \{n\}$.

Fix $\vec{a} \in V$. For every position $p = \langle \psi, \vec{b}, \varepsilon \rangle$ of $G(\mathfrak{A}, \phi, {}^N A)$ define $f(p) = (\vec{a} \restriction N \setminus J_\psi) \cup (\vec{b} \restriction J_\psi)$ and $F(p) = \langle \psi, f(p), \varepsilon \rangle$. Observe that $F(p)$ is a position of $G(\mathfrak{A}, \phi, V)$. Define a strategy T for Eloïse for $G(\mathfrak{A}, \phi, {}^N A)$ as follows:

(1) If $p = \langle \sim\chi, \vec{b}, \varepsilon \rangle$, then $T(p) = \langle \chi, \vec{b}, 1 - \varepsilon \rangle$.

(2) If $p = \langle \chi_1 \vee_{/K} \chi_2, \vec{b}, \varepsilon \rangle$ and $S(F(p)) = \langle \chi_i, f(p), \varepsilon \rangle$, then $T(p) = \langle \chi_i, \vec{b}, \varepsilon \rangle$.

(3) If $p = \langle \exists v_{n/K}\chi, \vec{b}, \varepsilon \rangle$ and $S(F(p)) = \langle \chi, f(p)(n : c), \varepsilon \rangle$, then $T(p) = \langle \chi, \vec{b}(n : c), \varepsilon \rangle$.

To show that T is a legal strategy, it suffices to observe that if $p = \langle \psi, \vec{b}, \varepsilon \rangle$ and $p' = \langle \psi, \vec{b}', \varepsilon \rangle$, where $\vec{b} \approx_K \vec{b}'$, then

$$f(p) = (\vec{a} \restriction N \setminus J_\psi) \cup (\vec{b} \restriction J_\psi) \approx_K (\vec{a} \restriction N \setminus J_\psi) \cup (\vec{b}' \restriction J_\psi) = f(p').$$

Hence $S(F(p)) = S(F(p'))$ because S is a legal strategy. Thus $T(p) = T(p')$.

To show that T is a winning strategy, let $\langle p_0, \ldots, p_q \rangle$ be a play of $G(\mathfrak{A}, \phi, {}^N A)$ in which Eloïse follows T. Then $\langle F(p_0), \ldots, F(p_q) \rangle$ is a play of $G(\mathfrak{A}, \phi, V)$ in which Eloïse follows S. Let $p_q = \langle \psi, \vec{b}, \varepsilon \rangle$ and $F(p_q) = \langle \psi, f(p_q), \varepsilon \rangle$. Note that $\vec{b}$ and $f(p_q)$ agree on the free variables of ψ because ϕ was a sentence. Thus $\mathfrak{A} \models \psi[\vec{b}]$ if and only if $\mathfrak{A} \models \psi[f(p_q)]$ if and only if $\varepsilon = 1$ because S is a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, V)$. $\qquad\square$

Definition. A position $\langle \psi, \vec{b}, \varepsilon \rangle$ of the game $G(\mathfrak{A}, \phi, V)$ is **reachable** if it occurs in some play of $G(\mathfrak{A}, \phi, V)$. Otherwise, it is **unreachable**.

For example, let ϕ be $\exists v_1 (v_0 = v_1)$, and let $\mathfrak{A}$ be the equality structure with universe $\{0, 1\}$. Then $\langle \phi, 00, 0 \rangle$ and $\langle v_0 = v_1, 10, 1 \rangle$ are both unreachable positions of the game $G(\mathfrak{A}, \phi, \{00, 01\})$. The position $\langle \phi, 00, 0 \rangle$ is unreachable because player 1 is always the initial verifier; the position $\langle v_0 = v_1, 11, 1 \rangle$ is unreachable because in any play of the game the initial valuation is either 00 or 01, and the players never have the opportunity to modify v_0.

Lemma 1.6. *Let ϕ be an IFG-formula, and let $\langle \psi, \vec{b}, \varepsilon \rangle$ be a reachable position of the game $G(\mathfrak{A}, \phi, V)$. Then*

    (a) $\psi \in \mathrm{Sub}^+(\phi)$ if and only if $\varepsilon = 1$.

    (b) $\psi \in \mathrm{Sub}^-(\phi)$ if and only if $\varepsilon = 0$.

Theorem 1.7. *Let ϕ be a first-order formula with N variables. We can treat ϕ as an IFG_N-formula if we interpret $\neg$ as $\sim$, $\vee$ as $\bigvee_{\emptyset}$, and $\exists v_n$ as $\exists v_{n/\emptyset}$. If we do so, then for every suitable structure $\mathfrak{A}$ and team $V \subseteq {}^N A$,*

    (a) ϕ is true in $\mathfrak{A}$ relative to V if and only if $\mathfrak{A} \models \phi[\vec{a}]$ for all $\vec{a} \in V$,

    (b) ϕ is false in $\mathfrak{A}$ relative to V if and only if $\mathfrak{A} \not\models \phi[\vec{a}]$ for all $\vec{a} \in V$.

Proof. We will prove the theorem by two simultaneous inductions on the complexity of ϕ. Suppose ϕ is atomic. Then the only strategy either player has for the game $G(\mathfrak{A}, \phi, V)$ is the empty

strategy. For every play $\langle\langle\phi, \vec{a}, 1\rangle\rangle$ in the game, Eloïse will win if and only if $\mathfrak{A} \models \phi[\vec{a}]$, and Abélard will win if and only if $\mathfrak{A} \not\models \phi[\vec{a}]$. Thus Eloïse has a winning strategy if and only if $\mathfrak{A} \models \phi[\vec{a}]$ for all $\vec{a} \in V$, and Abélard has a winning strategy if and only if $\mathfrak{A} \not\models \phi[\vec{a}]$ for all $\vec{a} \in V$.

Suppose ϕ is $\sim\psi$. Then $\sim\psi$ is true in $\mathfrak{A}$ relative to V if and only if ϕ is false in $\mathfrak{A}$ relative to V, which by inductive hypothesis holds if and only if $\mathfrak{A} \not\models \psi[\vec{a}]$ for all $\vec{a} \in V$, if and only if $\mathfrak{A} \models \neg\psi[\vec{a}]$ for all $\vec{a} \in V$. Similarly $\sim\psi$ is false in $\mathfrak{A}$ relative to V if and only if ψ is true in $\mathfrak{A}$ relative to V, which by inductive hypothesis holds if and only if $\mathfrak{A} \models \psi[\vec{a}]$ for all $\vec{a} \in V$, if and only if $\mathfrak{A} \not\models \neg\phi[\vec{a}]$ for all $\vec{a} \in V$.

Suppose ϕ is $\psi_1 \vee \psi_2$, and suppose $\psi_1 \vee_{/\emptyset} \psi_2$ is true in $\mathfrak{A}$ relative to V. Then Eloïse has a winning strategy S for $G(\mathfrak{A}, \psi \vee_{/\emptyset} \psi_2, V)$. Let $V_i = \{\, \vec{a} \in V \mid S(\langle\psi_1 \vee \psi_2, \vec{a}, 1\rangle) = \langle\psi_i, \vec{a}, 1\rangle \,\}$, and let $S_i(\langle\chi, \vec{b}, 1\rangle) = S(\langle\chi, \vec{b}, 1\rangle)$ for all $\chi \in \text{Sub}^+(\psi_i)$. Then S_i is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi_i, V)$. Hence ψ_i is true in $\mathfrak{A}$ relative to V_i, so by inductive hypothesis $\mathfrak{A} \models \psi_i[\vec{a}]$ for all $\vec{a} \in V_i$. Thus $\mathfrak{A} \models \psi_1 \vee \psi_2[\vec{a}]$ for all $\vec{a} \in V$. Conversely, suppose $\mathfrak{A} \models \psi_1 \vee \psi_2[\vec{a}]$ for all $\vec{a} \in V$. Let $V_1 = \{\, \vec{a} \in V \mid \mathfrak{A} \models \psi_1[\vec{a}] \,\}$, and let $V_2 = \{\, \vec{a} \in V \mid \mathfrak{A} \not\models \psi_1[\vec{a}] \,\}$. Then $\mathfrak{A} \models \psi_2[\vec{a}]$ for all $\vec{a} \in V_2$. By inductive hypothesis, Eloïse has a winning strategy S_1 for $G(\mathfrak{A}, \psi_1, V_1)$ and a winning strategy S_2 for $G(\mathfrak{A}, \psi_2, V_2)$. Define a strategy S for Eloïse for $G(\mathfrak{A}, \psi_1 \vee_{/\emptyset} \psi_2, V)$ by $S(\langle\psi_1 \vee_{/\emptyset} \psi_2, \vec{a}, 1\rangle) = \langle\psi_i, \vec{a}, 1\rangle$ if $\vec{a} \in V_i$, and $S(\langle\chi, \vec{b}, 1\rangle) = S_i(\langle\psi_i, \vec{b}, 1\rangle)$ for all $\chi \in \text{Sub}^+(\psi_i)$. Observe that S is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi_1 \vee_{/\emptyset} \psi_2, V)$. Hence $\psi_1 \vee_{/\emptyset} \psi_2$ is true in $\mathfrak{A}$ relative to V.

Now suppose $\psi_1 \vee_{/\emptyset} \psi_2$ is false in $\mathfrak{A}$ relative to V. Then Abélard has a winning strategy S for $G(\mathfrak{A}, \psi_1 \vee_{/\emptyset} \psi_2, V)$, which means that no matter which disjunct Eloïse chooses on her first move, Abélard can win the play by following S. Let S_i be S restricted to positions of the form $\langle\chi, \vec{b}, 0\rangle$, where $\chi \in \text{Sub}^-(\psi_i)$. Then S_i is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_i, V)$. Thus ψ_i is false in $\mathfrak{A}$ relative to V, so by inductive hypothesis $\mathfrak{A} \not\models \psi_i[\vec{a}]$ for all $\vec{a} \in V$. Hence $\mathfrak{A} \not\models \psi_1 \vee \psi_2[\vec{a}]$ for all $\vec{a} \in V$. Conversely, suppose $\mathfrak{A} \not\models \psi_1 \vee \psi_2[\vec{a}]$ for all $\vec{a} \in V$. Then $\mathfrak{A} \not\models \psi_1[\vec{a}]$ for all $\vec{a} \in V$ and $\mathfrak{A} \not\models \psi_2[\vec{a}]$ for all $\vec{a} \in V$, so by inductive hypothesis Abélard has a winning strategy S_1 for $G(\mathfrak{A}, \psi_1, V)$ and a winning strategy S_2 for $G(\mathfrak{A}, \psi_2, V)$. Observe that $S_1 \cup S_2$

is a winning strategy for Abélard for the game $G(\mathfrak{A}, \psi_1 \vee_{/\emptyset} \psi_2, V)$. Thus $\psi_1 \vee_{/\emptyset} \psi_2$ is false in $\mathfrak{A}$ relative to V.

Suppose ϕ is $\exists v_{n/\emptyset} \psi$, and suppose $\exists v_{n/\emptyset} \psi$ is true in $\mathfrak{A}$ relative to V. Then Eloïse has a winning strategy S' for $G(\mathfrak{A}, \exists v_{n/\emptyset} \psi, V)$. Define $f \colon V \to A$ by $f(\vec{a}) = c$, where

$$S'(\langle \exists v_{n/\emptyset} \psi, \vec{a}, 1 \rangle) = \langle \psi, \vec{a}(n:c), 1 \rangle.$$

Let $S(\langle \chi, \vec{b}, 1 \rangle) = S'(\langle \chi, \vec{b}, 1 \rangle)$ for all $\chi \in \mathrm{Sub}^+(\psi)$. Then S is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi, V(n:f))$. Therefore by inductive hypothesis $\mathfrak{A} \models \psi[\vec{a}(n:f(\vec{a}))]$ for all $\vec{a} \in V$. Hence $\mathfrak{A} \models \exists v_n \psi[\vec{a}]$ for all $\vec{a} \in V$. Conversely, suppose $\mathfrak{A} \models \exists v_n \psi[\vec{a}]$ for all $\vec{a} \in V$. Then for every $\vec{a} \in A$ there is a $c \in A$ such that $\mathfrak{A} \models \psi[\vec{a}(n:c)]$. Let $f \colon V \to A$ be a function that chooses one such c for each $\vec{a} \in V$. Then for all $\vec{a} \in V$, $\mathfrak{A} \models \psi[\vec{a}(n:f(\vec{a}))]$. Therefore by inductive hypothesis Eloïse has a winning strategy S for $G(\mathfrak{A}, \psi, V(n:f))$. Let $S'(\langle \exists v_{n/\emptyset} \psi, \vec{a}, 1 \rangle) = \langle \psi, \vec{a}(n:f(\vec{a})), 1 \rangle$, and let $S'(\langle \chi, \vec{b}, 1 \rangle) = S(\langle \chi, \vec{b}, 1 \rangle)$ for all $\chi \in \mathrm{Sub}^+(\psi)$. Observe that S' is a winning strategy for Eloïse for $G(\mathfrak{A}, \exists v_{n/\emptyset} \psi, V)$. Thus $\exists v_{n/\emptyset} \psi$ is true in $\mathfrak{A}$ relative to V.

Now suppose $\exists v_{n/\emptyset} \psi$ is false in $\mathfrak{A}$ relative to V. Then Abélard has a winning strategy S for $G(\mathfrak{A}, \exists v_{n/J} \psi, V)$, which means that no matter which element of the universe Eloïse assigns to v_n, Abélard can win the play by following S. Hence S is a winning strategy for him for the game $G(\mathfrak{A}, \psi, V(n:A))$. Therefore by inductive hypothesis $\mathfrak{A} \not\models \psi[\vec{a}(n:c)]$ for all $\vec{a} \in V$ and $c \in A$. Hence $\mathfrak{A} \not\models \exists v_n \psi[\vec{a}]$ for all $\vec{a} \in V$. Conversely, suppose that $\mathfrak{A} \not\models \exists v_n \psi[\vec{a}]$ for all $\vec{a} \in V$. Then for all $\vec{a} \in A$ and $c \in A$, $\mathfrak{A} \not\models \psi[\vec{a}(n:c)]$, so by inductive hypothesis Abélard has a winning strategy S for $G(\mathfrak{A}, \psi, V(n:A))$. Observe that S is winning strategy for Abélard for the game $G(\mathfrak{A}, \exists v_{n/\emptyset} \psi, V)$. $\qquad\square$

Corollary 1.8. *Let ϕ be a first-order sentence, and let $\mathfrak{A}$ be a suitable structure. Then*

    (a) ϕ is true in $\mathfrak{A}$ if and only if $\mathfrak{A} \models \phi$,

    (b) ϕ is false in $\mathfrak{A}$ if and only if $\mathfrak{A} \not\models \phi$.

1.4   IFG logic and Σ_1^1 logic

1.4.1   Translating IFG-sentences into Σ_1^1-sentences

The definitions and proofs in this subsection were inspired by Theorem 4.3 in [4].

Definition. Let σ be a first-order signature, and let ϕ be a formula in $\mathscr{L}_{IFG}^\sigma$ with N variables. Define by simultaneous recursion on the subformulas of ϕ two first-order **Skolem forms** ϕ^+ and ϕ^- in the expanded signature $\tau_\phi = \sigma \cup \{\, F_\psi \mid \psi \in \mathrm{Sub}(\phi) \,\}$. Each F_ψ is a new function symbol of the appropriate arity. In particular, if ψ is atomic or of the form $\sim\chi$, then F_ψ is a new constant symbol. Choose one such constant symbol (there will be at least one) and denote it by 0.

- If ϕ is atomic, then ϕ^+ is ϕ and ϕ^- is $\neg\phi$.

- If ϕ is $\sim\psi$, then $(\sim\psi)^+$ is ψ^- and $(\sim\psi)^-$ is ψ^+.

- If ϕ is $\psi_1 \vee_{/J} \psi_2$, then

$$(\psi_1 \vee_{/J} \psi_2)^+ \text{ is } (F_{\psi_1 \vee_{/J} \psi_2}(v_{N\setminus J}) = 0 \wedge \psi_1^+) \vee (F_{\psi_1 \vee_{/J} \psi_2}(v_{N\setminus J}) \neq 0 \wedge \psi_2^+) \text{ and}$$

$$(\psi_1 \vee_{/J} \psi_2)^- \text{ is } \psi_1^- \wedge \psi_2^-.$$

- If ϕ is $\exists v_{n/J}\psi$, then

$$(\exists v_{n/J}\psi)^+ \text{ is } \psi^+(v_n / F_{\exists v_{n/J}\psi}(v_{N\setminus J})) \text{ and}$$

$$(\exists v_{n/J}\psi)^- \text{ is } \forall v_n \psi^-.$$

Here $v_{N\setminus J}$ is an abbreviation for the sequence $\langle\, v_i \mid i \in N \setminus J \,\rangle$.

Definition. Let σ be a first-order signature, and let $\phi \in \mathscr{L}_{\mathrm{IFG}_N}^\sigma$. Let ϕ^+ and ϕ^- be the associated first-order Skolem forms in the expanded signature τ_ϕ. Let $\mathfrak{A}$ be a σ-structure, and let $\mathfrak{A}' = \langle \mathfrak{A}, \tau_\phi \rangle$ be an expansion of $\mathfrak{A}$ to τ_ϕ. Then for any $V \subseteq {}^N A$ we can define a pair of strategies $S_1 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \phi, V)$ and $S_0 = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}', \phi, V)$ as follows.

- $S_1(\langle \sim\psi, \vec{b}, 1 \rangle) = \langle \psi, \vec{b}, 0 \rangle$.

- $S_1(\langle \psi_1 \vee_{/J} \psi_2, \vec{b}, 1 \rangle) = \langle \psi_1, \vec{b}, 1 \rangle$ if $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{b} \restriction N \setminus J) = 0$ and

  $S_1(\langle \psi_1 \vee_{/J} \psi_2, \vec{b}, 1 \rangle) = \langle \psi_2, \vec{b}, 1 \rangle$ if $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{b} \restriction N \setminus J) \neq 0$.

- $S_1(\langle \exists v_{n/J} \psi, \vec{b}, 1 \rangle) = \langle \psi, \vec{b}(n:c), 1 \rangle$, where $c = F^{\mathfrak{A}'}_{\exists v_{n/J} \psi}(\vec{b} \restriction N \setminus J)$.

Similarly,

- $S_0(\langle \sim \psi, \vec{b}, 0 \rangle) = \langle \psi, \vec{b}, 1 \rangle$.

- $S_0(\langle \psi_1 \vee_{/J} \psi_2, \vec{b}, 0 \rangle) = \langle \psi_1, \vec{b}, 0 \rangle$ if $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{b} \restriction N \setminus J) = 0$ and

  $S_0(\langle \psi_1 \vee_{/J} \psi_2, \vec{b}, 0 \rangle) = \langle \psi_2, \vec{b}, 0 \rangle$ if $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{b} \restriction N \setminus J) \neq 0$.

- $S_0(\langle \exists v_{n/J} \psi, \vec{b}, 0 \rangle) = \langle \psi, \vec{b}(n:c), 0 \rangle$, where $c = F^{\mathfrak{A}'}_{\exists v_{n/J} \psi}(\vec{b} \restriction N \setminus J)$.

Lemma 1.9. *Let σ be a first-order signature. For every $\phi \in \mathscr{L}^{\sigma}_{IFG_N}$, every σ-structure $\mathfrak{A}$, every expansion $\langle \mathfrak{A}, \tau_\phi \rangle$ of $\mathfrak{A}$ to the signature of ϕ^+ and ϕ^-, and every $V \subseteq {}^N A$,*

(a) *$\langle \mathfrak{A}, \tau_\phi \rangle \models \phi^+[\vec{a}]$ for all $\vec{a} \in V$ if and only if $\mathrm{Strat}^+(\mathfrak{A}, \langle \mathfrak{A}, \tau_\phi \rangle, \phi, V)$ is a winning strategy for player 1 for $G(\mathfrak{A}, \phi, V)$, and*

(b) *$\langle \mathfrak{A}, \tau_\phi \rangle \models \phi^-[\vec{a}]$ for all $\vec{a} \in V$ if and only if $\mathrm{Strat}^-(\mathfrak{A}, \langle \mathfrak{A}, \tau_\phi \rangle, \phi, V)$ is a winning strategy for player 0 for $G(\mathfrak{A}, \phi, V)$.*

Proof. Let $\phi \in \mathscr{L}^{\sigma}_{IFG_N}$, let $\mathfrak{A}$ be a σ-structure, let $\mathfrak{A}' = \langle \mathfrak{A}, \tau_\phi \rangle$ be an expansion of $\mathfrak{A}$ to the signature of ϕ^+ and ϕ^-, and let $V \subseteq {}^N A$. We will prove the lemma by two simultaneous inductions on the subformulas of ϕ.

If ϕ is atomic, then ϕ^+ is ϕ, ϕ^- is $\neg\phi$, $\tau_\phi = \sigma \cup \{0\}$, and both $S_1 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \phi, V)$ and $S_0 = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}', \phi, V)$ are empty. Suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models \phi^+[\vec{a}]$. Then for all $\vec{a} \in V$, $\mathfrak{A} \models \phi[\vec{a}]$. Hence Eloïse wins every play of the game $G(\mathfrak{A}, \phi, V)$. Thus S_1 is a winning strategy for her. Conversely, suppose S_1 is a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, V)$. Then Eloïse must win every play of $G(\mathfrak{A}, \phi, V)$, so for all $\vec{a} \in V$, $\mathfrak{A} \models \phi[\vec{a}]$. Hence, for all $\vec{a} \in V$, $\mathfrak{A}' \models \phi^+[\vec{a}]$.

Now suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models \phi^-[\vec{a}]$. Then for all $\vec{a} \in V$, $\mathfrak{A} \models \neg\phi[\vec{a}]$. Hence for all $\vec{a} \in V$, $\mathfrak{A} \not\models \phi[\vec{a}]$. Thus Abélard wins every play of $G(\mathfrak{A}, \phi, V)$. Therefore S_0 is a winning

strategy for him. Conversely, suppose S_0 is a winning strategy for Abélard for $G(\mathfrak{A}, \phi, V)$. Then Abélard must win every play of $G(\mathfrak{A}, \phi, V)$, which means that for all $\vec{a} \in V$, $\mathfrak{A} \not\models \phi[\vec{a}]$. Hence for all $\vec{a} \in V$, $\mathfrak{A} \models \neg\phi[\vec{a}]$. Thus for all $\vec{a} \in V$, $\mathfrak{A}' \models \phi^-[\vec{a}]$.

Suppose ϕ is $\sim\psi$. Then $(\sim\psi)^+$ is ψ^- and $(\sim\psi)^-$ is ψ^+. Let $\mathfrak{A}'' = \langle \mathfrak{A}, \tau_\psi \rangle$ be the reduct of $\mathfrak{A}'$ to τ_ψ. Suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models (\sim\psi)^+[\vec{a}]$. Then for all $\vec{a} \in V$, $\mathfrak{A}'' \models \psi^-[\vec{a}]$, so by inductive hypothesis $S_0 = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}'', \psi, V)$ is a winning strategy for Abélard for the game $G(\mathfrak{A}, \psi, V)$. By Lemma 1.2, $\widetilde{S}_0$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \sim\psi, V)$. Observe that $\widetilde{S}_0 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \sim\psi, V)$, as desired. Conversely, suppose $\widetilde{S}_0 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \sim\psi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \sim\psi, V)$. Then by Lemma 1.2, S_0 is a winning strategy for Abélard for $G(\mathfrak{A}, \psi, V)$. Observe that $S_0 = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}'', \psi, V)$. Thus, by inductive hypothesis, for all $\vec{a} \in V$, $\mathfrak{A}'' \models \psi^-[\vec{a}]$. Hence for all $\vec{a} \in V$, $\mathfrak{A}' \models (\sim\psi)^+[\vec{a}]$.

Now suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models (\sim\psi)^-[\vec{a}]$. Then for all $\vec{a} \in V$, $\mathfrak{A}'' \models \psi^+[\vec{a}]$, so by inductive hypothesis $S_1 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}'', \psi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi, V)$. By Lemma 1.2, $\widetilde{S}_1$ is a winning strategy for Abélard for $G(\mathfrak{A}, \sim\psi, V)$. Observe that $\widetilde{S}_1 = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}', \sim\psi, V)$, as desired. Conversely, suppose $\widetilde{S}_1 = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}', \sim\psi, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \sim\psi, V)$. Then by Lemma 1.2, S_1 is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi, V)$. Observe that $S_1 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}'', \psi, V)$. Thus, by inductive hypothesis, for all $\vec{a} \in V$, $\mathfrak{A}'' \models \psi^+[\vec{a}]$. Hence for all $\vec{a} \in V$, $\mathfrak{A}' \models (\sim\psi)^-[\vec{a}]$.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. Then $(\psi_1 \vee_{/J} \psi_2)^+$ is

$$(F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) = 0 \wedge \psi_1^+) \vee (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) \neq 0 \wedge \psi_2^+)$$

and $(\psi_1 \vee_{/J} \psi_2)^-$ is $\psi_1^- \wedge \psi_2^-$. Let $\mathfrak{A}_1 = \langle \mathfrak{A}, \tau_{\psi_1} \rangle$ be the reduct of $\mathfrak{A}'$ to τ_{ψ_1}, and let $\mathfrak{A}_2 = \langle \mathfrak{A}, \tau_{\psi_2} \rangle$ be the reduct of $\mathfrak{A}'$ to τ_{ψ_2}. Suppose that for all $\vec{a} \in V$,

$$\mathfrak{A}' \models (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) = 0 \wedge \psi_1^+) \vee (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) \neq 0 \wedge \psi_2^+)[\vec{a}].$$

Let $V_1 = \{\, \vec{a} \in V \mid F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{a} \restriction N \setminus J) = 0^{\mathfrak{A}'} \,\}$, and let $V_2 = \{\, \vec{a} \in V \mid F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{a} \restriction N \setminus J) \neq 0^{\mathfrak{A}'} \,\}$. If $\vec{a} \in V_1$, then $\mathfrak{A}_1 \models \psi_1^+[\vec{a}]$, so by inductive hypothesis $S_1 = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}_1, \psi_1, V_1)$ is a winning strategy for Eloïse for the game $G(\mathfrak{A}, \psi_1, V_1)$. If $\vec{a} \in V_2$, then $\mathfrak{A}_2 \models \psi_2^+[\vec{a}]$, so by inductive

hypothesis $S_2 = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}_2, \psi_2, V_2)$ is a winning strategy for Eloïse for the game $G(\mathfrak{A}, \psi_2, V_2)$. It follows that $S = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}', \psi_1 \vee_{/J} \psi_2, V)$ is a winning strategy for Eloïse for the game $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$.

Conversely, suppose $S = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}', \psi_1 \vee_{/J} \psi_2, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$. Let $V_i = \{\, \vec{a} \in V \mid S(\langle \psi_1 \vee_{/J} \psi_2, \vec{a}, 1 \rangle) = \langle \psi_i, \vec{a}, 1 \rangle \,\}$. For all $\chi \in \text{Sub}(\psi_i)$, let $S_i(\langle \chi, \vec{b}, 1 \rangle) = S(\langle \chi, \vec{b}, 1 \rangle)$. Then S_i must be a winning strategy for Eloïse for $G(\mathfrak{A}, \psi_i, V_i)$, and $S_i = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}_i, \psi_i, V_i)$. Therefore, by inductive hypothesis, for all $\vec{a} \in V_i$, $\mathfrak{A}_i \models \psi_i^+[\vec{a}]$. Since S is a winning strategy for Eloïse it must be the case that for all $\vec{a} \in V_1$, $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{a} \restriction N \setminus J) = 0^{\mathfrak{A}'}$, and for all $\vec{a} \in V_2$, $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{a} \restriction N \setminus J) \neq 0^{\mathfrak{A}'}$. Hence $\mathfrak{A}' \models (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) = 0 \wedge \psi_1^+)[\vec{a}]$ if $\vec{a} \in V_1$, and $\mathfrak{A}' \models (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) \neq 0 \wedge \psi_2^+)[\vec{a}]$ if $\vec{a} \in V_2$. Thus, for all $\vec{a} \in V$,

$$\mathfrak{A}' \models (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) = 0 \wedge \psi_1^+) \vee (F_{\psi_1 \vee_{/J} \psi_2}(v_{N \setminus J}) \neq 0 \wedge \psi_2^+)[\vec{a}].$$

Now suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models \psi_1^- \wedge \psi_2^-[\vec{a}]$. Then for all $\vec{a} \in V$, $\mathfrak{A}' \models \psi_1^-[\vec{a}]$ and $\mathfrak{A}' \models \psi_2^-[\vec{a}]$. Hence, by inductive hypothesis, $S_1 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}_1, \psi_1, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_1, V)$, and $S_2 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}_2, \psi_2, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_2, V)$. Therefore $S_1 \cup S_2$ is a winning strategy for him for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$. Observe that $S_1 \cup S_2 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}', \psi_1 \vee_{/J} \psi_2, V)$, as desired.

Conversely, suppose $S = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}', \psi_1 \vee_{/J} \psi_2, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$. Define $S_1(\langle \chi, \vec{b}, 0 \rangle) = S(\langle \chi, \vec{b}, 0 \rangle)$ if $\chi \in \text{Sub}(\psi_1)$, and $S_2(\langle \chi, \vec{b}, 0 \rangle) = S(\langle \chi, \vec{b}, 0 \rangle)$ if $\chi \in \text{Sub}(\psi_2)$. Then $S_1 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}_1, \psi_1, V)$ and is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_1, V)$. Likewise, $S_2 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}_2, \psi_2, V)$ and is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_2, V)$. Thus, by inductive hypothesis, for all $\vec{a} \in V$, $\mathfrak{A}_1 \models \psi_1^-[\vec{a}]$ and $\mathfrak{A}_2 \models \psi_2^-[\vec{a}]$. Hence for all $\vec{a} \in V$, $\mathfrak{A}' \models \psi_1^- \wedge \psi_2^-[\vec{a}]$.

Suppose ϕ is $\exists v_{n/J} \psi$. Then $(\exists v_{n/J} \psi)^+$ is $\psi^+(v_n / F_{\exists v_{n/J} \psi}(v_{N \setminus J}))$, and $(\exists v_{n/J} \psi)^-$ is $\forall v_n \psi^-$. Let $\mathfrak{A}'' = \langle \mathfrak{A}, \tau_\psi \rangle$ be the reduct of $\mathfrak{A}'$ to τ_ψ. Suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models \psi^+(v_n / F_{\exists v_{n/J} \psi}(v_{N \setminus J}))[\vec{a}]$. Let $f : V \to A$ be defined by $f(\vec{a}) = F^{\mathfrak{A}'}_{\exists v_{n/J} \psi}(\vec{a} \restriction N \setminus J)$. Then for all $\vec{b} \in V(n : f)$, $\mathfrak{A}'' \models \psi^+[\vec{b}]$. Hence, by inductive hypothesis, $S = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}', \psi, V(n : f))$

is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi, V(n : f))$. Define a strategy S_1 for Eloïse for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$ by

$$S_1(\langle \exists v_{n/J}\psi, \vec{a}, 1 \rangle) = \langle \psi, \vec{a}(n : f(\vec{a})), 1 \rangle \quad \text{for all } \vec{a} \in V,$$

$$S_1(\langle \chi, \vec{b}, 1 \rangle) = S(\langle \chi, \vec{b}, 1 \rangle) \quad \text{for all } \chi \in \text{Sub}^+(\psi).$$

Then S_1 is a winning strategy for Eloïse, and $S_1 = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}', \exists v_{n/J}\psi, V)$.

Conversely, suppose $S_1 = \text{Strat}^+(\mathfrak{A}, \mathfrak{A}', \exists v_{n/J}\psi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Define $f : V \to A$ as before. Define a strategy S for Eloïse for $G(\mathfrak{A}, \psi, V(n : f))$ by $S(\langle \chi, \vec{b}, 1 \rangle) = S_1(\langle \chi, \vec{b}, 1 \rangle)$ for all $\chi \in \text{Sub}^+(\psi)$. Then S must be a winning strategy. Hence, by inductive hypothesis, for all $\vec{b} \in V(n : f)$, $\mathfrak{A}'' \models \psi^+[\vec{b}]$. It follows that for all $\vec{a} \in V$, $\mathfrak{A}' \models \psi^+(v_n/F_{\exists v_{n/J}\psi}(v_{N \setminus J}))[\vec{a}]$.

Now suppose that for all $\vec{a} \in V$, $\mathfrak{A}' \models \forall v_n \psi^-[\vec{a}]$. Then for all $\vec{b} \in V(n : A)$, $\mathfrak{A}'' \models \psi^-[\vec{b}]$. Hence, by inductive hypothesis, $S_0 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}'', \psi, V(n : A))$ is a winning strategy for Abélard for $G(\mathfrak{A}, \psi, V(n : A))$. Observe that S_0 is also a winning strategy for him for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$, and that $S_0 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}', \exists v_{n/J}\psi, V)$.

Conversely, suppose $S_0 = \text{Strat}^-(\mathfrak{A}, \mathfrak{A}', \exists v_{n/J}\psi, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Then S_0 must also be a winning strategy for him for $G(\mathfrak{A}, \psi, V(n : A))$. Hence, by inductive hypothesis, for all $\vec{b} \in V(n : A)$, $\mathfrak{A}'' \models \psi^-[\vec{b}]$. Thus, for all $\vec{a} \in V$, $\mathfrak{A}' \models \forall v_n \psi^-[\vec{a}]$. $\square$

Lemma 1.10. *Let σ be a first-order signature. For every $\phi \in \mathscr{L}_{IFG}^\sigma$ and every σ-structure $\mathfrak{A}$,*

(a) *if Eloïse has a winning strategy for $G(\mathfrak{A}, \phi, V)$, then there is an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_\phi \rangle$ of $\mathfrak{A}$ to the signature τ_ϕ of ϕ^+ and ϕ^- such that $\text{Strat}^+(\mathfrak{A}, \mathfrak{A}', \phi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, V)$,*

(b) *if Abélard has a winning strategy for $G(\mathfrak{A}, \phi, V)$, then there is an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_\phi \rangle$ of $\mathfrak{A}$ to the signature τ_ϕ of ϕ^+ and ϕ^- such that $\text{Strat}^-(\mathfrak{A}, \mathfrak{A}', \phi, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \phi, V)$.*

Proof. Let $\phi \in \mathscr{L}^{\sigma}_{IFG}$, and let $\mathfrak{A}$ be a σ-structure. We proceed by simultaneous induction on the subformulas of ϕ.

Suppose ϕ is atomic. Then $\tau_{\phi} = \sigma \cup \{0\}$. Suppose Eloïse has a winning strategy for $G(\mathfrak{A}, \phi, V)$. Then for all $\vec{a} \in V$, $\mathfrak{A} \models \phi[\vec{a}]$. Let $\mathfrak{A}' = \langle \mathfrak{A}, \tau_{\phi} \rangle$ be any expansion of $\mathfrak{A}$ to τ_{ϕ}. Then $\mathrm{Strat}^{+}(\mathfrak{A}, \mathfrak{A}', \phi, V) = \emptyset$, which is a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, V)$. Now suppose Abélard has a winning strategy for $G(\mathfrak{A}, \phi, V)$. Then for all $\vec{a} \in V$, $\mathfrak{A} \not\models \phi[\vec{a}]$. Let $\mathfrak{A}' = \langle \mathfrak{A}, \tau_{\phi} \rangle$ be any expansion of $\mathfrak{A}$ to τ_{ϕ}. Then $\mathrm{Strat}^{-}(\mathfrak{A}, \mathfrak{A}', \phi, V) = \emptyset$, which is a winning strategy for Abélard for $G(\mathfrak{A}, \phi, V)$.

Suppose ϕ is $\sim\psi$ and Eloïse has a winning strategy for $G(\mathfrak{A}, \sim\psi, V)$. Then by Lemma 1.2, Abélard has a winning strategy for $G(\mathfrak{A}, \psi, V)$. Hence, by inductive hypothesis, there is an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_{\phi} \rangle$ such that $S_0 = \mathrm{Strat}^{-}(\mathfrak{A}, \mathfrak{A}', \psi, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \psi, V)$. By Lemma 1.2, $\widetilde{S}_0$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \sim\psi, V)$. Observe that $\widetilde{S}_0 = \mathrm{Strat}^{+}(\mathfrak{A}, \mathfrak{A}', \sim\psi, V)$, as desired.

Now suppose Abélard has a winning strategy for $G(\mathfrak{A}, \sim\psi, V)$. Then by Lemma 1.2, Eloïse has a winning strategy for $G(\mathfrak{A}, \psi, V)$. Hence, by inductive hypothesis, there is an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_{\phi} \rangle$ such that $S_1 = \mathrm{Strat}^{+}(\mathfrak{A}, \mathfrak{A}', \psi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi, V)$. By Lemma 1.2, $\widetilde{S}_1$ is a winning strategy for Abélard for $G(\mathfrak{A}, \sim\psi, V)$. Observe that $\widetilde{S}_1 = \mathrm{Strat}^{-}(\mathfrak{A}, \mathfrak{A}', \sim\psi, V)$, as desired.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$ and Eloïse has a winning strategy S for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$. Let

$$V_1 = \{\, \vec{a} \in V \mid S(\langle \psi_1 \vee_{/J} \psi_2, \vec{a}, 1 \rangle) = \langle \psi_1, \vec{a}, 1 \rangle \,\},$$

$$V_2 = \{\, \vec{a} \in V \mid S(\langle \psi_1 \vee_{/J} \psi_2, \vec{a}, 1 \rangle) = \langle \psi_2, \vec{a}, 1 \rangle \,\}.$$

Let S_i be the winning strategy for $G(\mathfrak{A}, \psi_i, V_i)$ that Eloïse inherits from S. By inductive hypothesis there exist expansions $\mathfrak{A}_1 = \langle \mathfrak{A}, \tau_{\psi_1} \rangle$ and $\mathfrak{A}_2 = \langle \mathfrak{A}, \tau_{\psi_2} \rangle$ such that $S'_i = \mathrm{Strat}^{+}(\mathfrak{A}, \mathfrak{A}_i, \psi_i, V_i)$

is a winning strategy for $G(\mathfrak{A}, \psi_i, V_i)$. Define the expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_{\psi_1 \vee_{/J} \psi_2} \rangle$ as follows:

$$F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{a} \restriction N \setminus J) = 0 \quad \text{if } \vec{a} \in V_1,$$

$$F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}(\vec{a} \restriction N \setminus J) \neq 0 \quad \text{if } \vec{a} \notin V_1,$$

$$F^{\mathfrak{A}'}_\chi = F^{\mathfrak{A}_i}_\chi \quad \text{if } \chi \in \mathrm{Sub}(\psi_i).$$

Then $\mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \psi_1 \vee_{/J} \psi_2, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$.

Now suppose Abélard has a winning strategy S for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$. Let S_i be the winning strategy for $G(\mathfrak{A}, \psi_i, V)$ that Abélard inherits from S. By inductive hypothesis there are expansions $\mathfrak{A}_1 = \langle \mathfrak{A}, \tau_{\psi_1} \rangle$ and $\mathfrak{A}_2 = \langle \mathfrak{A}, \tau_{\psi_1} \rangle$ such that $S'_i = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}_i, \psi_i, V)$ is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_i, V)$. Define an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_{\psi_1 \vee_{/J} \psi_2} \rangle$ by letting $F^{\mathfrak{A}'}_{\psi_1 \vee_{/J} \psi_2}$ be arbitrary and setting $F^{\mathfrak{A}'}_\chi = F^{\mathfrak{A}_i}_\chi$ if $\chi \in \mathrm{Sub}(\psi_i)$. Let $S' = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}', \psi_1 \vee_{/J} \psi_2, V)$. Then $S' = S'_1 \cup S'_2$, which is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$.

Suppose ϕ is $\exists v_{n/J}\psi$ and Eloïse has a winning strategy S for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Define $f\colon V \to A$ by taking $f(\vec{a})$ to be the unique $c \in A$ such that $S(\langle \exists v_{n/J}\psi, \vec{a}, 1 \rangle) = \langle \psi, \vec{a}(n:c), 1 \rangle$. Then Eloïse has a winning strategy for $G(\mathfrak{A}, \psi, V(n:f))$, namely the one she inherits from S. By inductive hypothesis, there is an expansion $\mathfrak{A}'' = \langle \mathfrak{A}, \tau_\psi \rangle$ such that

$$S'' = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \psi, V(n:f))$$

is a winning strategy for her for $G(\mathfrak{A}, \psi, V(n:f))$. Define an expansion $\mathfrak{A}' = \langle \mathfrak{A}'', \tau_{\exists v_{n/J}\psi} \rangle$ by taking $F^{\mathfrak{A}'}_{\exists v_{n/J}\psi}(\vec{a} \restriction N \setminus J) = f(\vec{a})$ for all $\vec{a} \in V$. Observe that $F^{\mathfrak{A}'}_{\exists v_{n/J}\psi}$ is well defined because S is a legal strategy. Also note that $S' = \mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \exists v_{n/J}\psi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$.

Now suppose Abélard has a winning strategy S for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Then S is a winning strategy for him for $G(\mathfrak{A}, \psi, V(n:A))$. By inductive hypothesis, there is an expansion $\mathfrak{A}'' = \langle \mathfrak{A}, \tau_\psi \rangle$ such that

$$S'' = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}'', \psi, V(n:A))$$

is a winning strategy for him for $G(\mathfrak{A}, \psi, V(n:A))$. Observe that S'' is also a winning strategy

for him for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Define an expansion $\mathfrak{A}' = \langle \mathfrak{A}'', \tau_{\exists v_{n/J}\psi} \rangle$ by letting $F^{\mathfrak{A}'}_{\exists v_{n/J}\psi}$ be arbitrary. Then $S'' = \mathrm{Strat}^-(\mathfrak{A}, \mathfrak{A}', \exists v_{n/J}\psi, V)$, as desired. $\qquad\square$

Theorem 1.11. *For every IFG$_N$-formula ϕ there is a Σ^1_1-formula ϕ^* of the same signature such that for every suitable structure $\mathfrak{A}$ and every $V \subseteq {}^N\! A$, ϕ is true in $\mathfrak{A}$ relative to V if and only if for all $\vec{a} \in V$, $\mathfrak{A} \models_{\mathrm{sol}} \phi^*[\vec{a}]$.*

Proof. Given $\phi \in \mathscr{L}^\sigma_{IFG}$, let ϕ^* be ϕ^+ prefixed with a second-order existential quantifier $\exists F_\psi$ for every $\psi \in \mathrm{Sub}(\phi)$. Suppose ϕ is true in $\mathfrak{A}$ relative to V. Then Eloïse has a winning strategy for the game $G(\mathfrak{A}, \phi, V)$. Hence, by Lemma 1.10, there is an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_\phi \rangle$ such that $\mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \phi, V)$ is a winning strategy for her for $G(\mathfrak{A}, \phi, V)$. Thus by Lemma 1.9, for all $\vec{a} \in V$, $\mathfrak{A}' \models \phi^+[\vec{a}]$, which implies $\mathfrak{A} \models_{\mathrm{sol}} \phi^*[\vec{a}]$.

Conversely, suppose that for all $\vec{a} \in V$, $\mathfrak{A} \models_{\mathrm{sol}} \phi^*[\vec{a}]$. Then there is an expansion $\mathfrak{A}' = \langle \mathfrak{A}, \tau_\phi \rangle$ such that for all $\vec{a} \in V$, $\mathfrak{A}' \models \phi^+[\vec{a}]$. Thus, by Lemma 1.9, $\mathrm{Strat}^+(\mathfrak{A}, \mathfrak{A}', \phi, V)$ is a winning strategy for Eloïse for $G(\mathfrak{A}, \phi, V)$. Hence ϕ is true in $\mathfrak{A}$ relative to V. $\qquad\square$

Corollary 1.12. *For every IFG-sentence ϕ there is a Σ^1_1-sentence ϕ^* of the same signature such that for every suitable structure $\mathfrak{A}$,*

$$\phi \text{ is true in } \mathfrak{A} \quad \textit{iff} \quad \mathfrak{A} \models_{\mathrm{sol}} \phi^*.$$

Corollary 1.13. *For every IFG-sentence ϕ and every suitable structure $\mathfrak{A}$,*

$$\phi \text{ is false in } \mathfrak{A} \quad \textit{iff} \quad \mathfrak{A} \models_{\mathrm{sol}} (\sim\phi)^*.$$

Proof. By Proposition 1.3 and Corollary 1.12. $\qquad\square$

We can think of ϕ^* as a second-order truth condition for ϕ, and of $(\sim\phi)^*$ as a second-order falsity condition for ϕ. Thus, the meaning of an IFG-sentence ϕ can be expressed by a pair of Σ^1_1-sentences $\langle \phi^*, (\sim\phi)^* \rangle$. Notice that $(\sim\phi)^*$ is stronger in general than $\neg(\phi^*)$. That is, $\mathfrak{A} \models_{\mathrm{sol}} (\sim\phi)^*$ implies $\mathfrak{A} \models_{\mathrm{sol}} \neg(\phi^*)$, but the converse fails if ϕ is undetermined in $\mathfrak{A}$.

1.4.2 Translating Σ_1^1-sentences into IFG-sentences

We now show that the reverse translation from Σ_1^1-sentences to IFG-sentences also works.

Lemma 1.14. *Every Σ_1^1-sentence is equivalent to a Σ_1^1-sentence in which no second-order predicate variables occur.*

Proof. Let Φ be any Σ_1^1-sentence, and let 0 be a 0-ary function variable (i.e., a second-order constant) that does not occur in Φ. Suppose X is an n-ary predicate variable that occurs in Φ. Since Φ is a Σ_1^1-sentence, X must existentially quantified. Let F be a new n-ary function variable. In Φ replace the existential quantifier $\exists X$ with $\exists F$, and replace each occurrence of $X(t_0, \ldots, t_{n-1})$ with $F(t_0, \ldots, t_{n-1}) \neq 0$. Do this for every predicate variable that occurs in Φ, and then prefix the entire formula with the existential quantifier $\exists 0$. The result will be a Σ_1^1-sentence that is equivalent to Φ, but has no predicate variables. $\square$

Lemma 1.15. *Every Σ_1^1-sentence is equivalent to a Σ_1^1-sentence of the form*

$$\exists F_0 \ldots F_{M-1} \forall v_0 \ldots \forall v_{N-1} \psi,$$

where ψ is a quantifier-free first-order formula in which

    (a) the function variables F_i do not occur nested, and

    (b) each function variable F_i occurs with only one sequence of arguments.

Proof. Let Φ_0 be any Σ_1^1-sentence. By the previous lemma, we may assume that Φ_0 has the form $\exists F_0 \ldots \exists F_{L-1} \phi_0$, where ϕ_0 is a first-order formula. Let ϕ_1 be the prenex normal form of ϕ_0. Let ϕ_2 be the first-order Skolemization of ϕ_1, and let $F_L, \ldots, F_{M-1}$ be the Skolem functions introduced during the Skolemization. Then ϕ_2 has the form $\forall v_0 \ldots \forall v_{N-1} \psi$, where ψ is a quantifier-free first-order formula, and $\exists F_0 \ldots \exists F_{M-1} \forall v_0, \ldots, \forall v_{N-1} \psi$ is equivalent to Φ_0.

To satisfy conditions (a) and (b), we apply apply the necessary number of the following transformations to ψ.

(a) Suppose two function variables occur nested in ψ. For example, suppose

$$F_i(t_0, \ldots, t_k, \ldots, t_{m-1})$$

occurs in ψ, and t_k is $F_j(t'_0, \ldots, t'_{n-1})$. Let ψ' be the result of replacing t_k with a new individual variable v_N. Observe that

$$\exists F_0 \ldots \exists F_{M-1} \forall v_0 \ldots \forall v_{N-1} \forall v_N (v_N \neq F_j(t'_0, \ldots, t'_{n-1}) \vee \psi')$$

has the correct form and is equivalent to Φ_0.

(b) Suppose that a function variable occurs with two different sequences of variables as arguments. For example, suppose $F_k(v_{i_0}, \ldots, v_{i_{n-1}})$ and $F_k(v_{j_0}, \ldots, v_{j_{n-1}})$ both occur in ψ. Introduce a new function variable F_M, and let ψ' be the result of replacing each occurrence of $F_k(v_{j_0}, \ldots, v_{j_{n-1}})$ with $F_M(v_{j_0}, \ldots, v_{j_{n-1}})$. Let θ be the quantifier-free formula

$$\neg(v_{i_0} = v_{j_0} \wedge \cdots \wedge v_{i_{n-1}} = v_{j_{n-1}}) \vee F_k(v_{i_0}, \ldots, v_{i_{n-1}}) = F_M(v_{i_0}, \ldots, v_{i_{n-1}})$$

Observe that

$$\exists F_0 \ldots \exists F_{M-1} \exists F_M \forall v_0 \ldots \forall v_{N-1} (\theta \wedge \psi')$$

has the correct form and is equivalent to Φ_0. $\qquad \square$

Theorem 1.16. *For every Σ^1_1-sentence Φ, there is an IFG-sentence ϕ such that ϕ^* is equivalent to Φ.*

Proof. Let Φ be a Σ^1_1-sentence. By the previous lemma, we may assume that Φ has the form

$$\exists F_0 \ldots \exists F_{M-1} \forall v_0 \ldots \forall v_{N-1} \psi,$$

where ψ is a quantifier-free first-order formula that satisfies conditions (a) and (b). For each function variable F_k, let $v_{N \setminus J_k}$ be the (unique) set of variables that occur as arguments of F_k. Let ϕ be

$$\forall v_{0/\emptyset} \ldots \forall v_{N-1/\emptyset} \exists v_{N/J_0} \ldots \exists v_{N+M-1/J_{M-1}} \psi',$$

where ψ' is the result of replacing each occurrence of $F_k(v_{N \setminus J_k})$ in ψ with v_{N+k}.

For all $k \in M$, let ψ'_k be $\exists v_{N+k/J_k} \ldots \exists v_{N+M-1/J_{M-1}} \psi'$, and let G_k be the function symbol $F_{\psi'_k}$. Then ϕ^* is

$$\exists G_0 \ldots \exists G_{M-1} \forall v_0 \ldots \forall v_{N-1} \psi'',$$

where ψ'' is the result of replacing each occurrence of v_{N+k} in ψ' with $G_k(v_{N \setminus J_k})$. Observe that ϕ^* is equivalent to Φ. $\qquad\square$

Corollary 1.17. *For every Σ_1^1-sentence Φ, there is an IFG-sentence ϕ such that for every suitable structure $\mathfrak{A}$,*

$$\mathfrak{A} \models_{\mathrm{sol}} \Phi \quad \textit{iff} \quad \phi \textit{ is true in } \mathfrak{A}.$$

1.4.3 Orthogonality of truth and falsity in IFG logic

Definition. For every IFG-sentence ϕ, define

$$\mathrm{Mod}^+(\phi) = \{\, \mathfrak{A} \mid \phi \text{ is true in } \mathfrak{A} \,\},$$

$$\mathrm{Mod}^-(\phi) = \{\, \mathfrak{A} \mid \phi \text{ is false in } \mathfrak{A} \,\}.$$

We can deduce from Proposition 1.3 that

$$\mathrm{Mod}^+(\sim\psi) = \mathrm{Mod}^-(\psi),$$

$$\mathrm{Mod}^-(\sim\psi) = \mathrm{Mod}^+(\psi).$$

Furthermore, it follows from results in the next section that

$$\mathrm{Mod}^+(\psi_1 \vee_{/\emptyset} \psi_2) = \mathrm{Mod}^+(\psi_1) \cup \mathrm{Mod}^+(\psi_2),$$

$$\mathrm{Mod}^-(\psi_1 \vee_{/\emptyset} \psi_2) = \mathrm{Mod}^-(\psi_1) \cap \mathrm{Mod}^-(\psi_2),$$

$$\mathrm{Mod}^+(\psi_1 \wedge_{/\emptyset} \psi_2) = \mathrm{Mod}^+(\psi_1) \cap \mathrm{Mod}^+(\psi_2),$$

$$\mathrm{Mod}^-(\psi_1 \wedge_{/\emptyset} \psi_2) = \mathrm{Mod}^-(\psi_1) \cup \mathrm{Mod}^-(\psi_2).$$

Definition. For every second-order sentence Φ, define $\mathrm{Mod}_{\mathrm{sol}}(\Phi) = \{\, \mathfrak{A} \mid \mathfrak{A} \models_{\mathrm{sol}} \Phi \,\}$.

In ordinary first-order logic it is commonplace to ignore the empty structure. In IFG logic, it is common to only consider structures with more than one element because, if there is only one element in the universe, restricting information is pointless. In the next theorem, we assume that all structure have at least two elements.

Theorem 1.18 (Burgess [2]). *Let Φ and Ψ be incompatible Σ_1^1-sentences. Then there is an IFG-sentence θ such that for every suitable structure $\mathfrak{A}$,*

$$\mathrm{Mod}^+(\theta) = \mathrm{Mod}_{\mathrm{sol}}(\Phi) \quad and \quad \mathrm{Mod}^-(\theta) = \mathrm{Mod}_{\mathrm{sol}}(\Psi).$$

Proof. Let θ_0 be the IFG-sentence $\forall v_{0/\emptyset} \exists v_{1/\{0\}}(v_0 = v_1)$. Observe that

$$\mathrm{Mod}^+(\theta_0) = \mathrm{Mod}^-(\theta_0) = \emptyset.$$

By Corollary 1.17, there exist IFG-sentences ϕ' and ψ' such that $\mathrm{Mod}^+(\phi') = \mathrm{Mod}_{\mathrm{sol}}(\Phi)$ and $\mathrm{Mod}^+(\psi') = \mathrm{Mod}_{\mathrm{sol}}(\Psi)$. Let ϕ be $\phi' \vee_{/\emptyset} \theta_0$, and let ψ be $\psi' \vee_{/\emptyset} \theta_0$. Then $\mathrm{Mod}^+(\phi) = \mathrm{Mod}_{\mathrm{sol}}(\Phi)$ and $\mathrm{Mod}^+(\psi) = \mathrm{Mod}_{\mathrm{sol}}(\psi)$, while $\mathrm{Mod}^-(\phi) = \mathrm{Mod}^-(\psi) = \emptyset$.

By Craig's interpolation theorem for Σ_1^1 logic, there is a first-order sentence χ such that $\mathrm{Mod}_{\mathrm{sol}}(\Phi) \subseteq \mathrm{Mod}(\chi) = \mathrm{Mod}^+(\chi)$ and $\mathrm{Mod}_{\mathrm{sol}}(\Psi) \subseteq \mathrm{Mod}(\neg\chi) = \mathrm{Mod}^-(\chi)$. Hence, $\mathrm{Mod}^+(\phi) \subseteq \mathrm{Mod}^+(\chi)$ and $\mathrm{Mod}^+(\psi) \subseteq \mathrm{Mod}^-(\chi)$. Let θ be $\phi \wedge_{/\emptyset} (\sim\psi \vee_{/\emptyset} \chi)$. Then

$$\mathrm{Mod}^+(\theta) = \mathrm{Mod}^+(\phi) \cap (\mathrm{Mod}^+(\sim\psi) \cup \mathrm{Mod}^+(\chi))$$

$$= \mathrm{Mod}^+(\phi) \cap (\mathrm{Mod}^-(\psi) \cup \mathrm{Mod}^+(\chi))$$

$$= \mathrm{Mod}^+(\phi) \cap \mathrm{Mod}^+(\chi)$$

$$= \mathrm{Mod}^+(\phi),$$

$$\mathrm{Mod}^-(\theta) = \mathrm{Mod}^-(\phi) \cup (\mathrm{Mod}^-(\sim\psi) \cap \mathrm{Mod}^-(\chi))$$

$$= \emptyset \cup (\mathrm{Mod}^+(\psi) \cap \mathrm{Mod}^-(\chi))$$

$$= \mathrm{Mod}^+(\psi).$$

Thus $\mathrm{Mod}^+(\theta) = \mathrm{Mod}_{\mathrm{sol}}(\Phi)$ and $\mathrm{Mod}^-(\theta) = \mathrm{Mod}_{\mathrm{sol}}(\Psi)$. $\qquad\square$

1.5   Trump semantics

The main definition and theorem of this section are due to Hodges [15, 16]. See also related work by Cameron and Hodges [5], Caicedo and Krynicki [4], Väänänen [23], and Dechesne [7].

1.5.1   Preliminaries

Definition. Given any set V, a **cover** of V is a collection of sets $\mathscr{U}$ such that $V = \bigcup \mathscr{U}$. A **disjoint cover** of V is a cover of V whose members are pairwise disjoint. A **partition** of V is a disjoint cover of V whose members are all nonempty. Sometimes the members of a partition are called **cells**.

Definition. Let $V \subseteq {}^N A$ be a team, $J \subseteq N$, and $\mathscr{U}$ a cover of V. Call the cover J-**saturated** if every $U \in \mathscr{U}$ is closed under $\approx_J$. That is, for every $\vec{a}, \vec{b} \in V$, if $\vec{a} \approx_J \vec{b}$ and $\vec{a} \in U \in \mathscr{U}$, then $\vec{b} \in U$.

Note that every member of a J-saturated cover of V is a union of equivalence classes under $\approx_J$. Also note that every cover of V is $\emptyset$-saturated, and that the only N-saturated covers of V are those that include V as one of their members.

Definition. Define a partial operation $\bigcup_J$ on collections of sets of valuations by setting $\bigcup_J \mathscr{U} = \bigcup \mathscr{U}$ whenever $\mathscr{U}$ is a J-saturated disjoint cover of $\bigcup \mathscr{U}$ and letting $\bigcup_J \mathscr{U}$ be undefined otherwise. Thus the formula $V = \bigcup_J \mathscr{U}$ asserts that $\mathscr{U}$ is a J-saturated disjoint cover of V. We will use the notation $V_1 \cup_J V_2$ to abbreviate $\bigcup_J \{V_1, V_2\}$, the notation $V_1 \cup_J V_2 \cup_J V_3$ to abbreviate $\bigcup_J \{V_1, V_2, V_3\}$, et cetera.

Lemma 1.19. *Let $V \subseteq {}^N A$ and $J \subseteq N$. If $\mathscr{U}$ is a J-saturated cover of V, then there is a J-saturated disjoint cover $\mathscr{U}'$ of V such that every cell in $\mathscr{U}'$ is contained in some cell in $\mathscr{U}$.*

Proof. Let $\langle U_\alpha \mid \alpha < \mu \rangle$ be an enumeration of $\mathscr{U}$. Define $U'_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$, and let $\mathscr{U}' = \{ U'_\alpha \mid \alpha < \mu \}$. By construction, $\mathscr{U}'$ is a disjoint cover of V such that $U'_\alpha \subseteq U_\alpha$. To show $\mathscr{U}'$ is J-saturated, suppose $\vec{a}, \vec{b} \in V$. If $\vec{a} \approx_J \vec{b}$ and $\vec{a} \in U'_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$, then $\vec{b} \in U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta = U'_\alpha$ because $\mathscr{U}$ is J-saturated. $\qquad \square$

Lemma 1.20. *Let $V \subseteq {}^N A$ and $J \subseteq K \subseteq N$. If $V = \bigcup_K \mathscr{U}$, then $V = \bigcup_J \mathscr{U}$.*

Proof. Suppose $V = \bigcup_K \mathscr{U}$ and $\vec{a}, \vec{b} \in V$. If $\vec{a} \approx_J \vec{b}$ and $\vec{a} \in U \in \mathscr{U}$, then $\vec{a} \approx_K \vec{b}$, so $\vec{b} \in U$. Thus $V = \bigcup_J \mathscr{U}$. $\qquad\square$

Lemma 1.21. *Let $V' \subseteq V \subseteq {}^N A$. If $V = \bigcup_J \mathscr{U}$, then $V' = \bigcup_J \mathscr{U}'$, where*

$$\mathscr{U}' = \{ U \cap V' \mid U \in \mathscr{U} \}.$$

Proof. Suppose $\vec{a}, \vec{b} \in V'$. If $\vec{a} \approx_J \vec{b}$, and $\vec{a} \in U \cap V' \in \mathscr{U}'$, then we have $\vec{b} \in U \cap V'$ because $V = \bigcup_J \mathscr{U}$. $\qquad\square$

Lemma 1.22. *If $V = V_1 \cup_J V_2$ and $V_1 = \bigcup_J \mathscr{U}_1$, $V_2 = \bigcup_J \mathscr{U}_2$, then $V = \bigcup_J (\mathscr{U}_1 \cup \mathscr{U}_2)$.*

Proof. Suppose $\vec{a}, \vec{b} \in V$, $\vec{a} \approx_J \vec{b}$, and $\vec{a} \in U \in \mathscr{U}_1 \cup \mathscr{U}_2$. Without loss of generality, we may assume $U \in \mathscr{U}_1$, which implies that $\vec{a} \in V_1$, in which case $\vec{b} \in V_1$ because $V = V_1 \cup_J V_2$. Hence $\vec{b} \in U$ because $V_1 = \bigcup_J \mathscr{U}_1$. $\qquad\square$

Definition. Let $V \subseteq {}^N A$ and $J \subseteq N$. A function $f : V \to A$ is **independent of** J, denoted $f : V \xrightarrow[J]{} A$, if $f(\vec{a}) = f(\vec{b})$ whenever $\vec{a} \approx_J \vec{b}$.

Note that any function $f : V \to A$ is independent of $\emptyset$, and that $f : V \to A$ is independent N if and only if f is a constant function.

Lemma 1.23. *Let $V \subseteq {}^N A$ and $J \subseteq K \subseteq N$. If $f : V \xrightarrow[K]{} A$, then $f : V \xrightarrow[J]{} A$.*

Proof. Suppose $f : V \xrightarrow[K]{} A$. If $\vec{a} \approx_J \vec{b}$, then $\vec{a} \approx_K \vec{b}$, so $f(\vec{a}) = f(\vec{b})$. Thus $f : V \xrightarrow[J]{} A$. $\qquad\square$

Lemma 1.24. *Let $V \subseteq {}^N A$ and $J \subseteq N$. If $V = V_1 \cup_J V_2$ and $f : V_1 \xrightarrow[J]{} A$, $g : V_2 \xrightarrow[J]{} A$, then $(f \cup g) : V \xrightarrow[J]{} A$.*

Proof. Suppose $\vec{a} \approx_J \vec{b}$. Then because $V = V_1 \cup_J V_2$, either $\vec{a}, \vec{b} \in V_1$ or $\vec{a}, \vec{b} \in V_2$. In the first case, $(f \cup g)(\vec{a}) = f(\vec{a}) = f(\vec{b}) = (f \cup g)(\vec{b})$, and in the second case, $(f \cup g)(\vec{a}) = g(\vec{a}) = g(\vec{b}) = (f \cup g)(\vec{b})$, because f and g are both independent of J. Therefore $(f \cup g)$ is independent of J. $\qquad\square$

Lemma 1.25. *If $f : V \xrightarrow[J]{} A$, $V = V_1 \cup_K V_2$, and $n \in K$, then $V(n : f) = V_1(n : f) \cup_K V_2(n : f)$.*

Proof. Suppose $f\colon V \xrightarrow{J} A$, $V = V_1 \cup_K V_2$, and $n \in K$. Then $V_1(n : f)$ and $V_2(n : f)$ are both subsets of $V(n : f)$, so $V_1(n : f) \cup V_2(n : f) \subseteq V(n : f)$. Conversely, suppose $\vec{a}(n : f(\vec{a})) \in V(n : f)$, where $\vec{a} \in V$. If $\vec{a} \in V_1$, then $\vec{a}(n : f(\vec{a})) \in V_1(n : f)$, and if $\vec{a} \in V_2$, then $\vec{a}(n : f(\vec{a})) \in V_2(n : f)$. Hence $\vec{a}(n : f(\vec{a})) \in V_1(n : f) \cup V_2(n : f)$. Thus $V(n : f) \subseteq V_1(n : f) \cup V_2(n : f)$.

To show $V_1(n : f)$ and $V_2(n : f)$ are disjoint, suppose $\vec{a}(n : f(\vec{a})) \in V_1(n : f)$ and $\vec{b}(n : f(\vec{b})) \in V_2(n : f)$, where $\vec{a} \in V_1$ and $\vec{b} \in V_2$. If $\vec{a}(n : f(\vec{a})) = \vec{b}(n : f(\vec{b}))$, then $\vec{a} \approx_{\{n\}} \vec{b}$, hence $\vec{a} \approx_K \vec{b}$ because $n \in K$, contradicting $V = V_1 \cup_K V_2$.

Finally suppose $\vec{a}(n : f(\vec{a})) \approx_K \vec{b}(n : f(\vec{b}))$ and $\vec{a}(n : f(\vec{a})) \in V_i(n : f)$, where $\vec{a} \in V_i$. Then $\vec{a} \approx_K \vec{b}$ because $n \in K$. Hence $\vec{b} \in V_i$. Thus $\vec{b}(n : f(\vec{b})) \in V_i(n : f(\vec{b}))$. $\qquad\square$

Lemma 1.26. *Suppose $f\colon V \xrightarrow{J} A$ and $g\colon V(n : f) \xrightarrow{K} A$. Then there is a function $h\colon V \xrightarrow{J \cap K} A$ such that $V(n : f)(n : g) = V(n : h)$. If $n \in K$, then h is independent of K.*

Proof. Define $h\colon V \to A$ by $h(\vec{a}) = g(\vec{a}(n : f(\vec{a})))$. To show that h is independent of $J \cap K$, suppose $\vec{a} \approx_{J \cap K} \vec{b}$. Then $\vec{a} \approx_J \vec{b}$, so $f(\vec{a}) = f(\vec{b})$. Hence $\vec{a}(n : f(\vec{a})) \approx_{J \cap K} \vec{b}(n : f(\vec{b}))$, which implies $\vec{a}(n : f(\vec{a})) \approx_K \vec{b}(n : f(\vec{b}))$. Thus $h(\vec{a}) = g(\vec{a}(n : f(\vec{a}))) = g(\vec{b}(n : f(\vec{b}))) = h(\vec{b})$. Now suppose $n \in K$ and $\vec{a} \approx_K \vec{b}$. Then $\vec{a}(n : f(\vec{a})) \approx_K \vec{b}(n : f(\vec{b}))$. Thus $h(\vec{a}) = g(\vec{a}(n : f(\vec{a}))) = g(\vec{b}(n : f(\vec{b}))) = h(\vec{b})$.

Let $f^{(n)}\colon V \to V(n : f)$ be the function that maps $\vec{a}$ to $\vec{a}(n : f(\vec{a}))$. Let $g^{(n)}$ and $h^{(n)}$ be defined similarly. Then $h^{(n)} = g^{(n)} \circ f^{(n)}$ and

$$V(n : h) = h^{(n)}(V) = g^{(n)} \circ f^{(n)}(V) = g^{(n)}(f^{(n)}(V)) = V(n : f)(n : g). \qquad \square$$

Lemma 1.27. *Given two functions $f\colon V \xrightarrow{J} A$ and $h\colon V \xrightarrow{K} A$, with $n \in K$, there is a function $g\colon V(n : f) \xrightarrow{K} A$ such that $V(n : f)(n : g) = V(n : h)$.*

Proof. Define $g\colon V(n : f) \xrightarrow{K} A$ by $g(\vec{a}(n : f(\vec{a})) = h(\vec{a})$. To show that g is well defined and independent of K, suppose $\vec{a}, \vec{b} \in V$ and $\vec{a}(n : f(\vec{a})) \approx_K \vec{b}(n : f(\vec{b}))$. Then $\vec{a} \approx_K \vec{b}$ because $n \in K$. Hence $g(\vec{a}(n : f(\vec{a})) = h(\vec{a}) = h(\vec{b}) = g(\vec{b}(n : f(\vec{b})))$. By construction $h^{(n)} = g^{(n)} \circ f^{(n)}$, so $V(n : h) = V(n : f)(n : g)$. $\qquad\square$

Lemma 1.28. *Suppose $f \colon V \underset{J}{\to} A$ and $g \colon V(m : f) \underset{K}{\to} A$. If $m \in K$ and $n \in J$, where $m \neq n$, then there exist functions $G \colon V \underset{K}{\to} A$ and $F \colon V(n : G) \underset{J}{\to} A$ such that $V(m : f)(n : g) = V(n : G)(m : F)$.*

Proof. Define $G \colon V \underset{K}{\to} A$ by $G(\vec{a}) = g(\vec{a}(m : f(\vec{a})))$ and $F \colon V(n : G) \underset{J}{\to} A$ by $F(\vec{a}(n : G(\vec{a}))) = f(\vec{a})$. Observe that G is independent of K because $m \in K$ and g is independent of K. Also observe that F is well defined and independent of J because $n \in J$ and f is independent of J.

To show $V(m : f)(n : g) = V(n : G)(m : F)$ it suffices to show that $g^{(n)} \circ f^{(m)} = F^{(m)} \circ G^{(n)}$. So let $\vec{a} \in V$. Then

$$g^{(n)} \circ f^{(m)}(\vec{a}) = \vec{a}(m : f(\vec{a}))(n : g(\vec{a}(m : f(\vec{a}))))$$

$$= \vec{a}(m : f(\vec{a}))(n : G(\vec{a}))$$

$$= \vec{a}(n : G(\vec{a}))(m : F(\vec{a}(n : G(\vec{a}))))$$

$$= F^{(m)} \circ G^{(n)}(\vec{a}). \qquad \square$$

Lemma 1.29. *Suppose $n \in J$.*

    (a) If $V = V_1 \cup_J V_2$, then $V(n : A) = V_1(n : A) \cup_J V_2(n : A)$.

    (b) If $V(n : A) = V_1 \cup_J V_2$, then $V_1 = V_1(n : A)$ and $V_2 = V_2(n : A)$.

Proof. (a) Suppose $V = V_1 \cup_J V_2$. If $\vec{a}(n : c) \in V(n : A)$, then either $\vec{a} \in V_1$ or $\vec{a} \in V_2$. In the first case $\vec{a}(n : c) \in V_1(n : A)$, and in the second case $\vec{a}(n : c) \in V_2(n : A)$. Hence $V(n : A) = V_1(n : A) \cup V_2(n : A)$. To show J-saturation, suppose $\vec{a}, \vec{b} \in V$, $\vec{a}(n : c) \approx_J \vec{b}(n : d)$, and $\vec{a}(n : c) \in V_i(n : A)$. Then $\vec{a} \approx_J \vec{b}$ because $n \in J$, and $\vec{a} \in V_i$, so $\vec{b} \in V_i$. Hence $\vec{b}(n : d) \in V_i(n : A)$. To show $V_1(n : A)$ and $V_2(n : A)$ are disjoint, suppose $\vec{c} \in V_1(n : A) \cap V_2(n : A)$. Then there exist $\vec{a} \in V_1$, $\vec{b} \in V_2$, and $d \in A$ such that $\vec{a}(n : d) = \vec{c} = \vec{b}(n : d)$. But then $\vec{a} \approx_J \vec{b}$, which implies $\vec{a}, \vec{b} \in V_1 \cap V_2$, contradicting the fact that V_1 and V_2 are disjoint.

(b) Suppose $V(n : A) = V_1 \cup_J V_2$. By definition, $V_i \subseteq V_i(n : A)$. If $\vec{a}(n : c) \in V_i(n : A)$, then $\vec{a} \approx_J \vec{a}(n : c)$ and $\vec{a} \in V_i$, so $\vec{a}(n : c) \in V_i$. Hence $V_i(n : A) \subseteq V_i$. $\qquad \square$

1.5.2   Trump Semantics

We now define an Tarski-style satisfaction relation for IFG-formulas.

Definition. Let ϕ be an IFG_N-formula, let $\mathfrak{A}$ be a suitable structure, and let $V \subseteq {}^N A$.

- If ϕ is atomic, then

  (+) $\mathfrak{A} \models^+ \phi[V]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \models \phi[\vec{a}]$,

  (−) $\mathfrak{A} \models^- \phi[V]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \not\models \phi[\vec{a}]$.

- If ϕ is $\sim\psi$, then

  (+) $\mathfrak{A} \models^+ \sim\psi[V]$ if and only if $\mathfrak{A} \models^- \psi[V]$,

  (−) $\mathfrak{A} \models^- \sim\psi[V]$ if and only if $\mathfrak{A} \models^+ \psi[V]$.

- If $J \subseteq N$ and ϕ is $\psi_1 \vee_{/J} \psi_2$, then

  (+) $\mathfrak{A} \models^+ \psi_1 \vee_{/J} \psi_2[V]$ if and only if there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$

  such that $\mathfrak{A} \models^+ \psi_1[V_1]$ and $\mathfrak{A} \models^+ \psi_2[V_2]$,

  (−) $\mathfrak{A} \models^- \psi_1 \vee_{/J} \psi_2[V]$ if and only if $\mathfrak{A} \models^- \psi_1[V]$ and $\mathfrak{A} \models^- \psi_2[V]$.

- If $J \subseteq N$ and ϕ is $\exists v_{n/J}\psi$, then

  (+) $\mathfrak{A} \models^+ \exists v_{n/J}\psi[V]$ if and only if there is a function $f \colon V \underset{J}{\to} A$ independent of J such

  that $\mathfrak{A} \models^+ \psi[V(n:f)]$,

  (−) $\mathfrak{A} \models^- \exists v_{n/J}\psi[V]$ if and only if $\mathfrak{A} \models^- \psi[V(n:A)]$.

It is easy to check that the abbreviations $\bigwedge_{/J}$ and $\forall v_{n/J}$ behave as expected.

- If $J \subseteq N$ and ϕ is $\psi_1 \wedge_{/J} \psi_2$, then

  (+) $\mathfrak{A} \models^+ \psi_1 \wedge_{/J} \psi_2[V]$ if and only if $\mathfrak{A} \models^+ \psi_1[V]$ and $\mathfrak{A} \models^+ \psi_2[V]$,

  (−) $\mathfrak{A} \models^- \psi_1 \wedge_{/J} \psi_2[V]$ if and only if there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$

  such that $\mathfrak{A} \models^- \psi_1[V_1]$ and $\mathfrak{A} \models^- \psi_2[V_2]$.

- If $J \subseteq N$ and ϕ is $\forall v_{n/J}\psi$, then

  (+) $\mathfrak{A} \models^+ \forall v_{n/J}\psi[V]$ if and only if $\mathfrak{A} \models^+ \psi[V(n:A)]$,

  (−) $\mathfrak{A} \models^- \forall v_{n/J}\psi[V]$ if and only if there is a function $f\colon V \underset{J}{\to} A$ independent of J such that $\mathfrak{A} \models^- \psi[V(n:f)]$.

Definition. Let ϕ be an IFG_N-formula, and let $\mathfrak{A}$ be a suitable structure. Define $\mathfrak{A} \models^\pm \phi$ if and only if $\mathfrak{A} \models^\pm \phi[^N A]$.

Lemma 1.30. *Let ϕ be an IFG_N-formula, and let $\mathfrak{A}$ be a suitable structure. Then $\mathfrak{A} \models^\pm \phi[\emptyset]$.*

Proof. We proceed by induction on the complexity of ϕ. If ϕ is atomic the lemma holds vacuously.

Suppose ϕ is $\sim\psi$. Then $\mathfrak{A} \models^\pm \sim\psi[\emptyset]$ if and only if $\mathfrak{A} \models^\mp \psi[\emptyset]$, which holds by inductive hypothesis.

Suppose $J \subseteq N$ and ϕ is $\psi_1 \vee_{/J} \psi_2$. By inductive hypothesis, $\mathfrak{A} \models^\pm \psi_1[\emptyset]$ and $\mathfrak{A} \models^\pm \psi_2[\emptyset]$. Therefore $\mathfrak{A} \models^\pm \psi_1 \vee_{/J} \psi_2[\emptyset]$.

Suppose $J \subseteq N$ and ϕ is $\exists v_{n/J}\psi$. By inductive hypothesis, $\mathfrak{A} \models^\pm \psi[\emptyset]$. Let f be the empty function from $\emptyset$ to A. Then f is vacuously independent of J and $\emptyset(n:f) = \emptyset$. Therefore $\mathfrak{A} \models^+ \exists v_{n/J}\psi[\emptyset]$. Also $\emptyset(n:A) = \emptyset$, so $\mathfrak{A} \models^- \exists v_{n/J}\psi[\emptyset]$. $\qquad\square$

The previous result may seems anomalous, but it is necessary for technical reasons. Specifically, in the positive disjunction clause we wish to allow $V_1 = V$ and $V_2 = \emptyset$ or vice versa, which corresponds to the situation in the game $G(\mathfrak{A}, \phi, V)$ when the verifier always wishes to choose the same disjunct. Later we will see that the empty team is the only team that can be winning and losing for the same formula.

The next lemma records the fact that if a player has a winning strategy, given that he or she knows the current valuation belongs to V, then that strategy should win given that he or she knows the current valuation belongs to a subset of V.

Lemma 1.31. *Let ϕ be an IFG_N-formula, let $\mathfrak{A}$ be a suitable structure, and let $W \subseteq V \subseteq {}^N A$. Then $\mathfrak{A} \models^\pm \phi[V]$ implies $\mathfrak{A} \models^\pm \phi[W]$.*

Proof. We proceed by induction on the complexity of ϕ. The atomic case follows immediately from the definition.

Suppose ϕ is $\sim\psi$. Then $\mathfrak{A} \models^{\pm} \sim\psi[V]$ if and only if $\mathfrak{A} \models^{\mp} \psi[V]$, which by inductive hypothesis implies $\mathfrak{A} \models^{\mp} \psi[W]$, which holds if and only if $\mathfrak{A} \models^{\pm} \sim\psi[W]$.

Suppose $J \subseteq N$ and ϕ is $\psi_1 \vee_{/J} \psi_2$. If $\mathfrak{A} \models^{+} \psi_1 \vee_{/J} \psi_2[V]$, then there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$ such that $\mathfrak{A} \models^{+} \psi_1[V_1]$ and $\mathfrak{A} \models^{+} \psi_2[V_2]$. Let $W_1 = V_1 \cap W$ and $W_2 = V_2 \cap W$. Then by Lemma 1.21, $W = W_1 \cup_J W_2$ is a J-saturated disjoint cover, and by inductive hypothesis $\mathfrak{A} \models^{+} \psi_1[W_1]$ and $\mathfrak{A} \models^{+} \psi_2[W_2]$. Thus $\mathfrak{A} \models^{+} \psi_1 \vee_{/J} \psi_2[W]$.

If $\mathfrak{A} \models^{-} \psi_1 \vee_{/J} \psi_2[V]$, then $\mathfrak{A} \models^{-} \psi_1[V]$ and $\mathfrak{A} \models^{-} \psi_2[V]$. By inductive hypothesis, $\mathfrak{A} \models^{-} \psi_1[W]$ and $\mathfrak{A} \models^{-} \psi_2[W]$. Thus $\mathfrak{A} \models^{-} \psi_1 \vee_{/J} \psi_2[W]$.

Suppose $J \subseteq N$ and ϕ is $\exists v_{n/J}\psi$. If $\mathfrak{A} \models^{+} \exists v_{n/J}\psi[V]$, then there is a function $f\colon V \underset{J}{\to} A$ independent of J such that $\mathfrak{A} \models^{+} \psi[V(n:f)]$. The restriction of f to W is independent of J, and by inductive hypothesis $\mathfrak{A} \models^{+} \psi[W(n:f)]$. Thus $\mathfrak{A} \models^{+} \exists v_{n/J}\psi[W]$.

If $\mathfrak{A} \models^{-} \exists v_{n/J}\psi[V]$, then $\mathfrak{A} \models^{-} \psi[V(n:A)]$. By inductive hypothesis we have $\mathfrak{A} \models^{-} \psi[W(n:A)]$. Thus $\mathfrak{A} \models^{-} \exists v_{n/J}\psi[W]$. $\qquad\square$

The main result of this section is that trump semantics and game semantics are equivalent. This is significant because of Hintikka's claim in The Principles of Mathematics Revisited that independence-friendly logic does not have a compositional semantics.

Theorem 1.32. *Let ϕ be an IFG_N-formula, let $\mathfrak{A}$ be a suitable structure, and let $V \subseteq {}^N A$. Then*

(a) $\mathfrak{A} \models^{+} \phi[V]$ *if and only if ϕ is true in $\mathfrak{A}$ relative to V;*

(b) $\mathfrak{A} \models^{-} \phi[V]$ *if and only if ϕ is false in $\mathfrak{A}$ relative to V.*

Proof. We will prove the theorem by simultaneous induction on the complexity of ϕ. Suppose ϕ is atomic. Then the only strategy either player has for the game $G(\mathfrak{A}, \phi, V)$ is the empty strategy. For every play $\langle\langle \phi, \vec{a}, 1\rangle\rangle$, Eloïse will win if and only if $\mathfrak{A} \models \phi[\vec{a}]$, and Abélard will

win if and only if $\mathfrak{A} \not\models \phi[\vec{a}]$. Thus Eloïse has a winning strategy if and only if $\mathfrak{A} \models^+ \phi[V]$, and Abélard has a winning strategy if and only if $\mathfrak{A} \models^- \phi[V]$.

Suppose ϕ is $\sim\psi$. Then $\mathfrak{A} \models^+ \sim\psi[V]$ if and only if $\mathfrak{A} \models^- \psi[V]$, which by inductive hypothesis holds if and only if Abélard has a winning strategy S for $G(\mathfrak{A}, \psi, V)$. By Lemma 1.2, this holds if and only if Eloïse has a winning strategy for $G(\mathfrak{A}, \sim\psi, V)$. Similarly, $\mathfrak{A} \models^- \sim\psi[V]$ if and only if $\mathfrak{A} \models^+ \psi[V]$, which by inductive hypothesis holds if and only if Eloïse has a winning strategy for $G(\mathfrak{A}, \psi, V)$ if and only if Abélard has a winning strategy for $G(\mathfrak{A}, \sim\psi, V)$.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. If $\mathfrak{A} \models^+ \psi_1 \vee_{/J} \psi_2[V]$, then there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$ such that $\mathfrak{A} \models^+ \psi_1[V_1]$ and $\mathfrak{A} \models^+ \psi_2[V_2]$. Hence, by inductive hypothesis, Eloïse has a winning strategy S_1 for $G(\mathfrak{A}, \psi_1, V_1)$ and a winning strategy S_2 for $G(\mathfrak{A}, \psi_2, V_2)$. Define a winning strategy S for Eloïse for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$ as follows. $S(\langle \psi_1 \vee_{/J} \psi_2, \vec{a}, 1 \rangle) = \langle \psi_i, \vec{a}, 1 \rangle$ if $\vec{a} \in V_i$, and $S(\langle \chi, \vec{b}, 1 \rangle) = S_i(\langle \chi, \vec{b}, 1 \rangle)$ if χ is a subformula of ψ_i. Note that S is a legal strategy because the disjoint cover $V = V_1 \cup V_2$ is J-saturated.

Conversely, if Eloïse has a winning strategy S for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$, then for each $\vec{a} \in V$, let $\vec{a} \in V_i$ if $S(\langle \psi_1 \vee_{/J} \psi_2, \vec{a}, 1 \rangle) = \langle \psi_i, \vec{a}, 1 \rangle$. The resulting disjoint cover $V = V_1 \cup_J V_2$ will be J-saturated because S is a legal strategy. Let $S_i(\langle \chi, \vec{b}, 1 \rangle) = S(\langle \chi, \vec{b}, 1 \rangle$ for every subformula χ of ψ_i. Then S_1 is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi_1, V_1)$, and S_2 is a winning strategy for $G(\mathfrak{A}, \psi_2, V_2)$. Therefore, by inductive hypothesis, $\mathfrak{A} \models^+ \psi_1[V_1]$ and $\mathfrak{A} \models^+ \psi_2[V_2]$. Hence $\mathfrak{A} \models^+ \psi_1 \vee_{/J} \psi_2[V]$.

If $\mathfrak{A} \models^- \psi_1 \vee_{/J} \psi_2[V]$, then $\mathfrak{A} \models^- \psi_1[V]$ and $\mathfrak{A} \models^- \psi_2[V]$, so by inductive hypothesis Abélard has a winning strategy S_1 for $G(\mathfrak{A}, \psi_1, V)$ and a winning strategy S_2 for $G(\mathfrak{A}, \psi_2, V)$. Then $S_1 \cup S_2$ is a winning strategy for Abélard for the game $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$.

Conversely, suppose Abélard has a winning strategy S for $G(\mathfrak{A}, \psi_1 \vee_{/J} \psi_2, V)$. Let $S_i(\langle \chi, \vec{b}, 0 \rangle) = S(\langle \chi, \vec{b}, 0 \rangle)$ for every subformula χ of ψ_i. Then S_1 is a winning strategy for Abélard for $G(\mathfrak{A}, \psi_1, V)$, and S_2 is a winning strategy for him for $G(\mathfrak{A}, \psi_2, V)$. Therefore, by inductive hypothesis, $\mathfrak{A} \models^- \psi_1[V]$ and $\mathfrak{A} \models^- \psi_2[V]$. Hence $\mathfrak{A} \models^- \psi_1 \vee_{/J} \psi_2[V]$.

Now suppose ϕ is $\exists v_{n/J}\psi$. If $\mathfrak{A} \models^+ \exists v_{n/J}\psi[V]$, then there is a function $f \colon V \underset{J}{\to} A$

such that $\mathfrak{A} \models^+ \psi[V(n : f)]$. By inductive hypothesis, Eloïse has a winning strategy S for $G(\mathfrak{A}, \psi, V(n : f))$. Let $S'(\langle \exists v_{n/J}\psi, \vec{a}, 1 \rangle) = \langle \psi, \vec{a}(n : f(\vec{a})), 1 \rangle$, and let $S'(\langle \chi, \vec{b}, 1 \rangle) = S(\langle \chi, \vec{b}, 1 \rangle)$ for every subformula χ of ψ. Note that S' is a legal strategy because f is independent of J, and it is a winning strategy because $\vec{a}(n : f(\vec{a})) \in V(n : f)$.

Conversely, suppose Eloïse has a winning strategy S' for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Define a function $f \colon V \xrightarrow{J} A$ by $f(\vec{a}) = c$ where $S'(\langle \exists v_{n/J}\psi, \vec{a}, 1 \rangle) = \langle \psi, \vec{a}(n : c), 1 \rangle$. Note that f is independent of J because S' is a legal strategy. Let $S(\langle \chi, \vec{b}, 1 \rangle) = S'(\langle \chi, \vec{b}, 1 \rangle)$ for every subformula χ of ψ. Then S is a winning strategy for Eloïse for $G(\mathfrak{A}, \psi, V(n : f))$. Therefore, by inductive hypothesis, $\mathfrak{A} \models^+ \psi[V(n : f)]$. Hence $\mathfrak{A} \models^+ \exists v_{n/J}\psi[V]$.

If $\mathfrak{A} \models^- \exists v_{n/J}\psi[V]$, then $\mathfrak{A} \models^- \psi[V(n : A)]$, so by inductive hypothesis Abélard has a winning strategy S for $G(\mathfrak{A}, \psi, V(n : A))$. Observe that S is winning strategy for Abélard for the game $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$.

Conversely, suppose Abélard has a winning strategy S for $G(\mathfrak{A}, \exists v_{n/J}\psi, V)$. Then S is a winning strategy for him for the game $G(\mathfrak{A}, \psi, V(n : A))$. Therefore, by inductive hypothesis, $\mathfrak{A} \models^- \psi[V(n : A)]$. Hence $\mathfrak{A} \models^- \exists v_{n/J}\psi[V]$. $\qquad\square$

Corollary 1.33. (a) $\mathfrak{A} \models^+ \phi$ if and only if ϕ is true in $\mathfrak{A}$.

(b) $\mathfrak{A} \models^- \phi$ if and only if ϕ is false in $\mathfrak{A}$.

Next we show that a given team cannot be winning and losing for the same formula, which reflects the fact that Abélard and Eloïse cannot both have winning strategies for the same semantic game.

Lemma 1.34. Let ϕ be an IFG_N-formula, and let $\mathfrak{A}$ be a suitable structure. If $V \subseteq {}^N A$ is a nonempty team, then we cannot have

$$\mathfrak{A} \models^+ \phi[V] \quad and \quad \mathfrak{A} \models^- \phi[V].$$

Proof. We proceed by induction on the complexity of ϕ. Suppose $\mathfrak{A} \models^+ \phi[V]$. If ϕ is atomic, then because V is nonempty there is an $\vec{a} \in V$ such that $\mathfrak{A} \models \phi[\vec{a}]$. Hence $\mathfrak{A} \not\models^- \phi[V]$.

Suppose ϕ is $\sim\psi$. Then $\mathfrak{A} \models^- \psi[V]$, so by inductive hypothesis $\mathfrak{A} \not\models^+ \psi[V]$. Hence $\mathfrak{A} \not\models^- \sim\psi[V]$.

Suppose $J \subseteq N$ and ϕ is $\psi_1 \vee_{/J} \psi_2$. Then there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$ such that $\mathfrak{A} \models^+ \psi_1[V_1]$ and $\mathfrak{A} \models^+ \psi_2[V_2]$. By inductive hypothesis, $\mathfrak{A} \not\models^- \psi_1[V_1]$ and $\mathfrak{A} \not\models^- \psi_2[V_2]$. It follows from downward closure that $\mathfrak{A} \not\models^- \psi_1[V]$ and $\mathfrak{A} \not\models^- \psi_2[V]$. Hence $\mathfrak{A} \not\models^- \psi_1 \vee_{/J} \psi_2[V]$.

Suppose $n < N$, $J \subseteq N$, and ϕ is $\exists v_{n/J}\psi$. Then there is an $f \colon V \underset{J}{\to} A$ such that $\mathfrak{A} \models^+ \psi[V(n : f)]$. By inductive hypothesis, $\mathfrak{A} \not\models^- \psi[V(n : f)]$. It follows from downward closure that $\mathfrak{A} \not\models^- \psi[V(n : A)]$. Hence $\mathfrak{A} \not\models^- \exists v_{n/J}\psi[V]$. $\qquad\square$

The next two results show that IFG-formulas that differ only in their number of variables have essentially the same meaning, as long as we do not encode any information in the extra variables. The phenomenon of one player using extra variables to send (otherwise forbidden) information to him or herself is called **signaling**. For example, let ϕ be the formula

$$v_0 = v_1 \vee_{/\{0,1\}} v_0 \neq v_1,$$

and let $\mathfrak{A}$ be the equality structure over $\{0, 1\}$. If $V = \{00, 01, 10, 11\}$, then Eloïse does not have a winning strategy for $G(\mathfrak{A}, \phi, V)$. However, if ϕ' is the 3-variable version of ϕ, and $V' = \{001, 010, 100, 111\}$, then Eloïse does have a winning strategy for $G(\mathfrak{A}, \phi', V')$ because she can use the value of v_2 to signal whether the values of v_0 and v_1 are equal.

Lemma 1.35. *Let ϕ be an IFG-formula with M variables, and let ϕ' be the same formula but with N variables, where $M < N$. Let $\mathfrak{A}$ be a suitable structure, $V \subseteq {}^N A$, $\emptyset \neq W \subseteq {}^{N \setminus M} A$, and $V' = \{\, \vec{a} \cup \vec{b} \mid \vec{a} \in V,\ \vec{b} \in W \,\}$. Then $\mathfrak{A} \models^\pm \phi[V]$ if and only if $\mathfrak{A} \models^\pm \phi'[V']$.*

Proof. We proceed by induction on the complexity of ϕ. Suppose ϕ is atomic. Then $\mathfrak{A} \models^+ \phi[V]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \models \phi[\vec{a}]$, if and only if for every $\vec{a} \cup \vec{b} \in V'$, $\mathfrak{A} \models \phi'[\vec{a} \cup \vec{b}]$, if and only if $\mathfrak{A} \models^+ \phi'[V']$. Also, $\mathfrak{A} \models^- \phi[V]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \not\models \phi[\vec{a}]$, if and only if for every $\vec{a} \cup \vec{b} \in V'$, $\mathfrak{A} \not\models \phi'[\vec{a} \cup \vec{b}]$, if and only if $\mathfrak{A} \models^- \phi'[V']$.

Suppose ϕ is $\sim\psi$. Then $\mathfrak{A}\models^\pm \sim\psi[V]$ if and only if $\mathfrak{A}\models^\mp \psi[V]$, if and only if $\mathfrak{A}\models^\mp \psi'[V']$ (by inductive hypothesis), if and only if $\mathfrak{A}\models^\pm \sim\psi'[V']$. Here ψ' is the same formula as ψ but with N variables.

Suppose $J\subseteq M$ and ϕ is $\psi_1 \vee_{/J} \psi_2$. If $\mathfrak{A}\models^+ \psi_1 \vee_{/J} \psi_2$ then there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$ such that $\mathfrak{A}\models^+ \psi_1[V_1]$ and $\mathfrak{A}\models^+ \psi_2[V_2]$. Let $V_1' = \{\,\vec{a}\cup\vec{b} \mid \vec{a}\in V_1,\ \vec{b}\in W\,\}$ and $V_2' = \{\,\vec{a}\cup\vec{b}\mid \vec{a}\in V_2,\ \vec{b}\in W\,\}$. Then $V' = V_1'\cup_J V_2'$ is a J-saturated disjoint cover, and by inductive hypothesis $\mathfrak{A}\models^+ \psi_1'[V_1']$ and $\mathfrak{A}\models^+ \psi_2'[V_2']$. Here ψ_1' and ψ_2' are the N-variate versions of ψ_1 and ψ_2, respectively. Thus $\mathfrak{A}\models^+ \psi_1' \vee_{/J} \psi_2'[V']$.

Conversely, suppose $\mathfrak{A}\models^+ \psi_1' \vee_{/J} \psi_2'[V']$. Let $\vec{b}\in W$ and $V'' = \{\,\vec{a}\cup\vec{b}\mid \vec{a}\in V\,\}$. By downward closure, $\mathfrak{A}\models^+ \psi_1' \vee_{/J} \psi_2'[V'']$, so there is a J-saturated disjoint cover $V'' = V_1''\cup_J V_2''$ such that $\mathfrak{A}\models^+ \psi_1'[V_1'']$ and $\mathfrak{A}\models^+ \psi_2'[V_2'']$. Let $V_1 = \{\,\vec{a}\restriction M \mid \vec{a}\in V_1''\,\}$ and $V_2 = \{\,\vec{a}\restriction M\mid \vec{a}\in V_2''\,\}$. Then $V = V_1\cup_J V_2$, and by inductive hypothesis $\mathfrak{A}\models^+ \psi_1[V_1]$ and $\mathfrak{A}\models^+ \psi_2[V_2]$. Thus $\mathfrak{A}\models^+ \psi_1 \vee_{/J} \psi_2[V]$.

For the negative disjunctive case, observe that $\mathfrak{A}\models^- \psi_1 \vee_{/J} \psi_2[V]$ if and only if $\mathfrak{A}\models^- \psi_1[V]$ and $\mathfrak{A}\models^- \psi_2[V]$ if and only if $\mathfrak{A}\models^- \psi_1'[V']$ and $\mathfrak{A}\models^- \psi_2'[V']$ (by inductive hypothesis) if and only if $\mathfrak{A}\models^- \psi_1' \vee_{/J} \psi_2'[V']$.

Suppose $J\subseteq M$ and ϕ is $\exists v_{m/J}\psi[V]$. If $\mathfrak{A}\models^+ \exists v_{m/J}\psi[V]$, then there exists an $f\colon V\to A$ independent of J such that $\mathfrak{A}\models^+ \psi[V(m:f)]$. Define $g\colon V'\to A$ by $g(\vec{a}\cup\vec{b}) = f(\vec{a})$. Then g is independent of J, and $V'(m:g) = \{\,\vec{a}\cup b\mid \vec{a}\in V(m:f),\ \vec{b}\in W\,\}$, so by inductive hypothesis $\mathfrak{A}\models^+ \psi'[V(m:g)]$. Thus $\mathfrak{A}\models^+ \exists v_{m/J}\psi'[V']$.

Conversely, suppose $\mathfrak{A}\models^+ \exists v_{m/J}\psi'[V']$. Let $\vec{b}\in W$ and $V'' = \{\,\vec{a}\cup\vec{b}\mid \vec{a}\in V\,\}$. By downward closure, $\mathfrak{A}\models^+ \exists v_{m/J}\psi'[V'']$, so there is a $g\colon V''\underset{J}{\to} A$ such that $\mathfrak{A}\models^+ \psi'[V''(m:g)]$. Define $f\colon V\underset{J}{\to} A$ by $f(\vec{a}) = g(\vec{a}\cup\vec{b})$. Observe that f is independent of J, and $V''(m:g) = \{\,\vec{a}\cup\vec{b}\mid \vec{a}\in V(m:f)\,\}$, so by inductive hypothesis $\mathfrak{A}\models^+ \psi[V(m:f)]$. Thus $\mathfrak{A}\models^+ \exists v_{m/J}\psi[V]$.

For the negative existential case, observe that $\mathfrak{A}\models^- \exists v_{m/J}\psi[V]$ if and only if $\mathfrak{A}\models^- \psi[V(m:A)]$ if and only if $\mathfrak{A}\models^- \psi'[V'(m:A)]$ (by inductive hypothesis) if and only if $\mathfrak{A}\models^- \exists v_{m/J}\psi'[V']$. $\qquad\square$

Corollary 1.36. *Let ϕ be an IFG-formula with M variables, let ϕ' be the same formula but with N variables, where $M < N$, and let $\mathfrak{A}$ be a suitable structure. Then $\mathfrak{A} \models^{\pm} \phi$ if and only if $\mathfrak{A} \models^{\pm} \phi'$.*

Proof. By Lemma 1.35, $\mathfrak{A} \models^{\pm} \phi$ if and only if $\mathfrak{A} \models^{\pm} \phi[^{M}A]$ if and only if $\mathfrak{A} \models^{\pm} \phi'[^{N}A]$ if and only if $\mathfrak{A} \models^{\pm} \phi'$. $\qquad\qquad\square$

Henceforth, we will use the same symbol for formulas that differ only in the number of variables they have. For example, we will write $\phi[V']$ instead of $\phi'[V']$.

1.5.3 Properties of IFG-formulas

In this subsection we present some useful properties of IFG-formulas. Usually, it is easier to state and prove the properties using trump semantics, but it is better to think about them in terms of games. For example, parts (a) and (c) of the following proposition say that if Eloïse has a winning strategy given less information, then she has a winning strategy given more information. Parts (b) and (d) say that if Abélard has a winning strategy, then the amount of information available to Eloïse is irrelevant.

Proposition 1.37. *Let ϕ and ψ be IFG$_N$-formulas, $\mathfrak{A}$ a suitable structure, and $V \subseteq {}^{N}A$. If $J \subseteq K \subseteq N$ and $n < N$, then*

(a) $\mathfrak{A} \models^{+} \phi \vee_{/K} \psi[V]$ *implies* $\mathfrak{A} \models^{+} \phi \vee_{/J} \psi[V]$,

(b) $\mathfrak{A} \models^{-} \phi \vee_{/J} \psi[V]$ *if and only if* $\mathfrak{A} \models^{-} \phi \vee_{/K} \psi[V]$,

(c) $\mathfrak{A} \models^{+} \exists v_{n/K}\phi[V]$ *implies* $\mathfrak{A} \models^{+} \exists v_{n/J}\phi[V]$,

(d) $\mathfrak{A} \models^{-} \exists v_{n/J}\phi[V]$ *if and only if* $\mathfrak{A} \models^{-} \exists v_{n/K}\phi[V]$.

Proof. (a) Suppose $\mathfrak{A} \models^{+} \phi \vee_{/K} \psi[V]$. Then there is a K-saturated disjoint cover $V = V_1 \cup_K V_2$ such that $\mathfrak{A} \models^{+} \phi[V_1]$ and $\mathfrak{A} \models^{+} \psi[V_2]$. Since $J \subseteq K$, $V = V_1 \cup_J V_2$ by Lemma 1.20. Hence $\mathfrak{A} \models^{+} \phi \vee_{/J} \psi[V]$.

(b) $\mathfrak{A} \models^- \phi \vee_{/J} \psi[V]$ if and only if $\mathfrak{A} \models^- \phi[V]$ and $\mathfrak{A} \models^- \psi[V]$ if and only if $\mathfrak{A} \models^-$ $\phi \vee_{/K} \psi[V]$.

(c) Suppose $\mathfrak{A} \models^+ \exists v_{n/K}\phi[V]$. Then there is a function $f\colon V \xrightarrow{K} A$ such that $\mathfrak{A} \models^+$ $\phi[V(n:f)]$. Since $J \subseteq K$, f is independent of J. Hence $\mathfrak{A} \models^+ \exists v_{n/J}\phi[V]$.

(d) $\mathfrak{A} \models^- \exists v_{n/J}\phi[V]$ if and only if $\mathfrak{A} \models^- \phi[V(n:A)]$ if and only if $\mathfrak{A} \models^- \exists v_{n/K}\phi[V]$. $\quad\square$

In ordinary first-order logic, quantifying over the same variable more than once has no effect: $\exists x \exists x \phi$ is logically equivalent to $\exists x \phi$. In IFG logic, a player might be able to pass information from one quantifier to the next, gaining an advantage.

Proposition 1.38. *Let ϕ be an IFG_N-formula, let $J, K \subseteq N$, and let $n < N$. Let $\mathfrak{A}$ be a suitable structure and $V \subseteq {}^N A$.*

(a) $\mathfrak{A} \models^+ \exists v_{n/J} \exists v_{n/K}\phi[V]$ *implies* $\mathfrak{A} \models^+ \exists v_{n/J \cap K}\phi[V]$.

(b) *If $n \in K$, then* $\mathfrak{A} \models^+ \exists v_{n/J} \exists v_{n/K}\phi[V]$ *if and only if* $\mathfrak{A} \models^+ \exists v_{n/K}\phi[V]$.

(c) $\mathfrak{A} \models^- \exists v_{n/J} \exists v_{n/K}\phi[V]$ *if and only if* $\mathfrak{A} \models^- \exists v_{n/L}\phi[V]$.

Proof. (a) Suppose $\mathfrak{A} \models^+ \exists v_{n/J} \exists v_{n/K}\phi[V]$. Then there is an $f\colon V \xrightarrow{J} A$ and a $g\colon V(n:f) \xrightarrow{K} A$ such that $\mathfrak{A} \models^+ \phi[V(n:f)(n:g)]$. By Lemma 1.26 there is a function $h\colon V \xrightarrow{J \cap K} A$ such that $V(n:f)(n:g) = V(n:h)$. Therefore $\mathfrak{A} \models^+ \exists v_{n/J \cap K}\phi[V]$.

(b) Suppose $n \in K$ and $\mathfrak{A} \models^+ \exists v_{n/J} \exists v_{n/K}\phi[V]$. Then there is an $f\colon V \xrightarrow{J} A$ and a $g\colon V(n:f) \xrightarrow{K} A$ such that $\mathfrak{A} \models^+ \phi[V(n:f)(n:g)]$. By Lemma 1.26 there is a function $h\colon V \xrightarrow{K} A$ such that $V(n:f)(n:g) = V(n:h)$. Therefore $\mathfrak{A} \models^+ \exists v_{n/K}\phi[V]$.

Conversely, suppose $\mathfrak{A} \models^+ \exists v_{n/K}\phi[V]$. Then there is an $h\colon V \xrightarrow{K} A$ such that $\mathfrak{A} \models^+$ $\phi[V(n:h)]$. Let $f\colon V \xrightarrow{J} A$ be any function independent of J. By Lemma 1.27 there is a function $g\colon V(n:f) \xrightarrow{K} A$ such that $V(n:f)(n:g) = V(n:h)$. Therefore $\mathfrak{A} \models^+ \exists v_{n/J} \exists v_{n/K}\phi[V]$.

(c) $\mathfrak{A} \models^- \exists v_{n/J} \exists v_{n/K}\phi[V]$ if and only if $\mathfrak{A} \models^- \exists v_{n/K}\phi[V(n:A)]$ if and only if $\mathfrak{A} \models^-$ $\phi[V(n:A)(n:A)]$ if and only if $\mathfrak{A} \models^- \phi[V(n:A)]$ if and only if $\mathfrak{A} \models^- \exists v_{n/L}\phi[V]$. $\quad\square$

Example. The converse of Proposition 1.38(a) does not hold. Consider the two formulas

$$\exists v_{1/v_0 v_1} \exists v_{1/v_0}(v_0 = v_1) \quad \text{and} \quad \exists v_{1/v_0}(v_0 = v_1).$$

Let $\mathfrak{A} = \{0, 1\}$ and $V = \{00, 11\}$. Then $\mathfrak{A} \models^+ \exists v_{1/v_0}(v_0 = v_1)[V]$ because the function $f: V \to A : \vec{a} \mapsto a_1$ is independent of $\{v_0\}$, $V(1: f) = V$, and $\mathfrak{A} \models^+ (v_0 = v_1)[V]$. However, $\mathfrak{A} \not\models^+ \exists v_{1/v_0 v_1} \exists v_{1/v_0}(v_0 = v_1)[V]$. To show this, let $f: V \to A$ be independent of $\{v_0, v_1\}$, and let $g: V(n: f) \to A$ be independent of $\{v_0\}$. Any function from V that is independent of $\{v_0, v_1\}$ is constant. Hence $V(n: f) = \{00, 10\}$ or $V(n: f) = \{01, 11\}$. In either case, g must be constant as well since g depends only on v_1. Thus the only two possibilities for $V(n: f)(n: g)$ are $\{00, 10\}$ or $\{01, 11\}$, and in either case $\mathfrak{A} \not\models^+ (v_0 = v_1)[V(n: f)(n: g)]$.

In the previous proposition, the same player (Eloïse) chooses the value of v_n twice in succession. If opponents choose the value of v_n each in turn, the latter choice dominates. However, in certain situations the first player may be able to interfere with the strategy of the second player by erasing information encoded in the variable. That is, the first player may be able to block a signal the second player had sent to him or herself from a prior move.

Proposition 1.39. *(a) $\mathfrak{A} \models^+ \exists v_{n/J} \forall v_{n/K} \phi[V]$ if and only if $\mathfrak{A} \models^+ \forall v_{n/K} \phi[V]$.*

(b) $\mathfrak{A} \models^- \exists v_{n/J} \forall v_{n/K} \phi[V]$ implies $\mathfrak{A} \models^- \forall v_{n/K} \phi[V]$.

(c) $\mathfrak{A} \models^- \forall v_{n/K} \phi[V(n: A)]$ implies $\mathfrak{A} \models^- \exists v_{n/J} \forall v_{n/K} \phi[V]$.

(d) $\mathfrak{A} \models^\pm \exists v_{n/J} \forall v_{n/K} \phi$ if and only if $\mathfrak{A} \models^\pm \forall v_{n/K} \phi$.

(e) $\mathfrak{A} \models^\pm \forall v_{n/J} \exists v_{n/K} \phi$ if and only if $\mathfrak{A} \models^\pm \exists v_{n/K} \phi$.

Proof. (a) $\mathfrak{A} \models^+ \exists v_{n/J} \forall v_{n/K} \phi[V]$ if and only if $\mathfrak{A} \models^+ \phi[V(n: f)(n: A)]$ for some $f: V \xrightarrow{J} A$ if and only if $\mathfrak{A} \models^+ \phi[V(n: A)]$ if and only if $\mathfrak{A} \models^+ \forall v_{n/K} \phi[V]$.

(b) Suppose $\mathfrak{A} \models^- \exists v_{n/J} \forall v_{n/K} \phi[V]$. Then $\mathfrak{A} \models^- \phi[V(n: A)(n: f)]$ for some function $f: V(n: A) \xrightarrow{K} A$. Hence $\mathfrak{A} \models^- \phi[V(n: f \restriction V)]$ because $V(n: f \restriction V) \subseteq V(n: A)(n: f)$. Also, $f \restriction V$ is independent of K, so $\mathfrak{A} \models^- \forall v_{n/K} \phi[V]$.

(c) Suppose $\mathfrak{A} \models^- \forall v_{n/K}\phi[V(n:A)]$. Then $\mathfrak{A} \models^- \phi[V(n:A)(n:f)]$ for some $f\colon V \xrightarrow[K]{} A$, which implies $\mathfrak{A} \models^- \exists v_{n/J}\forall v_{n/K}\phi[V]$.

(d) Follows from parts (a)–(c) and the fact that $^N A = {}^N A(n:A)$.

(e)

$$
\begin{aligned}
\mathfrak{A} \models^\pm \forall v_{n/J}\exists v_{n/K}\phi \quad &\text{iff} \quad \mathfrak{A} \models^\pm (\exists v_{n/J}(\exists v_{n/K}\phi)^\cup)^\cup \\
&\text{iff} \quad \mathfrak{A} \models^\mp \exists v_{n/J}(\exists v_{n/K}\phi)^\cup \\
&\text{iff} \quad \mathfrak{A} \models^\mp \exists v_{n/J}(\forall v_{n/K}\phi^\cup) \\
&\text{iff} \quad \mathfrak{A} \models^\mp \forall v_{n/K}\phi^\cup \\
&\text{iff} \quad \mathfrak{A} \models^\pm \exists v_{n/K}\phi. \qquad\qquad \square
\end{aligned}
$$

To see how Eloïse's choice of the value of v_n might hinder Abélard, consider the formula $\forall v_{1/v_0}(v_0 = v_1)$. If $\mathfrak{A}$ is the equality structure with universe $\{0,1\}$, then

$$\mathfrak{A} \models^- \forall v_{1/v_0}(v_0 = v_1)[\{01, 10\}]$$

because Abélard can choose to leave the value of v_1 as it is. However,

$$\mathfrak{A} \not\models^- \exists v_{1/\emptyset}\forall v_{1/v_0}(v_0 = v_1)[\{01, 10\}]$$

because $\mathfrak{A} \not\models^- \forall v_{1/v_0}(v_0 = v_1)[\{00, 01, 10, 11\}]$. To see why, suppose $f\colon \{00, 01, 10, 11\} \xrightarrow[v_0]{} A$. Then $f(00) = f(10) = a$ and $f(01) = f(11) = b$, so $\{00, 01, 10, 11\}(1:f) = \{0a, 0b, 1a, 1b\}$, where $a = 0$ or $a = 1$.

In ordinary first-order logic, adjacent quantifiers of the same type commute. That is, the formula $\exists v_0 \exists v_1 \phi$ is logically equivalent to $\exists v_1 \exists v_0 \phi$. Because of signaling, this is no longer the case in IFG logic. For example, the IFG$_2$-sentence $\exists v_{0/v_0 v_1}\exists v_{1/\emptyset}(v_0 = v_1)$ is true in all structures, while $\exists v_{1/\emptyset}\exists v_{0/v_0 v_1}(v_0 = v_1)$ is not. However, if neither quantifier depends upon the other, then the quantifiers do commute.

Proposition 1.40. *Suppose $m \in K$ and $n \in J$, where $m \neq n$. Then*

$$\mathfrak{A} \models^\pm \exists v_{m/J}\exists v_{n/K}\phi[V] \quad \textit{iff} \quad \mathfrak{A} \models^\pm \exists v_{n/K}\exists v_{m/J}\phi[V].$$

Proof. Suppose $\mathfrak{A} \models^+ \exists v_{m/J}\exists v_{n/K}\phi[V]$. Then there exist $f\colon V \xrightarrow{J} A$ and $g\colon V(m:f) \xrightarrow{K} A$ such that $\mathfrak{A} \models^+ \phi[V(m:f)(n:g)]$. By Lemma 1.28 there exist functions $G\colon V \xrightarrow{K} A$ and $F\colon V(n:G) \xrightarrow{J} A$ such that $V(m:f)(n:g) = V(n:G)(m:F)$. Hence $\mathfrak{A} \models^+ \exists v_{n/K}\exists v_{m/J}\phi[V]$. The converse follows by symmetry. Also,

$$\mathfrak{A} \models^- \exists v_{m/J}\exists v_{n/K}\phi[V] \quad \text{iff} \quad \mathfrak{A} \models^- \phi[V(m:A)(n:A)]$$
$$\text{iff} \quad \mathfrak{A} \models^- \phi[V(n:A)(m:A)]$$
$$\text{iff} \quad \mathfrak{A} \models^- \exists v_{n/K}\exists v_{m/J}\phi[V]. \qquad \square$$

1.5.4 Perfect IFG-formulas

Definition. An IFG-formula ϕ is **perfect** if all of its independence sets are empty, that is, if $G(\mathfrak{A}, \phi)$ is a game of perfect information. Every perfect IFG-formula is equivalent to the ordinary first-order formula obtained by omitting the empty subscripts.

Definition. Given any IFG-formula ϕ, the **perfection** of ϕ, denoted $\phi_\emptyset$, is defined recursively as follows.

- If ϕ is atomic, then $\phi_\emptyset$ is ϕ.

- $(\sim\psi)_\emptyset$ is $\sim(\psi_\emptyset)$.

- $(\psi_1 \vee_{/J} \psi_2)_\emptyset$ is $(\psi_1)_\emptyset \vee_{/\emptyset} (\psi_2)_\emptyset$.

- $(\exists v_{n/J}\psi)_\emptyset$ is $\exists v_{n/\emptyset}\psi$.

Thus $\phi_\emptyset$ is just ϕ with all of the independence sets changed to $\emptyset$.

By Theorem 1.7, for any IFG-formula ϕ, $\mathfrak{A} \models^+ \phi_\emptyset[V]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \models \phi_\emptyset[\vec{a}]$, and $\mathfrak{A} \models^- \phi_\emptyset[V]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \not\models \phi_\emptyset[\vec{a}]$. Technically the statement $\mathfrak{A} \models \phi_\emptyset[\vec{a}]$ is an abuse of notation since $\phi_\emptyset$ is not an ordinary first-order formula. Henceforth we will not distinguish between perfect IFG-formulas and their ordinary first-order counterparts.

An important feature of the perfection process is that no winning strategies are lost.

Proposition 1.41. *If* $\mathfrak{A} \models^{\pm} \phi[V]$*, then* $\mathfrak{A} \models^{\pm} \phi_{\emptyset}[V]$*.*

Proof. If ϕ is atomic, then ϕ is $\phi_{\emptyset}$. If ϕ is $\sim\psi$, then $\mathfrak{A} \models^{\pm} \sim\psi[V]$ if and only if $\mathfrak{A} \models^{\mp} \psi[V]$, which implies (by inductive hypothesis) $\mathfrak{A} \models^{\mp} \psi_{\emptyset}[V]$, which holds if and only if $\mathfrak{A} \models^{\pm} \sim(\psi_{\emptyset})[V]$.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. By Proposition 1.37, if $\mathfrak{A} \models^{+} \psi_1 \vee_{/J} \psi_2[V]$, then $\mathfrak{A} \models^{+} \psi_1 \vee_{/\emptyset} \psi_2[V]$, which means there is a disjoint cover $V = V_1 \cup V_2$ such that $\mathfrak{A} \models^{+} \psi_1[V_1]$ and $\mathfrak{A} \models^{+} \psi_2[V_2]$. By inductive hypothesis, $\mathfrak{A} \models^{+} (\psi_1)_{\emptyset}[V_1]$ and $\mathfrak{A} \models^{+} (\psi_2)_{\emptyset}[V_2]$. Hence $\mathfrak{A} \models^{+} (\psi_1)_{\emptyset} \vee_{/\emptyset} (\psi_2)_{\emptyset}[V]$. Thus $\mathfrak{A} \models^{+} (\psi_1 \vee_{/J} \psi_2)_{\emptyset}[V]$. If $\mathfrak{A} \models^{-} \psi_1 \vee_{/J} \psi_2[V]$, then $\mathfrak{A} \models^{-} \psi_1[V]$ and $\mathfrak{A} \models^{-} \psi_2[V]$. By inductive hypothesis, $\mathfrak{A} \models^{-} (\psi_1)_{\emptyset}[V]$ and $\mathfrak{A} \models^{-} (\psi_2)_{\emptyset}[V]$. Hence $\mathfrak{A} \models^{-} (\psi_1)_{\emptyset} \vee_{/\emptyset} (\psi_2)_{\emptyset}[V]$. Thus $\mathfrak{A} \models^{-} (\psi_1 \vee_{/J} \psi_2)_{\emptyset}[V]$.

Suppose ϕ is $\exists v_{n/J}\psi$. If $\mathfrak{A} \models^{+} \exists v_{n/J}\psi[V]$, then $\mathfrak{A} \models^{+} \exists v_{n/\emptyset}\psi[V]$, which means $\mathfrak{A} \models^{+} \psi[V(n : f)]$ for some function $f: V \rightarrow A$. By inductive hypothesis, $\mathfrak{A} \models^{+} \psi_{\emptyset}[V(n : f)]$. Hence $\mathfrak{A} \models^{+} \exists v_{n/\emptyset}(\psi_{\emptyset})[V]$. Thus $\mathfrak{A} \models^{+} (\exists v_{n/J}\psi)_{\emptyset}[V]$. If $\mathfrak{A} \models^{-} \exists v_{n/J}\psi[V]$, then $\mathfrak{A} \models^{-} \psi[V(n : A)]$. By inductive hypothesis, $\mathfrak{A} \models^{-} \psi_{\emptyset}[V(n : A)]$. Hence $\mathfrak{A} \models^{-} \exists v_{n/\emptyset}(\psi_{\emptyset})[V]$. Thus $\mathfrak{A} \models^{-} (\exists v_{n/J}\psi)_{\emptyset}[V]$. $\square$

<center>

Chapter 2

Independence-Friendly Cylindric Set Algebras

</center>

2.1   Introduction

Definition. An **independence-friendly cylindric power set algebra** is an algebra whose universe is $\mathscr{P}(\mathscr{P}(^N A)) \times \mathscr{P}(\mathscr{P}(^N A))$, where A is a set and N is a natural number. The set A is called the **base set**, and the number N is called the **dimension** of the algebra. Since each element X is an ordered pair, we will use the notation X^+ to refer to the first coordinate of the pair, and X^- to refer to the second coordinate. There are a finite number of operations:

- the constant $0 = \langle \{\emptyset\}, \mathscr{P}(^N A) \rangle$;

- the constant $1 = \langle \mathscr{P}(^N A), \{\emptyset\} \rangle$;

- for all $i, j < N$, the constant D_{ij} is defined by

  (+) $D_{ij}^+ = \mathscr{P}(\{ \vec{a} \in {}^N A \mid a_i = a_j \})$,

  (−) $D_{ij}^- = \mathscr{P}(\{ \vec{a} \in {}^N A \mid a_i \neq a_j \})$;

- if $X = \langle X^+, X^- \rangle$, then $X^\cup = \langle X^-, X^+ \rangle$;

- for every $J \subseteq N$, the binary operation $+_J$ is defined by

  (+) $V \in (X +_J Y)^+$ if and only if $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$,

  (−) $(X +_J Y)^- = X^- \cap Y^-$;

- for every $J \subseteq N$, the binary operation $\cdot_J$ is defined by

(+) $(X \cdot_J Y)^+ = X^+ \cap Y^+$,

(−) $W \in (X \cdot_J Y)^-$ if and only if $W = W_1 \cup_J W_2$ for some $W_1 \in X^-$ and $W_2 \in Y^-$;

- for every $n < N$ and $J \subseteq N$, the unary operation $C_{n,J}$ is defined by

(+) $V \in C_{n,J}(X)^+$ if and only if $V(n : f) \in X^+$ for some $f : V \underset{J}{\rightarrow} A$,

(−) $W \in C_{n,J}(X)^-$ if and only if $W(n : A) \in X^-$.

Definition. An **independence-friendly cylindric set algebra** is any subalgebra of an independence-friendly cylindric power set algebra. An $\mathbf{IFG}_N$**-cylindric set algebra** is an independence-friendly cylindric set algebra of dimension N.

We will use the notation $X =^+ Y$ to abbreviate $X^+ = Y^+$ and $X =^- Y$ to abbreviate $X^- = Y^-$. Furthermore, $X \leq^+ Y$ abbreviates $X^+ \subseteq Y^+$, and $X \leq^- Y$ abbreviates $X^- \subseteq Y^-$. Define $X \leq Y$ if and only if $X \leq^+ Y$ and $Y \leq^- X$. It follows immediately from the definitions that $X \leq^\pm Y$ if and only if $X^\cup \leq^\mp Y^\cup$. Hence, $X \leq Y$ if and only if $Y^\cup \leq X^\cup$.

2.1.1 Duality

We include the operations $\cdot_J$ in the signature of independence-friendly cylindric set algebras for the sake of compatibility with De Morgan algebra. We could have omitted $\cdot_J$ from the definition and instead defined it in terms of $+_J$ and $^\cup$.

Lemma 2.1. $X \cdot_J Y = (X^\cup +_J Y^\cup)^\cup$.

Proof. It follows immediately from the definition that $(X^\cup)^\cup = X$ and $(X^\cup)^\pm = X^\mp$. Hence

$$(X \cdot_J Y)^+ = X^+ \cap Y^+$$
$$= (X^\cup)^- \cap (Y^\cup)^-$$
$$= (X^\cup +_J Y^\cup)^-$$
$$= ((X^\cup +_J Y^\cup)^\cup)^+.$$

Also $W \in (X \cdot_J Y)^-$ if and only if $W = W_1 \cup_J W_2$ for some $W_1 \in X^- = (X^\cup)^+$ and

$W_2 \in Y^- = (Y^\cup)^+$ if and only if $W \in (X^\cup +_J Y^\cup)^+ = ((X^\cup +_J Y^\cup)^\cup)^-$. $\square$

The cylindrifications $C_{n,J}$ also have their corresponding dual operations, defined by

$C^\partial_{n,J}(X) = C_{n,J}(X^\cup)^\cup$. We do not include the dual cylindrifications in the signature of IFG$_N$-

cylindric set algebras for the sake of compatibility with cylindric algebra.

Every IFG$_N$-cylindric set algebra

$$\mathfrak{C} = \langle U; \, 0, \, 1, \, D_{ij}, \, {}^\cup, \, +_J, \, \cdot_J, \, C_{n,J}\rangle$$

has a dual algebra

$$\mathfrak{C}^\partial = \langle U; \, 1, \, 0, \, D^\cup_{ij}, \, {}^\cup, \, \cdot_J, \, +_J, \, C^\partial_{n,J}\rangle.$$

Furthermore, the operation ${}^\cup$ is an isomorphism from $\mathfrak{C}$ to $\mathfrak{C}^\partial$. Therefore any algebraic statement

about independence-friendly cylindric set algebras can be dualized by systematically exchanging

the symbols 0 with 1, D_{ij} with $D^\cup_{ij}$, $+_J$ with $\cdot_J$, and $C_{n,J}$ with $C^\partial_{n,J}$. Statements involving the

superscripts $^+$ and $^-$ can be dualized by their exchange, even though the symbols $^+$ and $^-$ do

not belong to the signature of independence-friendly cylindric set algebras. In particular, the

dual of $\leq$ is $\geq$. Henceforth we will often state a theorem and its dual together, but we will not

prove the dual.

2.1.2   Fixed points

An element X of an independence-friendly cylindric set algebra is a **fixed point** if

$X^\cup = X$. Clearly X is a fixed point if and only if $X^+ = X^-$. Two fixed points in partic-

ular deserve special mention. In a given IFG$_N$-cylindric set algebra, let $\Omega = \langle\{\emptyset\}, \{\emptyset\}\rangle$ and

$\mho = \langle \mathscr{P}(^N A), \mathscr{P}(^N A)\rangle$. Neither the symbol Ω nor the symbol $\mho$ belong to the signature of

independence-friendly cylindric set algebras. However, we will see that Ω is definable in most

independence-friendly cylindric set algebras, and it is often present even when it is not definable.

Lemma 2.2. *In any IFG$_N$-cylindric set algebra $\mathfrak{C}$, if there exists an element $X \in \mathfrak{C}$ such that*

$X^+ \cap X^- = \{\emptyset\}$ *and* $^N A \notin X^+ \cup X^-$*, then* $\Omega \in \mathfrak{C}$.

Proof. Suppose X is such an element, and consider $C_{0,\emptyset}\cdots C_{N-1,\emptyset}(X \cdot_N X^{\cup})$. Observe that $V \in C_{0,\emptyset}\cdots C_{N-1,\emptyset}(X \cdot_N X^{\cup})^+$ if and only if the exist functions $f_0,\ldots,f_{N-1}$ such that $V(0 : f_0)\cdots(N-1 : f_{N-1}) \in (X \cdot_N X^{\cup})^+ = X^+ \cap X^- = \{\emptyset\}$ if and only if $V = \emptyset$. Also, $W \in C_{0,\emptyset}\cdots C_{N-1,\emptyset}(X \cdot_N X^{\cup})^-$ if and only if $W(0 : A)\cdots(N-1 : A) \in (X \cdot_N X^{\cup})^-$. Note that if $W = \emptyset$ then $W(0 : A)\cdots(N-1 : A) = \emptyset$, and if $W \neq \emptyset$ then $W(0 : A)\cdots(N-1 : A) = {}^N A$. Also note that $\emptyset \in (X \cdot_N X^{\cup})^-$ because $\emptyset \in X^+ \cap X^-$, while ${}^N A \notin (X \cdot_N X^{\cup})^-$ because if it were we would have ${}^N A \in X^+$ or ${}^N A \in X^-$. Thus $W \in C_{N-1,\emptyset}\cdots C_{0,\emptyset}(X \cdot_N X^{\cup})^-$ if and only if $W = \emptyset$. $\square$

In any independence-friendly cylindric set algebra of dimension at least two and whose base set has at least two elements we have $D_{01}^+ \cap D_{01}^- = \emptyset$ and ${}^N A \notin D_{01}^+ \cup D_{01}^-$. Hence $C_{0,\emptyset}\cdots C_{N-1,\emptyset}(D_{01} \cdot_N D_{01}^{\cup}) = \Omega$. In the next section we will show that if $|A| = 1$ or $N = 1$ then there are examples of independence-friendly cylindric set algebras that do not include Ω.

2.2 $\quad \mathfrak{Cs}_{IFG_N}(\mathfrak{A})$

The motivating examples of independence-friendly cylindric set algebras are algebras whose elements are the meanings of IFG-formulas.

Definition. Given a structure $\mathfrak{A}$ and an IFG_N-formula ϕ, define

$$\|\phi\|_{\mathfrak{A}}^+ = \{\, V \subseteq {}^N A \mid \mathfrak{A} \models^+ \phi[V]\,\},$$

$$\|\phi\|_{\mathfrak{A}}^- = \{\, W \subseteq {}^N A \mid \mathfrak{A} \models^- \phi[W]\,\},$$

$$\|\phi\|_{\mathfrak{A}} = \langle \|\phi\|_{\mathfrak{A}}^+, \|\phi\|_{\mathfrak{A}}^- \rangle.$$

By Theorem 1.32, $\|\phi\|_{\mathfrak{A}}^+$ is the set of all winning teams for ϕ in $\mathfrak{A}$, and $\|\phi\|_{\mathfrak{A}}^-$ is the set of all losing teams for ϕ in $\mathfrak{A}$. We will call $\|\phi\|_{\mathfrak{A}}$ the **meaning** of ϕ in $\mathfrak{A}$. If the structure is clear from context we will omit the subscript and simply write $\|\phi\|^+$, $\|\phi\|^-$, and $\|\phi\|$.

Definition. Given an σ-structure $\mathfrak{A}$, the **IFG_N-cylindric set algebra over** $\mathfrak{A}$, denoted $\mathfrak{Cs}_{IFG_N}(\mathfrak{A})$, is the IFG_N-cylindric set algebra whose universe is the set $\{\, \|\phi\|_{\mathfrak{A}} \mid \phi \in \mathscr{L}_{IFG_N}^{\sigma}\,\}$.

Observe that if $X = \|\phi\|$ and $Y = \|\psi\|$, then

$$0 = \|v_0 \neq v_0\|,$$

$$1 = \|v_0 = v_0\|,$$

$$D_{ij} = \|v_i = v_j\|,$$

$$X^{\cup} = \|\sim\phi\|,$$

$$X +_J Y = \|\phi \vee_{/J} \psi\|,$$

$$X \cdot_J Y = \|\phi \wedge_{/J} \psi\|,$$

$$C_{n,J}(X) = \|\exists v_{n/J}\phi\|.$$

Lemma 2.3. *For any formula ϕ and suitable structure $\mathfrak{A}$ we have*

$$\|\phi\|_{\mathfrak{A}}^{+} \cap \|\phi\|_{\mathfrak{A}}^{-} = \{\emptyset\}.$$

Proof. By Lemma 1.30 and Lemma 1.34. □

Lemma 2.4. *For any formula ϕ and suitable structure $\mathfrak{A}$, if $V' \subseteq V \in \|\phi\|_{\mathfrak{A}}^{\pm}$, then $V' \in \|\phi\|_{\mathfrak{A}}^{\pm}$.*

Proof. By Lemma 1.31. □

In ordinary first-order logic a sentence can have one of only two possible truth-values: **true** or **false**. Thus if ϕ is a sentence and $\mathfrak{A}$ is a suitable structure either $\phi^{\mathfrak{A}} = 0$ or $\phi^{\mathfrak{A}} = 1$. In IFG logic a sentence ϕ may be neither true nor false, so $\|\phi\|_{\mathfrak{A}}$ can take values other than 0 or 1. Interestingly, the only other possible value is Ω.

Proposition 2.5. *If ϕ is an IFG_N-sentence and $\mathfrak{A}$ is a suitable structure, then $\|\phi\|_{\mathfrak{A}} \in \{0, \Omega, 1\}$.*

Proof. If $\|\phi\|_{\mathfrak{A}} \neq \Omega$, then there is a nonempty $V \subseteq {}^{N}A$ such that $V \in \|\phi\|_{\mathfrak{A}}^{+}$ or $V \in \|\phi\|_{\mathfrak{A}}^{-}$. By Proposition 1.5, in the first case $\|\phi\|_{\mathfrak{A}} = 1$, and in the second case $\|\phi\|_{\mathfrak{A}} = 0$. □

Lemma 2.6. *If $|\mathfrak{A}| = 1$, then for every IFG_N-formula ϕ, either $\mathfrak{A} \models^{+} \phi$ or $\mathfrak{A} \models^{-} \phi$.*

Proof. If $|\mathfrak{A}| = 1$, then $|{}^{N}A| = 1$. Let ${}^{N}A = \{\vec{a}\}$.

Suppose ϕ is atomic. Then either $\mathfrak{A} \models \phi[\vec{a}]$ or $\mathfrak{A} \not\models \phi[\vec{a}]$. In the first case $\mathfrak{A} \models^+ \phi[\{\vec{a}\}]$, and in the second case $\mathfrak{A} \models^- \phi[\{\vec{a}\}]$.

Suppose ϕ is $\sim\psi$. Then by inductive hypothesis $\mathfrak{A} \models^+ \psi[\{\vec{a}\}]$ or $\mathfrak{A} \models^- \psi[\{\vec{a}\}]$, in which case $\mathfrak{A} \models^- \sim\psi[\{\vec{a}\}]$ or $\mathfrak{A} \models^+ \sim\psi[\{\vec{a}\}]$.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. If $\mathfrak{A} \models^+ \psi_1[\{\vec{a}\}]$, then $\mathfrak{A} \models^+ \psi_1 \vee_{/J} \psi_2[\{\vec{a}\}]$ because $\{\vec{a}\} = \{\vec{a}\} \cup_J \emptyset$ and it is always the case that $\mathfrak{A} \models^+ \psi_2[\emptyset]$. Similarly, if $\mathfrak{A} \models^+ \psi_2[\{\vec{a}\}]$, then $\mathfrak{A} \models^+ \psi_1 \vee_{/J} \psi_2[\{\vec{a}\}]$. If $\mathfrak{A} \models^- \psi_1[\{\vec{a}\}]$ and $\mathfrak{A} \models^- \psi_2[\{\vec{a}\}]$, then $\mathfrak{A} \models^- \psi_1 \vee_{/J} \psi_2[\{\vec{a}\}]$.

Suppose ϕ is $\exists v_{n/J}\psi$. If $\mathfrak{A} \models^+ \psi[\{\vec{a}\}]$, let $f \colon \{\vec{a}\} \underset{J}{\to} A$ be the function that sends $\vec{a}$ to a_0. Then $\{\vec{a}\}(n : f) = \{\vec{a}\}$, so $\mathfrak{A} \models^+ \psi[\{\vec{a}\}(n : f)]$. Hence $\mathfrak{A} \models^+ \exists v_{n/J}\psi[\{\vec{a}\}]$. If $\models^- \psi[\{\vec{a}\}]$, then $\{\vec{a}\}(n : A) = \{\vec{a}\}$, so $\mathfrak{A} \models^- \psi[\{\vec{a}\}(n : A)]$. Hence $\mathfrak{A} \models^- \exists v_{n/J}\psi[\{\vec{a}\}]$. $\qquad\square$

Thus if $|\mathfrak{A}| = 1$, $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ is essentially the same structure as the two-element Boolean algebra.

Lemma 2.7. *Let $\mathfrak{A}$ be a structure. The element $\Omega \in \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ if and only if there is an atomic IFG$_N$-formula ϕ such that $\mathfrak{A} \not\models^+ \phi$ and $\mathfrak{A} \not\models^- \phi$.*

Proof. Suppose $\mathfrak{A} \not\models^+ \phi[^NA]$ and $\mathfrak{A} \not\models^- \phi[^NA]$. Then $\|\phi\|^+_{\mathfrak{A}} \cap \|\phi\|^-_{\mathfrak{A}} = \{\emptyset\}$ by Lemma 2.3, and $^NA \notin \|\phi\|^+_{\mathfrak{A}} \cup \|\phi\|^-_{\mathfrak{A}}$ by hypothesis. Hence $\Omega \in \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ by Lemma 2.2. Conversely, suppose that for every atomic IFG$_N$-formula, either $\mathfrak{A} \models^+ \phi[^NA]$ or $\mathfrak{A} \models^- \phi[^NA]$. We wish to show that for every IFG$_N$-formula, either $\mathfrak{A} \models^+ \phi[^NA]$ or $\mathfrak{A} \models^- \phi[^NA]$.

Suppose ϕ is $\sim\psi$. By inductive hypothesis, either $\mathfrak{A} \models^+ \psi[^NA]$ or $\mathfrak{A} \models^- \psi[^NA]$. In the first case $\mathfrak{A} \models^- \sim\psi[^NA]$, and in the second case $\mathfrak{A} \models^+ \sim\psi[^NA]$.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. Then by inductive hypothesis $\mathfrak{A} \models^{\pm} \psi_1[^NA]$ and $\mathfrak{A} \models^{\pm} \psi_2[^NA]$. If $\mathfrak{A} \models^+ \psi_1[^NA]$ or $\mathfrak{A} \models^+ \psi_2[^NA]$, then $\mathfrak{A} \models^+ \psi_1 \vee_{/J} \psi_2[^NA]$. If $\mathfrak{A} \models^- \psi_1[^NA]$ and $\mathfrak{A} \models^- \psi_2[^NA]$, then $\mathfrak{A} \models^- \psi_1 \vee_{/J} \psi_2[^NA]$.

Suppose ϕ is $\exists v_{n/J}\psi$. Then by inductive hypothesis $\mathfrak{A} \models^{\pm} \psi[^NA]$. If $\mathfrak{A} \models^+ \psi[^NA]$, let $f \colon {}^NA \underset{J}{\to} A$ be any function independent of J (e.g., a constant function). Then $^NA(n : f) \subseteq {}^NA$, so $\mathfrak{A} \models^+ \psi[^NA(n : f)]$. Hence $\mathfrak{A} \models^+ \exists v_{n/J}\psi[^NA]$. If $\mathfrak{A} \models^- \psi[^NA]$, then $\mathfrak{A} \models^- \psi[^NA(n : A)]$

because $^NA(n:A) = {}^NA$. Hence $\mathfrak{A} \models^- \exists v_{n/J} \psi[^NA]$. □

Proposition 2.8. *Let $\mathfrak{A}$ be a nonempty structure. Then $\Omega \in \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ if and only if at least one of the following conditions is satisfied:*

(a) $|\mathfrak{A}| \geq 2$ *and* $N \geq 2$;

(b) $|\mathfrak{A}| \geq 2$, $N = 1$, *and there exist terms s and t and elements $x, y \in \mathfrak{A}$ such that $s^{\mathfrak{A}}(x) = t^{\mathfrak{A}}(x)$ and $s^{\mathfrak{A}}(y) \neq t^{\mathfrak{A}}(y)$;*

(c) $|\mathfrak{A}| \geq 2$, $N = 1$, *and there exist an m-ary relation symbol R, terms $\langle\, s_i \mid i < m \,\rangle$, and elements $x, y \in \mathfrak{A}$ such that $\langle s_0^{\mathfrak{A}}(x), \dots, s_{m-1}^{\mathfrak{A}}(x)\rangle \in R^{\mathfrak{A}}$ and $\langle s_0^{\mathfrak{A}}(y), \dots, s_{m-1}^{\mathfrak{A}}(y)\rangle \notin R^{\mathfrak{A}}$.*

Proof. If (a) is true, then $\mathfrak{A} \not\models^+ v_0 = v_1[^NA]$ and $\mathfrak{A} \not\models^- v_0 = v_1[^NA]$. If (b) is true, then $\mathfrak{A} \not\models^+ s = t[A]$ and $\mathfrak{A} \not\models^- s = t[A]$. If (c) is true, then $\mathfrak{A} \not\models^+ Rs_0 \dots s_{m-1}[A]$ and $\mathfrak{A} \not\models^- Rs_0 \dots s_{m-1}[A]$.

Conversely, suppose (a), (b), and (c) are all false. If $|\mathfrak{A}| = 1$ then $\Omega \notin \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ by Lemma 2.6. Suppose $|\mathfrak{A}| \geq 2$ and $N = 0$. If $\mathfrak{A}$ is a relational structure then $\mathfrak{Cs}_{\mathrm{IFG}_0}(\mathfrak{A}) = \emptyset$ because there are no IFG_0-formulas in a purely relational language. Otherwise every atomic IFG_0-formula has the form $s = t$ or $Rs_0 \dots s_{m-1}$, where all of the terms involved are closed. In the first case either $\mathfrak{A} \models^+ s = t[\{\emptyset\}]$ or $\mathfrak{A} \models^- s = t[\{\emptyset\}]$, and in the second case either $\mathfrak{A} \models^+ Rs_0 \dots s_{m-1}[\{\emptyset\}]$ or $\mathfrak{A} \models^- Rs_0 \dots s_{m-1}[\{\emptyset\}]$.

Suppose $|\mathfrak{A}| \geq 2$, $N = 1$, every pair of terms s and t has the property that either for every $x \in \mathfrak{A}$, $s^{\mathfrak{A}}(x) = t^{\mathfrak{A}}(x)$, or for every $x \in \mathfrak{A}$, $s^{\mathfrak{A}}(x) \neq t^{\mathfrak{A}}(x)$, and every relation symbol R and sequence of terms $\langle\, s_i \mid i < m \,\rangle$ has the property that either for every $x \in \mathfrak{A}$, $\langle s_0^{\mathfrak{A}}(x), \dots, s_{m-1}^{\mathfrak{A}}(x)\rangle \in R^{\mathfrak{A}}$ or for every $x \in \mathfrak{A}$, $\langle s_0^{\mathfrak{A}}(x), \dots, s_{m-1}^{\mathfrak{A}}(x)\rangle \notin R^{\mathfrak{A}}$. Every atomic IFG_1-formula has the form $s = t$ or $Rs_0 \dots s_{m-1}$, where all of the terms involve at most one variable. In the first case, either $\mathfrak{A} \models^+ s = t[A]$ or $\mathfrak{A} \models^- s = t[A]$. In the second case, $\mathfrak{A} \models^+ Rs_0 \dots s_{m-1}[A]$ or $\mathfrak{A} \models^- Rs_0 \dots s_{m-1}[A]$. □

2.2.1   Rooted, suited, and double-suited algebras

Lemma 2.3 and Lemma 2.4 inspire the following definitions.

Definition. A set of teams X^* is **rooted** if $\emptyset \in X^*$. A pair $\langle X^+, X^- \rangle$ of sets of teams is **rooted** if both of its coordinates are rooted. An independence-friendly cylindric set algebra is **rooted** if all of its elements are rooted.

Proposition 2.9. *The subalgebra of an independence-friendly cylindric set algebra generated by a set of rooted elements is a rooted independence-friendly cylindric set algebra.*

Proof. The constant elements 0, 1, and D_{ij} are all rooted, and if X is rooted, then $X^\cup$ is.

Suppose X and Y are rooted. Then $\emptyset \in (X +_J Y)^+$ because $\emptyset = \emptyset \cup_J \emptyset$ and $\emptyset \in X^+$, $\emptyset \in Y^+$. Also, $\emptyset \in X^- \cap Y^- = (X +_J Y)^-$.

Suppose X is rooted. Then $\emptyset \in C_{n,J}(X)^+$ because the empty function f from $\emptyset$ to A is vacuously independent of J and $\emptyset(n : f) = \emptyset \in X^+$. Also, $\emptyset \in C_{n,J}(X)^-$ because $\emptyset(n : A) = \emptyset \in X^-$. □

Given a set A, let $\mathrm{Root}_N(A)$ denote the IFG$_N$-cylindric set algebra whose universe is the set of all rooted elements in $\mathscr{P}(\mathscr{P}(^N A)) \times \mathscr{P}(\mathscr{P}(^N A))$.

Definition. A nonempty subset of $\mathscr{P}(^N A)$ is called a **suit** if it is closed under subsets. That is, a suit X^* is a nonempty collection of subsets of $^N A$ such that $V' \subseteq V \in X^*$ implies $V' \in X^*$. A **double suit** is a pair $\langle X^+, X^- \rangle$ of suits such that $X^+ \cap X^- = \{\emptyset\}$.

Definition. An independence-friendly cylindric set algebra is **suited** if all of its elements are pairs of suits. It is **double-suited** if all of its elements are double suits.

Note that suits, double suits, and suited independence-friendly cylindric set algebras are all rooted.

Proposition 2.10. *The subalgebra of an IFG$_N$-cylindric set algebra generated by a set of pairs of suits is a suited IFG$_N$-cylindric set algebra.*

Proof. The constant elements 0, 1, and D_{ij} are all pairs of suits, and if X is a pair of suits, then so is $X^\cup$.

Suppose X and Y are pairs of suits. If $V' \subseteq V \in (X +_J Y)^+$, then $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$, $V_2 \in Y^+$. Let $V_1' = V_1 \cap V'$ and $V_2' = V_2 \cap V'$. By Lemma 1.21, $V' = V_1' \cup_J V_2'$, and $V_1' \in X^+$, $V_2' \in Y^+$. Hence $V' \in (X +_J Y)^+$. Thus $(X +_J Y)^+$ is a suit. If $W' \subseteq W \in (X +_J Y)^- = X^- \cap Y^-$, then $W' \subseteq W \in X^-$ and $W' \subseteq W \in Y^-$. Hence $W' \in X^- \cap Y^- = (X +_J Y)^-$. Thus $(X +_J Y)^-$ is a suit.

Suppose X is a pair of suits. If $V' \subseteq V \in C_{n,J}(X)^+$, then $V(n : f) \in X^+$ for some $f : V \underset{J}{\to} A$. Let $g = f \restriction V'$. Then $g : V' \underset{J}{\to} A$, and $V'(n : g) \subseteq V(n : f)$, so $V'(n : g) \in X^+$. Hence $V' \in C_{n,J}(X)^+$. Thus $C_{n,J}(X)^+$ is a suit. If $W' \subseteq W \in C_{n,J}(X)^-$, then $W'(n : A) \subseteq W(n : A) \in X^-$, so $W'(n : A) \in X^-$. Hence $C_{n,J}(X)^-$ is a suit. $\square$

Proposition 2.11. *The subalgebra of an IFG$_N$-cylindric set algebra generated by a set of double suits is a double-suited IFG$_N$-cylindric set algebra.*

Proof. The constant elements 0, 1, and D_{ij} are all double suits, and if X is a double suit, then so is $X^\cup$.

Suppose X and Y are double suits. We already know by the previous proposition that $X +_J Y$ is a pair of suits. To show that $(X +_J Y)^+ \cap (X +_J Y)^- = \{\emptyset\}$, suppose $V \in (X +_J Y)^+ \cap (X +_J Y)^- = (X +_J Y)^+ \cap X^- \cap Y^-$. Then $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$, $V_2 \in Y^+$. Since $V_1 \subseteq V \in X^-$ and $V_2 \subseteq V \in Y^-$, we have $V_1 \in X^+ \cap X^-$ and $V_2 \in Y^+ \cap Y^-$. Hence $V_1 = \emptyset = V_2$. Thus $V = \emptyset$. Therefore $X +_J Y$ is a double suit.

Suppose X is a double suit. We already know by the previous proposition that $C_{n,J}(X)$ is a pair of suits. To show that $C_{n,J}(X)^+ \cap C_{n,J}(X)^- = \{\emptyset\}$, suppose $V \in C_{n,J}(X)^+ \cap C_{n,J}(X)^-$. Then $V(n : f) \in X^+$ for some $f : V \underset{J}{\to} A$, and $V(n : A) \in X^-$. But $V(n : f) \subseteq V(n : A)$, so $V(n : f) \in X^+ \cap X^-$. Hence $V(n : f) = \emptyset$, which implies $V = \emptyset$. Therefore $C_{n,J}(X)$ is a double suit. $\square$

Given a set A, let $\mathrm{Suit}_N(A)$ denote the IFG$_N$-cylindric set algebra whose universe is the

set of all pairs of suits in $\mathscr{P}(\mathscr{P}(^NA)) \times \mathscr{P}(\mathscr{P}(^NA))$. Let $\mathrm{DSuit}_N(A)$ denote the IFG_N-cylindric set algebra whose universe is the set of all double suits in $\mathscr{P}(\mathscr{P}(^NA)) \times \mathscr{P}(\mathscr{P}(^NA))$.

2.2.2 Embedding $\mathfrak{Cs}_N(\mathfrak{A})$ into $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$

Since IFG logic is a conservative extension of ordinary first-order logic, we should expect the ordinary cylindric set algebra $\mathfrak{Cs}_N(\mathfrak{A})$ to be embeddable into some reduct of $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$.

Definition. A double suit X is **flat** if there is a $V \subseteq {}^NA$ such that $X^+ = \mathscr{P}(V)$.

Definition. A double suit X is **perfect** if there is a $V \subseteq {}^NA$ such that

$$X = \langle \mathscr{P}(V), \mathscr{P}(^NA \setminus V) \rangle.$$

Proposition 2.12. *A double suit X is perfect if and only if $X +_\emptyset X^\cup = 1$.*

Proof. Suppose $X = \langle \mathscr{P}(V), \mathscr{P}(^NA \setminus V) \rangle$ is perfect. Then ${}^NA = V \cup_\emptyset (^NA \setminus V)$, where $V \in X^+$ and ${}^NA \setminus V \in (X^\cup)^+$. Hence ${}^NA \in (X +_\emptyset X^\cup)^+$. Since $X +_\emptyset X^\cup$ is a double suit we have $X +_\emptyset X^\cup = \langle \mathscr{P}(^NA), \mathscr{P}(\emptyset) \rangle = 1$. Conversely, suppose $X +_\emptyset X^\cup = 1$. Then there exist $V \in X^+$ and $V' \in X^-$ such that ${}^NA = V \cup_\emptyset V'$. But then $V' = {}^NA \setminus V$. Since X is a double suit, $X = \langle \mathscr{P}(V), \mathscr{P}(^NA \setminus V) \rangle$. $\square$

Definition. The reduct of an independence-friendly cylindric set algebra to the signature $\langle 0, 1, D_{ij}, {}^\cup, +_\emptyset, \cdot_\emptyset, C_{n,\emptyset} \rangle$ is called the $\emptyset$-**reduct** of the algebra. A subalgebra of the $\emptyset$-reduct is called a **perfect subalgebra** if all of its members are perfect. The subalgebra of the $\emptyset$-reduct of $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ generated by the meanings of atomic formulas is denoted $\mathfrak{Cs}_{\mathrm{IFG}_{N,\emptyset}}(\mathfrak{A})$.

It is worth noting that $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ is generated by the set of its perfect elements because it is generated by the meanings of atomic formulas, which are all perfect.

Lemma 2.13. *Suppose $X = \langle \mathscr{P}(V), \mathscr{P}(^NA \setminus V) \rangle$ and $Y = \langle \mathscr{P}(W), \mathscr{P}(^NA \setminus W) \rangle$. Then*

(a) $X +_\emptyset Y = \langle \mathscr{P}(V \cup W), \mathscr{P}(^NA \setminus (V \cup W)) \rangle$,

(b) $C_{n,\emptyset}(X) = \langle \mathscr{P}(V(n:A)), \mathscr{P}(^NA \setminus V(n:A)) \rangle$.

Proof. (a) Suppose $U \in (X +_\emptyset Y)^+$. Then $U = U_1 \cup_\emptyset U_2$ for some $U_1 \in X^+ = \mathscr{P}(V)$ and

$U_2 \in Y^+ = \mathscr{P}(W)$. Thus $U_1 \subseteq V$ and $U_2 \subseteq W$, so $U = U_1 \cup U_2 \subseteq V \cup W$. Hence $U \in \mathscr{P}(V \cup W)$.

Conversely, suppose $U \in \mathscr{P}(V \cup W)$. Let $U_1 = U \cap V$ and $U_2 = U \setminus U_1$. Then $U = U_1 \cup_\emptyset U_2$

and $U_1 \in \mathscr{P}(V) = X^+$, $U_2 \in \mathscr{P}(W) = Y^+$. Hence $U \in (X +_\emptyset Y)^+$.

Observe that $(X +_\emptyset Y)^- = \mathscr{P}(^NA \setminus V) \cap \mathscr{P}(^NA \setminus W) = \mathscr{P}((^NA \setminus V) \cap (^NA \setminus W)) = $

$\mathscr{P}(^NA \setminus (V \cup W))$.

(b) Suppose $U \in C_{n,\emptyset}(X)^+$. Then $U(n : f) \in \mathscr{P}(V)$ for some $f \colon U \to A$. Hence

$U(n : f) \subseteq V$, which implies $U \subseteq V(n : A)$. Thus $U \in \mathscr{P}(V(n : A))$. Conversely, suppose

$U \in \mathscr{P}(V(n : A))$. Then $U \subseteq V(n : A)$, which means that for every $\vec{a} \in U$ there is a $\vec{b} \in V$

such that $\vec{a} = \vec{b}(n : a_n)$. Let $f \colon U \to V$ be a function that chooses one such $\vec{b}$ for every $\vec{a}$. Then

$\mathrm{pr}_n \circ f \colon V \to A$ and $U(n : \mathrm{pr}_n \circ f) \subseteq V$. Hence $U \in C_{n,\emptyset}(X)^+$.

Observe that $U \in C_{n,\emptyset}(X)^-$ if and only if $U(n : A) \in \mathscr{P}(^NA \setminus V)$ if and only if

$U(n : A) \subseteq {}^NA \setminus V$ if and only if $U \subseteq {}^NA \setminus V(n : A)$ if and only if $U \in \mathscr{P}(^NA \setminus V(n : A))$. $\square$

Proposition 2.14. *Let $\mathfrak{C}$ be the $\emptyset$-reduct of an independence-friendly cylindric set algebra, and*

let $\mathfrak{D}$ be any subalgebra generated by a set of perfect elements. Every element in $\mathfrak{D}$ is perfect.

Proof. The constants 0, 1, and D_{ij} are all perfect. If $X = \langle \mathscr{P}(V), \mathscr{P}(^NA \setminus V) \rangle$ is perfect, then

so is $X^\cup = \langle \mathscr{P}(^NA \setminus V), \mathscr{P}(V) \rangle = \langle \mathscr{P}(^NA \setminus V), \mathscr{P}(^NA \setminus (^NA \setminus V)) \rangle$. By the previous lemma, if

X and Y are perfect, then so are $X +_\emptyset Y$ and $C_{n,\emptyset}(X)$. $\square$

Corollary 2.15. $\mathfrak{Cs}_{\mathrm{IFG}_{N,\emptyset}}(\mathfrak{A})$ *is perfect.*

Proof. The meanings of atomic formulas are all perfect. $\square$

Corollary 2.16. *If ϕ is a perfect IFG-formula, then $\|\phi\|$ is perfect.*

It is conceivable that $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ includes perfect elements that cannot be generated by

perfect operations from the meanings of atomic formulas. The next proposition shows that this

is in fact not the case.

Proposition 2.17. *Every perfect element in $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ belongs to $\mathfrak{Cs}_{\mathrm{IFG}_{N,\emptyset}}(\mathfrak{A})$.*

Proof. Let $X = \|\phi\| = \langle \mathscr{P}(V), \mathscr{P}(^N A \setminus V) \rangle$, and consider $\|\phi_\emptyset\|$. We know $\|\phi_\emptyset\|$ is perfect, so let $\|\phi_\emptyset\| = \langle \mathscr{P}(V_\emptyset), \mathscr{P}(^N A \setminus V_\emptyset) \rangle$. It suffices to show that $V = V_\emptyset$. By Proposition 1.41, $\mathfrak{A} \models^+ \phi[V]$ implies $\mathfrak{A} \models^+ \phi_\emptyset[V]$, so $V \subseteq V_\emptyset$. Conversely, $\mathfrak{A} \models^- \phi[^N A \setminus V]$ implies $\mathfrak{A} \models^- \phi_\emptyset[^N A \setminus V]$, so $^N A \setminus V \subseteq {}^N A \setminus V_\emptyset$. Hence $V_\emptyset \subseteq V$. $\qquad\square$

Theorem 2.18. $\mathfrak{Cs}_N(\mathfrak{A}) \cong \mathfrak{Cs}_{\mathrm{IFG}_{N,\emptyset}}(\mathfrak{A})$.

Proof. Define two functions

$$F\colon \mathfrak{Cs}(\mathfrak{A}) \to \mathfrak{Cs}_{\mathrm{IFG}_{N,\emptyset}}(\mathfrak{A}), \qquad\qquad G\colon \mathfrak{Cs}_{\mathrm{IFG}_{N,\emptyset}}(\mathfrak{A}) \to \mathfrak{Cs}(\mathfrak{A}).$$

$$V \mapsto \langle \mathscr{P}(V), \mathscr{P}(^N A \setminus V) \rangle \qquad\qquad X \mapsto \bigcup X^+$$

First observe that

$$G \circ F(V) = \bigcup \mathscr{P}(V) = V,$$

$$F \circ G(X) = \langle \mathscr{P}\left(\bigcup X^+\right), \mathscr{P}\left(^N A \setminus \bigcup X^+\right) \rangle = X.$$

Hence F is bijective.

To show that F is a homomorphism, observe that

$$F(0) = F(\emptyset) = \langle \mathscr{P}(\emptyset), \mathscr{P}(^N A) \rangle = \langle \{\emptyset\}, \mathscr{P}(^N A) \rangle = 0,$$

$$F(1) = F(^N A) = \langle \mathscr{P}(^N A), \mathscr{P}(\emptyset) \rangle = \langle \mathscr{P}(^N A), \{\emptyset\} \rangle = 1,$$

$$F(D_{ij}) = F(\{\, \vec{a} \in {}^N A \mid a_i = a_j \,\})$$

$$= \langle \mathscr{P}(\{\, \vec{a} \in {}^N A \mid a_i = a_j \,\}), \mathscr{P}(\{\, \vec{a} \in {}^N A \mid a_i \neq a_j \,\}) \rangle$$

$$= D_{ij},$$

$$F(-V) = F(^N A \setminus V) = \langle \mathscr{P}(^N A \setminus V), \mathscr{P}(V) \rangle = F(V)^\cup,$$

$$F(V + W) = F(V \cup W) = \langle \mathscr{P}(V \cup W), \mathscr{P}(^N A \setminus (V \cup W)) \rangle = F(V) +_\emptyset F(W),$$

$$F(C_n(V)) = F(V(n : A)) = \langle \mathscr{P}(V(n : A)), \mathscr{P}(^N A \setminus V(n : A)) \rangle = C_{n,\emptyset}(F(V)).$$

Therefore F is an isomorphism. $\qquad\square$

2.3   IFG$_N$-cylindric set algebras and De Morgan algebra

Definition. A **bounded distributive lattice** is an algebra $\mathfrak{L} = \langle L; 0, 1, \vee, \wedge \rangle$ such that

$$(x \vee y) \vee z = x \vee (y \vee z), \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z),$$

$$x \vee y = y \vee x, \qquad x \wedge y = y \wedge x,$$

$$x \vee (x \wedge y) = x, \qquad x \wedge (x \vee y) = x,$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$0 \vee x = x, \qquad 1 \wedge x = x.$$

We will refer to each pair of axioms except the last as associativity, commutativity, absorption, and distributivity, respectively. We can define a partial order $\leq$ on $\mathfrak{L}$ by $x \leq y$ if and only if $x \vee y = y$ (if and only if $x \wedge y = x$). Hence the last pair of axioms asserts that $0 \leq x \leq 1$.

Definition. A **De Morgan algebra** $\mathfrak{A} = \langle A; 0, 1, \sim, \vee, \wedge \rangle$ is a bounded distributive lattice with an additional unary operation $\sim$ that satisfies $\sim\sim x = x$ and $\sim(x \vee y) = \sim x \wedge \sim y$.

Proposition 2.19. *Let $\mathfrak{A}$ be a De Morgan algebra. Then $\mathfrak{A}$ satisfies*

(a) $\sim(x \wedge y) = \sim x \vee \sim y$,

(b) $\sim 0 = 1$,

(c) $x \leq y$ if and only if $\sim y \leq \sim x$.

Proof. (a) $\sim(x \wedge y) = \sim(\sim\sim x \wedge \sim\sim y) = \sim\sim(\sim x \vee \sim y) = \sim x \vee \sim y$.

(b) $\sim 0 = \sim 0 \wedge 1 = \sim 0 \wedge \sim\sim 1 = \sim(0 \vee \sim 1) = \sim\sim 1 = 1$.

(c) $x \leq y$ if and only if $x \vee y = y$ if and only if $\sim x \wedge \sim y = \sim y$ if and only if $\sim y \leq \sim x$. $\square$

Definition. A **Kleene algebra** $\mathfrak{A} = \langle A; 0, 1, \sim, \vee, \wedge \rangle$ is a De Morgan algebra that satisfies the additional axiom $x \wedge \sim x \leq y \vee \sim y$.

Definition. A **Boolean algebra** $\mathfrak{A} = \langle A; 0, 1, -, \vee, \wedge \rangle$ is a De Morgan algebra that satisfies the complementation axioms $x \wedge -x = 0$ and $x \vee -x = 1$.

We refer the reader to [1] for the elementary theory of De Morgan algebras.

Unlike cylindric set algebras, independence-friendly cylindric set algebras do not have an underlying Boolean algebra structure. The complementation axioms fail. However, the reduct of a rooted IFG$_N$-cylindric set algebra to the signature $\langle 0, 1, \cup, +_N, \cdot_N \rangle$ is a De Morgan algebra, and the same reduct of a double-suited IFG$_N$-cylindric set algebra is a Kleene algebra.

Let $\mathfrak{C}$ be an IFG$_N$-cylindric set algebra with base set A. Let X, Y, Z be elements in $\mathfrak{C}$, let $i, j, k, \ell, m, n < N$, let $J, K, L \subseteq N$, and let $V, W \subseteq {}^N A$.

2.3.1 Associativity and commutativity

Proposition 2.20. $X +_J Y = Y +_J X$ and $X \cdot_J Y = Y \cdot_J X$.

Proof. $V \in (X +_J Y)^+$ if and only if $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$ if and only if $V = V_2 \cup_J V_1$ for some $V_2 \in Y^+$ and $V_1 \in X^+$ if and only if $V \in (Y +_J X)^+$. Also, $(X +_J Y)^- = X^- \cap Y^- = Y^- \cap X^- = (Y +_J X)^-$. $\qquad \square$

Proposition 2.21. $(X +_J Y) +_J Z = X +_J (Y +_J Z)$ and $(X \cdot_J Y) \cdot_J Z = X \cdot_J (Y \cdot_J Z)$.

Proof. Suppose $V \in ((X +_J Y) +_J Z)^+$. Then $V = (V_1 \cup_J V_2) \cup_J V_3$ for some $V_1 \in X^+$, $V_2 \in Y^+$, and $V_3 \in Z^+$. By Lemma 1.22,

$$V = (V_1 \cup_J V_2) \cup_J V_3$$

$$= V_1 \cup_J V_2 \cup_J V_3$$

$$= V_1 \cup_J (V_2 \cup_J V_3).$$

Hence $V \in (X +_J (Y +_J Z))^+$. By the same argument, if $V \in (X +_J (Y +_J Z))^+$, then $V \in ((X +_J Y) +_J Z)^+$. Therefore, $((X +_J Y) +_J Z)^+ = (X +_J (Y +_J Z))^+$.

In addition,

$$((X +_J Y) +_J Z)^- = (X^- \cap Y^-) \cap Z^-$$

$$= X^- \cap (Y^- \cap Z^-)$$

$$= (X +_J (Y +_J Z))^-. \qquad \square$$

Associativity can fail if the two operations are not the same. For example, let $\mathfrak{A}$ be the structure with universe $A = \{0, 1, 2\}$ in which each element is named by a constant symbol. Consider the IFG$_1$-cylindric set algebra over $\mathfrak{A}$ (that is, $\mathfrak{Cs}_{IFG_1}(\mathfrak{A})$). Let $X = \|v_0 = 0\|$, $Y = \|v_0 = 1\|$, and $Z = \|v_0 = 2\|$. Then

$$X = \langle \{\emptyset, \{0\}\}, \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \rangle,$$

$$Y = \langle \{\emptyset, \{1\}\}, \{\emptyset, \{0\}, \{2\}, \{0, 2\}\} \rangle,$$

$$Z = \langle \{\emptyset, \{2\}\}, \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \rangle.$$

We claim that $(X +_\emptyset Y) +_N Z \neq X +_\emptyset (Y +_N Z)$. On the left,

$$(X +_\emptyset Y)^+ = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\},$$

$$((X +_\emptyset Y) +_N Z)^+ = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{2\}\},$$

while on the right,

$$(Y +_N Z)^+ = \{\emptyset, \{1\}, \{2\}\},$$

$$(X +_\emptyset (Y +_N Z))^+ = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}\}.$$

Of course, on the falsity-axis, $((X +_\emptyset Y) +_N Z)^- = X^- \cap Y^- \cap Z^- = (X +_\emptyset (Y +_N Z))^-$.

Notice that $((X +_\emptyset Y) +_N Z)^+ \subseteq (X +_\emptyset (Y +_N Z))^+$. This suggests the following strengthening of Proposition 2.21.

Lemma 2.22. *Let* $J \subseteq K$. *Then*

(a) $(X +_J Y) +_K Z \leq^+ X +_J (Y +_K Z)$ *and* $(X +_J Y) +_K Z =^- X +_J (Y +_K Z)$;

(b) $(X \cdot_J Y) \cdot_K Z =^+ X \cdot_J (Y \cdot_K Z)$ *and* $(X \cdot_J Y) \cdot_K Z \leq^- X \cdot_J (Y \cdot_K Z)$.

Proof. (a) Suppose $V \in ((X +_J Y) +_K Z)^+$. Then $V = (V_1 \cup_J V_2) \cup_K V_3$ for some $V_1 \in X^+$, $V_2 \in Y^+$, and $V_3 \in Z^+$. By Lemma 1.20 and Lemma 1.22,

$$V = (V_1 \cup_J V_2) \cup_K V_3$$

$$= (V_1 \cup_J V_2) \cup_J V_3$$

$$= V_1 \cup_J V_2 \cup_J V_3$$

$$= V_1 \cup_J (V_2 \cup_J V_3).$$

Let $V_5 = V_2 \cup V_3$. Since V_1, V_2, and V_3 are pairwise disjoint, we have by Lemma 1.21 that $V_5 = V \cap V_5 = ((V_1 \cup_J V_2) \cap V_5) \cup_K (V_3 \cap V_5) = V_2 \cup_K V_3$. Hence $V_5 \in (Y +_K Z)^+$. Thus $V \in (X +_J (Y +_K Z))^+$.

(b) This is just the dual of part (a). $\qquad\square$

2.3.2 The elements 0, Ω, $\mho$, and 1

Proposition 2.23. *(a)* $X +_J 0 = X = X \cdot_J 1$.

(b) $X \cdot_J 0 = 0$ *and* $X +_J 1 = 1$ *if and only if* X *is rooted.*

(c) $\Omega^\cup = \Omega +_J \Omega = \Omega \cdot_J \Omega = \Omega$.

(d) $\mho^\cup = \mho +_J \mho = \mho \cdot_J \mho = \mho$.

Proof. (a) Suppose $V \in (X +_J 0)^+$. Then $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$ and $V_2 \in 0^+$. But then $V_2 = \emptyset$, so $V_1 = V$. Thus $V \in X^+$. Conversely, suppose $V \in X^+$. Then $V = V \cup_J \emptyset$, $V \in X^+$, and $\emptyset \in 0^+$. Thus $V \in (X +_J 0)^+$. Therefore $(X +_J 0)^+ = X^+$. Also, $(X +_J 0)^- = X^- \cap 0^- = X^-$.

(b) Suppose $X \cdot_J 0 = 0$ and $X +_J 1 = 1$. Then $\emptyset \in (X \cdot_J 0)^+ = X^+ \cap 0^+ \subseteq X^+$, and $\emptyset \in (X +_J 1)^- = X^- \cap 1^- \subseteq X^-$. Hence X is rooted. Conversely, suppose X is rooted. Then $(X \cdot_J 0)^+ = X^+ \cap 0^+ = \{\emptyset\}$, and for every $V \subseteq {}^N A$, $V = \emptyset \cup_J V$, where $\emptyset \in X^-$ and $V \in 0^-$. Hence $V \in (X \cdot_J 0)^-$. Thus $(X \cdot_J 0)^- = \mathscr{P}({}^N A)$. Therefore $X \cdot_J 0 = 0$. Similarly, $X +_J 1 = 1$.

(c) That $\Omega^{\cup} = \Omega$ follows immediately from the definitions. To show that $\Omega +_J \Omega = \Omega$, observe that $\emptyset = \emptyset \cup_J \emptyset$. Hence $\emptyset \in (\Omega +_J \Omega)^+$. Conversely, if $V \in (\Omega +_J \Omega)^+$, then $V = V_1 \cup_J V_2$ for some $V_1, V_2 \in \Omega^+$. Hence $V = V_1 = V_2 = \emptyset$. Thus $(\Omega +_J \Omega)^+ = \Omega^+$. Also $(\Omega +_J \Omega)^- = \Omega^- \cap \Omega^- = \Omega^-$.

(d) That $\mho^{\cup} = \mho$ follows immediately from the definitions. To show that $\mho +_J \mho = \mho$, observe that for any $V \subseteq {}^N A$, $V = V \cup_J \emptyset$. Hence $V \in (\mho +_J \mho)^+$. Thus $(\mho +_J \mho)^+ = \mho^+$. Also $(\mho +_J \mho)^- = \mho^- \cap \mho^- = \mho^-$. $\qquad\square$

Proposition 2.24. *If X and Y are double suits, and $X \leq \Omega \leq Y$, then $X \cdot_J Y = X$ and $X +_J Y = Y$.*

Proof. Suppose $X \leq \Omega \leq Y$. Then $X^+ = \{\emptyset\} = Y^-$, so $(X \cdot_J Y)^+ = X^+ \cap Y^+ = \{\emptyset\} = X^- \cap Y^- = (X +_J Y)^-$. Also, $V \in (X +_J Y)^+$ if and only if $V \in Y^+$, and $W \in (X \cdot_J Y)^-$ if and only if $W \in X^-$. Therefore $X \cdot_J Y = X$ and $X +_J Y = Y$. $\qquad\square$

2.3.3   Absorption

The absorption axioms do not hold in general, but they do hold in important special cases. For example, if X is not rooted then $(X +_N X)^+ = \emptyset = (X \cdot_N X)^-$. Hence $X +_N (X \cdot_K 1) \neq X$ whenever X is a nonempty, non-rooted element. In contrast, if X and Y are rooted, then the absorption axioms hold partially for every pair of addition and multiplication operations, and they hold fully when the "outside" operation is $+_N$ or $\cdot_N$.

Lemma 2.25. *If Y is rooted, then $X^+ \subseteq (X +_J Y)^+$ and $X^- \subseteq (X \cdot_J Y)^-$.*

Proof. Suppose $V \in X^+$. Then $V = V \cup_J \emptyset$, where $V \in X^+$ and $\emptyset \in Y^+$. Hence $V \in (X +_J Y)^+$. $\qquad\square$

Lemma 2.26. *If X and Y are rooted, then $(X +_N Y)^+ = X^+ \cup Y^+$ and $(X \cdot_N Y)^- = X^- \cup Y^-$.*

Proof. By definition, $V \in (X +_N Y)^+$ if and only if $V = V_1 \cup_N V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$, which holds if and only if $V_1 = V$ and $V_2 = \emptyset$ or vice versa. Hence $V \in (X +_N Y)^+$ if and only if $V \in X^+$ or $V \in Y^+$. $\qquad\square$

Lemma 2.27. *Suppose X and Y are rooted. Then*

(a) $X \leq^+ X +_J (X \cdot_K Y)$ *and* $X =^- X +_J (X \cdot_K Y)$;

(b) $X =^+ X \cdot_J (X +_K Y)$ *and* $X \leq^- X \cdot_J (X +_K Y)$.

Thus, $X \cdot_J (X +_K Y) \leq X \leq X +_J (X \cdot_K Y)$.

Proof. (a) Suppose $V \in X^+$. Then $V = V \cup_J \emptyset$ where $V \in X^+$ and $\emptyset \in X^+ \cap Y^+ = (X \cdot_K Y)^+$. Hence $V \in (X +_J (X \cdot_K Y))^+$.

Suppose $W \in X^-$. Then $W = W \cup_K \emptyset$, where $W \in X^-$ and $\emptyset \in Y^-$. Hence $W \in X^- \cap (X \cdot_K Y)^- = (X +_J (X \cdot_K Y))^-$. Conversely, suppose $W \in (X +_J (X \cdot_K Y))^-$. Then $W \in X^- \cap (X \cdot_K Y)^- \subseteq X^-$. $\qquad\square$

To show that absorption can fail even when X and Y are rooted, consider the IFG_2-cylindric set algebra over the equality structure $\mathfrak{A}$ whose universe is $A = \{0, 1\}$. Let $X = D_{01} +_N D_{01}^{\cup}$. Then

$$X^+ = \{\emptyset, \{00\}, \{01\}, \{10\}, \{11\}, \{00, 11\}, \{01, 10\}\},$$

$$X^- = \{\emptyset\}.$$

However, $\{00, 01, 10, 11\} \in (X +_\emptyset (X +_N X))^+$ because $\{00, 01, 10, 11\} = \{00, 11\} \cup_\emptyset \{01, 10\}$, where $\{00, 11\} \in X^+$ and $\{01, 10\} \in X^+ = (X +_N X)^+$. Thus $(X +_\emptyset (X +_N X))^+ \supset X^+$.

One can obtain a similar example where $(X \cdot_\emptyset (X +_N X))^- \supset X^-$ by taking $X = D_{01} \cdot_N D_{01}^{\cup}$.

Notice that $D_{01} +_N D_{01}^{\cup} \neq 1$, which demonstrates that the complementation axioms fail in any independence-friendly cylindric set algebra of dimension greater than 1.

Lemma 2.28. *If X is flat, then $X +_J (X \cdot_K Y) = X$. If $X^{\cup}$ is flat, then $X \cdot_J (X +_K Y) = X$.*

Proof. Suppose $X^+ = \mathscr{P}(V)$. If $V' \in (X +_J (X \cdot_K Y))^+$, then $V' = V_1 \cup_J V_2$ for some $V_1 \in X^+$ and $V_2 \in (X \cdot_K Y)^+ = X^+ \cap Y^+$. Hence $V' = V_1 \cup_J V_2 \subseteq V$. Thus $V' \in \mathscr{P}(V) = X^+$. $\qquad\square$

Lemma 2.29. *Suppose X and Y are rooted. Then*

(a) $X +_N (X \cdot_J Y) = X$;

(b) $X \cdot_N (X +_J Y) = X$.

Proof. (a) First, observe that $(X +_N (X \cdot_J Y))^+ = X^+ \cup (X^+ \cap Y^+) = X^+$. Next, suppose $W \in (X +_N (X \cdot_J Y))^-$. Then $W \in X^- \cap (X \cdot_J Y)^- \subseteq X^-$. Conversely, suppose $W \in X^-$. Then $W = W \cup_J \emptyset$ and $W \in X^-, \emptyset \in Y^-$. Hence $W \in (X \cdot_J Y)^-$. Thus $W \in X^- \cap (X \cdot_J Y)^- = (X +_N (X \cdot_J Y))^-$. $\qquad\square$

Thus, the reduct of a rooted IFG_N-cylindric set algebra to the signature $\langle +_N, \cdot_N \rangle$ is a lattice. As with all lattices, we can define a partial order $\leq'$ by declaring $X \leq' Y$ if and only if $X +_N Y = Y$ (or equivalently, $X \cdot_N Y = X$). The next proposition shows that in a rooted IFG_N-cylindric set algebra, our two partial orders $\leq$ and $\leq'$ agree.

Proposition 2.30. *Suppose X and Y are rooted. Then $X \leq Y$ if and only if $X +_N Y = Y$ if and only if $X \cdot_N Y = X$.*

Proof. First, $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $Y^- \subseteq X^-$ if and only if $X^+ \cup Y^+ = Y^+$ and $X^- \cap Y^- = Y^-$ if and only if $(X +_N Y)^+ = Y^+$ and $(X +_N Y)^- = Y^-$.

Second, $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $Y^- \subseteq X^-$ if and only if $X^+ \cap Y^+ = X^+$ and $X^- \cup Y^- = X^-$ if and only if $(X \cdot_N Y)^+ = X^+$ and $(X \cdot_N Y)^- = X^-$. $\qquad\square$

Proposition 2.31. *If X is rooted, then $0 \leq X \leq 1$.*

Proof. By Proposition 2.23 and Proposition 2.30. $\qquad\square$

Proposition 2.32. *If $J \subseteq K$, then $X +_K Y \leq X +_J Y$ and $X \cdot_J Y \leq X \cdot_K Y$.*

Proof. Suppose $J \subseteq K$. If $V \in (X +_K Y)^+$, then $V = V_1 \cup_K V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$. But $V = V_1 \cup_K V_2$ implies $V = V_1 \cup_J V_2$, so $V \in (X +_J Y)^+$. Thus $(X +_K Y)^+ \subseteq (X +_J Y)^+$. In addition, $(X +_J Y)^- = X^- \cap Y^- = (X +_K Y)^-$. $\qquad\square$

Therefore, when X is rooted, we have the following string of inequalities:

$$X \cdot_\emptyset X \leq X \cdot_J X \leq X \cdot_N X = X = X +_N X \leq X +_J X \leq X +_\emptyset X.$$

Proposition 2.33. *If $X \leq X'$ and $Y \leq Y'$, then $X +_J Y \leq X' +_J Y'$ and $X \cdot_J Y \leq X' \cdot_J Y'$.*

Proof. Suppose $X \leq X'$ and $Y \leq Y'$. If $V \in (X +_J Y)^+$, then $V = V_1 \cup_J V_2$ for some $V_1 \in X^+ \subseteq (X')^+$ and $V_2 \in Y^+ \subseteq (Y')^+$. Hence $V \in (X' +_J Y')^+$. Also, $(X' +_J Y')^- = (X')^- \cap (Y')^- \subseteq X^- \cap Y^- = (X +_J Y)^-$. Thus $X +_J Y \leq X' +_J Y'$. □

2.3.4 Distributivity

Lemma 2.34. *Suppose X is a double suit. Then*

(a) $(X \cdot_J (Y +_K Z))^\pm \subseteq ((X \cdot_J Y) +_K (X \cdot_J Z))^\pm$;

(b) $(X +_J (Y \cdot_K Z))^\pm \subseteq ((X +_J Y) \cdot_K (X +_J Z))^\pm$.

Proof. (a) Suppose $V \in (X \cdot_J (Y +_K Z))^+$. Then $V \in X^+ \cap (Y +_K Z)^+$, which implies that $V = V_1 \cup_K V_2$ for some $V_1 \in Y^+$ and $V_2 \in Z^+$. Observe that since $V_1, V_2 \subseteq V \in X^+$ we have $V_1, V_2 \in X^+$. Hence $V_1 \in X^+ \cap Y^+ = (X \cdot_J Y)^+$ and $V_2 \in X^+ \cap Z^+ = (X \cdot_J Z)^+$. Thus $V \in ((X \cdot_J Y) +_K (X \cdot_J Z))^+$.

Now suppose $W \in (X \cdot_J (Y +_K Z))^-$. Then $W = W_1 \cup_J W_2$ for some $W_1 \in X^-$ and $W_2 \in Y^- \cap Z^-$. It follows that $W \in (X \cdot_J Y)^- \cap (X \cdot_J Z)^- = ((X \cdot_J Y) +_K (X \cdot_J Z))^-$. □

To show that distributivity can fail, consider the IFG_2-cylindric set algebra over the structure $\mathfrak{A}$ whose universe is $A = \{0, 1\}$ and in which each element is named by a constant. Let $X = \|v_0 = 0\| +_N \|v_0 = 1\|$, and let

$$V = \{00, 01, 10, 11\},$$

$$V_1 = \{00, 01\},$$

$$V_2 = \{10, 11\}.$$

Observe that $V = V_1 \cup_{\{1\}} V_2$, $V_1 \in X^+$ and $V_2 \in X^+$. Hence

$$V \in (X +_{\{1\}} X)^+ = ((X \cdot_J 1) +_{\{1\}} (X \cdot_J 1))^+.$$

However, $V \notin X^+ = (X \cdot_J (1 +_{\{1\}} 1))^+$.

Lemma 2.35. *Suppose X, Y, and Z are rooted.*

(a) $X \cdot_J (Y +_N Z) =^+ (X \cdot_J Y) +_N (X \cdot_J Z).$

(b) $X \cdot_N (Y +_K Z) =^- (X \cdot_N Y) +_K (X \cdot_N Z).$

(c) $X +_N (Y \cdot_K Z) =^+ (X +_N Y) \cdot_K (X +_N Z).$

(d) $X +_J (Y \cdot_N Z) =^- (X +_J Y) \cdot_N (X +_J Z).$

Proof.

$$(X \cdot_J (Y +_N Z))^+ = X^+ \cap (Y^+ \cup Z^+)$$
$$= (X^+ \cap Y^+) \cup (X^+ \cap Z^+)$$
$$= ((X \cdot_J Y) +_N (X \cdot_J Z))^+.$$

$$(X \cdot_N (Y +_K Z))^- = X^- \cup (Y^- \cap Z^-)$$
$$= (X^- \cup Y^-) \cap (X^- \cup Z^-)$$
$$= ((X \cdot_N Y) +_K (X \cdot_N Z))^-.$$

$$(X +_N (Y \cdot_K Z))^+ = X^+ \cup (Y^+ \cap Z^+)$$
$$= (X^+ \cup Y^+) \cap (X^+ \cup Z^+)$$
$$= ((X +_N Y) \cdot_K (X +_N Z))^+.$$

$$(X +_J (Y \cdot_N Z))^- = X^- \cap (Y^- \cup Z^-)$$
$$= (X^- \cap Y^-) \cup (X^- \cap Z^-)$$
$$= ((X +_J Y) \cdot_N (X +_J Z))^-. \qquad \square$$

The previous lemmas and propositions combine to yield the following theorem.

Theorem 2.36. *The reduct of a rooted IFG_N-cylindric set algebra to the signature $\langle 0, 1, {}^{\cup}, +_N, \cdot_N \rangle$ is a De Morgan algebra.*

2.3.5   Complementation

We have already remarked that the complementation axioms fail in independence-friendly cylindric set algebras. The failure of the complementation axioms stems from the failure of the law of excluded middle in IFG logic. However sentences of the form $\phi \wedge_{/J} \sim \phi$ are never true, while sentences of the form $\phi \vee_{/K} \sim \phi$ are never false.

Proposition 2.37. *If X and Y are double suits, then $X \cdot_J X^{\cup} \leq Y +_K Y^{\cup}$.*

Proof. Suppose X and Y are double suits. Then $(X \cdot_J X^{\cup})^+ = X^+ \cap X^- = \{\emptyset\} \subseteq (Y +_K Y^{\cup})^+$, and $(Y +_K Y^{\cup})^- = Y^+ \cap Y^- = \{\emptyset\} \subseteq (X +_J X^{\cup})^-$. $\qquad\square$

Thus if a double-suited IFG_N-cylindric set algebra includes Ω, then

$$X \cdot_J X^{\cup} \leq \Omega \leq X +_K X^{\cup}.$$

Theorem 2.38. *The reduct of a double-suited IFG_N-cylindric set algebra to the signature $\langle 0, 1, {}^{\cup}, +_N, \cdot_N \rangle$ is a Kleene algebra.*

Even though the complementation axioms do not hold universally, it might still be possible for particular elements to satisfy them. In a double-suited IFG_N-cylindric set algebra, the only elements that satisfy the complementation axioms are 0 and 1.

Lemma 2.39. $X +_N X^{\cup} = 1$ *(equivalently, $X \cdot_N X^{\cup} = 0$) if and only if $X^+ \cup X^- = \mathscr{P}(^N A)$ and $X^+ \cap X^- = \{\emptyset\}$.*

Proof. Suppose $X +_N X^{\cup} = 1$. Then $X^- \cap X^+ = (X +_N X^{\cup})^- = \{\emptyset\}$. Hence X is rooted. Thus $X^+ \cup X^- = (X +_N X^{\cup})^+ = \mathscr{P}(^N A)$.

Conversely, suppose $X^+ \cup X^- = \mathscr{P}(^N A)$ and $X^+ \cap X^- = \{\emptyset\}$. Then X is rooted, so $(X +_N X^{\cup}) = X^+ \cup X^- = 1^+$ and $(X +_N X^{\cup})^- = X^- \cap X^+ = 1^-$. $\qquad\square$

Lemma 2.40. *Suppose X is a double suit. Then $X +_N X^\cup = 1$ if and only if $X = 0$ or $X = 1$.*

Proof. Suppose $X +_N X^\cup = 1$. Then, by the previous lemma, $(X +_N X^\cup)^+ = X^+ \cup X^- = \mathscr{P}(^N A)$. In particular, $^N A \in X^+ \cup X^-$. If $^N A \in X^+$ then, since X^+ is a suit, $X^+ = \mathscr{P}(^N A)$ and $X^- = \{\emptyset\}$. Hence $X = 1$. Similarly, if $^N A \in X^-$, then $X = 0$.

Conversely, suppose $X = 0$ or $X = 1$. In either case, $X +_N X^\cup = 0 +_N 1 = 1$. $\qquad\square$

Definition. Let L be a bounded lattice. Two elements $x, y \in L$ are **complements** if $x \vee y = 1$ and $x \wedge y = 0$. An element $x \in L$ is **complemented** if it has a complement.

For example, Ω and $\mho$ are complements in any IFG$_N$-cylindric set algebra that includes both. However in a double-suited IFG$_N$-cylindric set algebra the only complemented elements are 0 and 1.
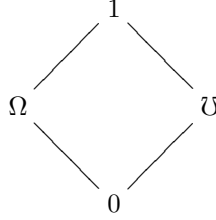
Lemma 2.41. *Suppose $X +_N Y = 1$ and $X \cdot_N Y = 0$. Then $X^+ \cup Y^+ = \mathscr{P}(^N A) = X^- \cup Y^-$ and $X^+ \cap Y^+ = \{\emptyset\} = X^- \cap Y^-$.*

Proof. First, $X^+ \cap Y^+ = (X \cdot_N Y)^+ = \{\emptyset\} = (X +_N Y)^- = X^- \cap Y^-$. Hence X and Y are both rooted. Thus $X^+ \cup Y^+ = (X +_N Y)^+ = \mathscr{P}(^N A) = (X \cdot_N Y)^- = X^- \cup Y^-$. $\qquad\square$

Proposition 2.42. *Suppose X and Y are double suits. Then $X +_N Y = 1$ and $X \cdot_N Y = 0$ if and only if $X = 1$ and $Y = 0$, or vice versa.*

Proof. Suppose $X +_N Y = 1$ and $X \cdot_N Y = 0$. By the previous lemma, $X^+ \cup Y^+ = \mathscr{P}(^N A) = X^- \cup Y^-$ and $X^+ \cap Y^+ = \{\emptyset\} = X^- \cap Y^-$. In particular, $^N A \in X^+ \cup Y^+$. Since X and Y are both double suits, if $^N A \in X^+$ then $X = 1$, and if $^N A \in Y^+$ then $Y = 1$. In the first case, $X^+ \cap Y^+ = \{\emptyset\}$ and $X^- \cup Y^- = \mathscr{P}(^N A)$ imply that $Y = 0$, and in the second case, that $Y = 1$.

Conversely, suppose $X = 1$ and $Y = 0$, or vice versa. In either case, $X +_N Y = 0 +_N 1 = 1$ and $X \cdot_N Y = 0 \cdot_N 1 = 0$. $\square$

2.4 IFG$_N$-cylindric set algebras and cylindric algebra

In addition to the axioms of Boolean algebra, the axioms of cylindric algebra are [11]:

(C1) $c_n(0) = 0$.

(C2) $x \cdot c_n(x) = x$.

(C3) $c_n(x \cdot c_n(y)) = c_n(x) \cdot c_n(y)$.

(C4) $c_m c_n(x) = c_n c_m(x)$.

(C5) $c_i(d_{ij}) = 1$.

(C6) $c_k(d_{ik} \cdot d_{kj}) = d_{ij}$ provided $k \neq i, j$.

(C7) $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$ provided $i \neq j$.

We will show that analogs of the axioms (C1)–(C6) hold in all double-suited independence-friendly cylindric set algebras. Axiom (C7) does not hold in general, even in double-suited independence-friendly cylindric set algebras.

2.4.1 Axiom C1

Proposition 2.43. $C_{n,J}(0) = 0$ and $C_{n,J}^{\partial}(1) = 1$.

Proof. Suppose $V \in C_{n,J}(0)^+$. Then $V(n : f) \in 0^+$ for some $f \colon V \underset{J}{\to} A$. But then $V(n : f) = \emptyset$, which holds if and only if $V = \emptyset$. Thus $V \in 0^+$. Conversely, suppose $V \in 0^+$. Then $V = \emptyset$. Observe that the empty function f from $\emptyset$ to A is vacuously independent of J and $\emptyset(n : f) = \emptyset \in 0^+$. Thus $\emptyset \in C_{n,J}(0)^+$. Therefore $C_{n,J}(0)^+ = 0^+$.

$C_{n,J}(0)^- \subseteq \mathscr{P}(^N A) = 0^-$ is immediate. To show $0^- \subseteq C_{n,J}(0)^-$, suppose $W \in 0^-$. Then $W \subseteq {}^N A$, so $W(n : A) \subseteq {}^N A$. Hence $W(n : A) \in 0^-$. Thus $W \in C_{n,J}(0)^-$. Therefore $C_{n,J}(0)^- = 0^-$. $\square$

Proposition 2.44. *If X is a double suit, then $C_{n,J}(X) = 0$ if and only if $X = 0$.*

Proof. Observe that ${}^N A \in C_{n,J}(X)^-$ if and only if ${}^N A = {}^N A(n : A) \in X^-$. Thus if $X \neq 0$, then ${}^N A \notin X^-$, so ${}^N A \notin C_{n,J}(X)^-$, hence $C_{n,J}(X) \neq 0$. $\square$

Proposition 2.45. $C_{n,J}(\Omega) = \Omega = C^{\partial}_{n,J}(\Omega)$ *and* $C_{n,J}(\mho) = \mho = C^{\partial}_{n,J}(\mho)$.

Proof. $\emptyset \in C_{n,J}(\Omega)^+$ because Ω is rooted. Conversely, suppose $V \in C_{n,J}(\Omega)^+$. Then $V(n : f) \in \Omega^+$ for some $f : V \underset{J}{\to} A$. Hence $V(n : f) = \emptyset$, which implies $V = \emptyset$. Thus $C_{n,J}(\Omega)^+ = \Omega^+$. Also, $W \in C_{n,J}(X)^-$ if and only if $W(n : A) \in \Omega^-$ if and only if $W = \emptyset$. Hence $C_{n,J}(X)^- = \Omega^-$.

We know that $C_{n,J}(\mho)$ is a pair of suits because $\mho$ is. Thus it suffices to show that ${}^N A \in C_{n,J}(\mho)^+$ and ${}^N A \in C_{n,J}(\mho)^-$. Observe that for any $f : {}^N A \underset{J}{\to} A$ we have ${}^N A(n : f) \subseteq {}^N A \in \mho^+$. Hence ${}^N A \in C_{n,J}(\mho)^+$. Also, ${}^N A(n : A) = {}^N A \in \mho^-$. Hence ${}^N A \in C_{n,J}(\mho)^-$. $\square$

Proposition 2.46. $C_{n,J}(1) = 1$ *and* $C^{\partial}_{n,J}(0) = 0$.

Proof. That $C_{n,J}(1)^+ \subseteq 1^+$ is immediate. To show $1^+ \subseteq C_{n,J}(1)^+$, suppose $V \in 1^+$. Define $f : V \underset{J}{\to} A$ by $f(\vec{a}) = c$ for some arbitrary $c \in A$. Then $V(n : f) \in 1^+$, so $V \in C_{n,J}(1)^+$. Therefore $C_{n,J}(1)^+ = 1^+$.

That $1^- \subseteq C_{n,J}(1)^-$ is immediate. To show $C_{n,J}(1)^- \subseteq 1^-$, suppose $W \in C_{n,J}(1)^-$. Then $W(n : A) \in 1^-$. Hence $W(n : A) = \emptyset$, which implies $W = \emptyset$. Thus $W \in 1^-$. Therefore $C_{n,J}(1)^- = 1^-$. $\square$

2.4.2 Axiom C2

Proposition 2.47. *(a) If $n \notin J$, then $X \cdot_K C_{n,J}(X) =^+ X$.*

(b) If X^- is a suit, then $X \cdot_N C_{n,J}(X) =^- X$.

In particular, if X is a double suit then $X \cdot_N C_{n,\emptyset}(X) = X$.

Proof. (a) Observe that $(X \cdot_K C_{n,J}(X))^+ = X^+ \cap C_{n,J}(X)^+ \subseteq X^+$. Conversely, suppose $V \in X^+$. If $n \notin J$, the projection $\mathrm{pr}_n : V \to A$ that maps $\vec{a}$ to a_n is independent of J, and $V(n : \mathrm{pr}_n) = V \in X^+$. Hence $V \in X^+ \cap C_{n,J}(X)^+ = (X \cdot_K C_{n,J}(X))^+$.

(b) Suppose X^- is a suit and $W \in (X \cdot_N C_{n,J}(X))^-$. Then $W = W \cup_N \emptyset$, where $W \in X^-$ and $\emptyset \in C_{n,J}(X)^-$, or vice versa. If $W \in X^-$ we are done, so suppose $W \in C_{n,J}(X)^-$. Then $W \subseteq W(n : A) \in X^-$; hence $W \in X^-$. Conversely, suppose $W \in X^-$. Since X^- is a suit, $C_{n,J}(X)^-$ is also a suit. In particular, $\emptyset \in C_{n,J}(X)^-$. Hence $W = W \cup_N \emptyset$, where $W \in X^-$ and $\emptyset \in C_{n,J}(X)^-$. Thus $W \in (X \cdot_N C_{n,J}(X))^-$. $\qquad\square$

To give an example where $n \in J$ and $X \cdot_K C_{n,J}(X) \neq^+ X$, let $\mathfrak{A}$ be the equality structure with universe $A = \{0, 1\}$, and consider $\mathfrak{Cs}_{\mathrm{IFG}_2}(\mathfrak{A})$. Observe that

$$D_{01} = \langle \mathscr{P}(\{00, 11\}), \mathscr{P}(\{01, 10\}) \rangle,$$

$$C_{0,N}(D_{01}) = \langle \{\emptyset, \{00\}, \{11\}\}, \{\emptyset\} \rangle,$$

$$(D_{01} \cdot_K C_{0,N}(D_{01}))^+ = \{\emptyset, \{00\}, \{11\}\} \neq D_{01}^+.$$

In the previous section we showed that the partial order $\leq$ interacts nicely with the operations $\cup$, $+_J$ and $\cdot_J$ (see Proposition 2.32, and Proposition 2.33). It interacts equally well with the operations $C_{n,J}$ and $C_{n,J}^\partial$. In particular, the previous proposition shows that in a double-suited IFG$_N$-cylindric set algebra, $C_{n,J}^\partial(X) \leq X \leq C_{n,J}(X)$ whenever $n \notin J$.

2.4.3 Axiom C3

Proposition 2.48. *If $J \subseteq K$, then $C_{n,K}(X) \leq C_{n,J}(X)$ and $C_{n,J}^\partial(X) \leq C_{n,K}^\partial(X)$.*

Proof. Suppose $J \subseteq K$ and $V \in C_{n,K}(X)^+$. Then $V(n : f) \in X^+$ for some $f : V \xrightarrow{K} A$. But then $f : V \xrightarrow{J} A$; hence $V \in C_{n,J}(X)^+$. Also, $C_{n,J}(X)^- = C_{n,K}(X)^-$ by definition. $\qquad\square$

Proposition 2.49. *If $X \leq Y$, then $C_{n,J}(X) \leq C_{n,J}(Y)$ and $C_{n,J}^\partial(X) \leq C_{n,J}^\partial(Y)$.*

Proof. Suppose $X \leq Y$ and $V \in C_{n,J}(X)^+$. Then $V(n : f) \in X^+ \subseteq Y^+$ for some $f : V \xrightarrow{J} A$. Hence $V \in C_{n,J}(Y)^+$. Now suppose $W \in C_{n,J}(Y)^-$. Then $W(n : A) \in Y^- \subseteq X^-$. Hence $W \in C_{n,J}(X)^-$. $\qquad\square$

Proposition 2.50.   *(a) If $J \subseteq K$, then $C_{n,J}(X \cdot_L C_{n,K}(Y)) \leq^+ C_{n,J}(X) \cdot_L C_{n,J}(Y)$.*

(b) If $n \in K$, then $C_{n,J}(X \cdot_L C_{n,K}(Y)) =^+ C_{n,J}(X) \cdot_L C_{n,K}(Y)$.

(c) If $n \in L$, then $C_{n,J}(X) \cdot_L C_{n,K}(Y) \leq^- C_{n,J}(X \cdot_L C_{n,K}(Y))$ and $C_{n,J}(X \cdot_L C_{n,K}(Y)) \leq^- C_{n,P}(C_{n,J}(X) \cdot_L C_{n,K}(Y))$.

(d) If $n \in L$ and $(C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$ is a suit, then $C_{n,J}(X \cdot_L C_{n,K}(Y)) =^- C_{n,J}(X) \cdot_L C_{n,K}(Y)$.

Thus, if X and Y are double suits, and $n \in K \cap L$, then

$$C_{n,J}(X \cdot_L C_{n,K}(Y)) = C_{n,J}(X) \cdot_L C_{n,K}(Y).$$

Proof. (a) Suppose $V \in (C_{n,J}(X \cdot_L C_{n,K}(Y)))^+$. Then there is a function $f \colon V \underset{J}{\to} A$ such that $V(n : f) \in (X \cdot_L C_{n,K}(Y))^+ = X^+ \cap C_{n,K}(Y)^+$, which implies that there is a $g \colon V(n : f) \underset{K}{\to} A$ such that $V(n : f)(n : g) \in Y^+$. By Lemma 1.26, if $J \subseteq K$ there is an $h \colon V \underset{J}{\to} A$ such that $V(n : f)(n : g) = V(n : h)$. Hence $V \in C_{n,J}(X)^+ \cap C_{n,J}(Y)^+ = (C_{n,J}(X) \cdot_L C_{n,J}(Y))^+$.

(b) Suppose $V \in C_{n,J}(X \cdot_L C_{n,K}(Y))^+$. Then there is a function $f \colon V \underset{J}{\to} A$ such that $V(n : f) \in (X \cdot_L C_{n,K}(Y))^+ = X^+ \cap C_{n,K}(Y)^+$, which implies that there is a function $g \colon V(n : f) \underset{K}{\to} A$ such that $V(n : f)(n : g) \in Y^+$. If $n \in K$, then by Lemma 1.26 there is a function $h \colon V \underset{K}{\to} A$ such that $V(n : f)(n : g) = V(n : h)$. Hence $V \in C_{n,J}(X)^+ \cap C_{n,K}(Y)^+ = (C_{n,J}(X) \cdot_L C_{n,K}(Y))^+$.

Conversely, suppose $V \in (C_{n,J}(X) \cdot_L C_{n,K}(Y))^+ = C_{n,J}(X)^+ \cap C_{n,J}(Y)^+$. Then there exist $f \colon V \underset{J}{\to} A$ and $h \colon V \underset{J}{\to} A$ such that $V(n : f) \in X^+$ and $V(n : h) \in Y^+$. If $n \in K$, then by Lemma 1.27 there is a $g \colon V(n : f) \underset{K}{\to} A$ such that $V(n : f)(n : g) = V(n : h)$. It follows that $V(n : f) \in X^+ \cap C_{n,K}(Y)^+ = (X \cdot_L C_{n,K}(Y))^+$. Hence $V \in (C_{n,J}(X \cdot_L C_{n,K}(Y)))^+$.

(c) Suppose $n \in L$ and $W \in (C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$. Then $W = W_1 \cup_L W_2$ for some $W_1 \in C_{n,J}(X)^-$ and $W_2 \in C_{n,K}(Y)^-$, which implies $W_1(n : A) \in X^-$ and $W_2(n : A) \in Y^-$. By Lemma 1.29, $W(n : A) = W_1(n : A) \cup_L W_2(n : A)$. Also, $W_2(n : A)(n : A) = W_2(n : A) \in Y^-$, so $W_2(n : A) \in C_{n,K}(Y)^-$. Thus, $W(n : A) \in (X \cdot_L C_{n,K}(Y))^-$. Therefore we have $W \in (C_{n,J}(X \cdot_L C_{n,K}(Y)))^-$.

Suppose $W \in (C_{n,J}(X \cdot_L C_{n,K}(Y)))^-$. Then $W(n : A) \in (X \cdot_L C_{n,K}(Y))^-$, which means $W(n : A) = W_1 \cup_L W_2$ for some $W_1 \in X^-$ and $W_2 \in C_{n,K}(Y)^-$. By Lemma 1.29, $W_1 = W_1(n : A)$, hence $W_1 \in C_{n,J}(X)^-$. Thus $W(n : A) \in (C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$. Therefore $W \in C_{n,P}(C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$.

(d) If $(C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$ is a suit, then $W \subseteq W(n : A) \in (C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$ implies $W \in (C_{n,J}(X) \cdot_L C_{n,K}(Y))^-$. $\qquad\square$

To give an example where (b) fails, again let $\mathfrak{A}$ be the equality structure with universe $A = \{0, 1\}$, and consider $\mathfrak{Cs}_{\mathrm{IFG}_2}(\mathfrak{A})$. Setting $X = 1$ and $Y = D_{01}$, observe that

$$D_{01} \leq C_{1,\{0\}}(D_{01}) = C_{1,N}(1) \cdot_L C_{1,\{0\}}(D_{01}).$$

In particular, $\{00, 11\} \in (C_{1,N}(1) \cdot_L C_{1,\{0\}}(D_{01}))^+$. However, $\{00, 11\} \notin C_{1,N}(1 \cdot_L C_{1,\{0\}}(D_{01}))$. To see why, suppose the contrary. Then there is an $f \colon \{00, 11\} \xrightarrow{N} A$ such that $\{00, 11\}(1 : f) \in (1 \cdot_L C_{1,\{0\}}(D_{01}))^+ = C_{1,\{0\}}(D_{01})^+$. Hence f is a constant function. Thus, either $\{00, 11\}(1 : f) = \{00, 10\} \in C_{1,\{0\}}(D_{01})^+$ or $\{00, 11\}(1 : f) = \{01, 11\} \in C_{1,\{0\}}(D_{01})^+$. In the first case, $\{00, 10\}(1, g) \in D_{01}^+$ for some $g \colon \{00, 10\} \xrightarrow{\{0\}} A$. Observe that g is a constant function, so either $\{00, 10\}(1, g) = \{00, 10\}$ or $\{00, 10\}(1, g) = \{01, 11\}$. Note that neither set belongs to D_{01}^+. The second case is similar.

2.4.4 Axiom C4

Proposition 2.51. *(a) $C_{n,J}(C_{n,K}(X)) \leq C_{n,J \cap K}(X)$.*

(b) If $n \in K$, then $C_{n,J}(C_{n,K}(X)) = C_{n,K}(X)$.

Proof. (a) Suppose $V \in C_{n,J}(C_{n,K}(X))^+$. Then $V(n : f)(n : g) \in X^+$ for some $f \colon V \xrightarrow{J} A$ and $g \colon V(n : f) \xrightarrow{K} A$. By Lemma 1.26 there is an $h \colon V \xrightarrow{J \cap K} A$ such that $V(n : f)(n : g) = V(n : h)$. Hence $V \in C_{n,J \cap K}(X)^+$. Also, $W \in C_{n,J}(C_{n,K}(X))^-$ if and only if $W(n : A)(n : A) \in X^-$ if and only if $W(n : A) \in X^-$ if and only if $W \in C_{n,J \cap K}(X)^-$.

(b) If $n \in K$, then the h from above is independent of K, so $V \in C_{n,K}(X)^+$. Conversely, suppose $V \in C_{n,K}(X)^+$. Then there is an $h \colon V \xrightarrow{K} A$ such that $V(n : h) \in X^+$. Let $f \colon V \xrightarrow{J} A$

be any function independent of J. By Lemma 1.27 there is a $g \colon V(n : f) \xrightarrow[K]{} A$ such that $V(n : f)(n : g) = V(n : h)$. Hence $V \in C_{n,J}(C_{n,K}(X))^+$. $\qquad \square$

Proposition 2.52. *If $m \in K$ and $n \in J$, where $m \neq n$, then*

$$C_{m,J}(C_{n,K}(X)) = C_{n,K}(C_{m,J}(X)).$$

Proof. Suppose $V \in C_{m,J}(C_{n,K}(X))^+$. Then there exist $f \colon V \xrightarrow[J]{} A$ and $g \colon V(m : f) \xrightarrow[K]{} A$ such that $V(m : f)(n : g) \in X^+$. By Lemma 1.28 there exist two functions $G \colon V \xrightarrow[K]{} A$ and $F \colon V(n : G) \xrightarrow[J]{} A$ such that $V(m : f)(n : g) = V(n : G)(m : F)$. Hence $V \in C_{n,K}(C_{m,J}(X))^+$. Thus $C_{m,J}(C_{n,K}(X))^+ \subseteq C_{n,K}(C_{m,J}(X))^+$. The reverse containment follows by symmetry. Therefore $C_{m,J}(C_{n,K}(X))^+ = C_{n,K}(C_{m,J}(X))^+$.

Observe that $W \in C_{m,J}(C_{n,K}(X))^-$ if and only if $W(m : A)(n : A) \in X^-$ if and only if $W(n : A)(m : A) \in X^-$ if and only if $W \in C_{n,K}(C_{m,J}(X))^-$. $\qquad \square$

2.4.5 Axiom C5

Proposition 2.53. *If $j \notin J$, then $C_{i,J}(D_{ij}) = 1$.*

Proof. That $C_{i,J}(D_{ij})^+ \subseteq 1^+$ is immediate. To show $1^+ \subseteq C_{i,J}(D_{ij})^+$, suppose $V \in 1^+$. Define $f \colon V \xrightarrow[J]{} A$ by $f(\vec{a}) = a_j$. Observe that f is independent of J because $j \notin J$. Also note that $V(i : f) = \{\, \vec{a}(i : a_j) \mid \vec{a} \in V \,\} \in D_{ij}^+$. Hence $V \in C_{i,J}(D_{ij})^+$. Therefore $C_{i,J}(D_{ij})^+ = 1^+$.

We know $\emptyset \in C_{i,J}(D_{ij})^-$, so to show $C_{i,J}(D_{ij})^- = \{\emptyset\} = 1^-$ it suffices to show that any $W \in C_{i,J}(D_{ij})^-$ must be empty. Suppose $W \in C_{i,J}(D_{ij})^-$. Then by definition $W(i : A) \in D_{ij}^-$. It follows that $W = \emptyset$ because if $\vec{a} \in W$, then $\vec{a}(i : a_j) \in W(i : A)$, which contradicts $W(i : A) \in D_{ij}^-$. Therefore $C_{i,J}(D_{ij})^- = 1^-$. $\qquad \square$

To show that the hypothesis $j \notin J$ is necessary, let $\mathfrak{A}$ be the equality structure $\mathfrak{A}$ with universe is $A = \{0, 1\}$, and consider $\mathfrak{Cs}_{\mathrm{IFG}_2}(\mathfrak{A})$. Observe that

$$C_{0,N}(D_{01}) = \langle \mathscr{P}(\{00, 10\}) \cup \mathscr{P}(\{01, 11\}), \{\emptyset\} \rangle.$$

2.4.6 Axiom C6

Proposition 2.54. *If $i \notin J$ or $j \notin J$, and $i \neq k \neq j$, then*

$$C_{k,J}(D_{ik} \cdot_\emptyset D_{kj}) = D_{ij}.$$

Proof. Without loss of generality, suppose $i \notin J$ and $i \neq k \neq j$. Suppose $V \in C_{k,J}(D_{ik} \cdot_\emptyset D_{kj})^+$. Then there is an $f \colon V \underset{J}{\to} A$ such that $V(k : f) \in (D_{ik} \cdot_\emptyset D_{kj})^+ = D_{ik}^+ \cap D_{kj}^+$. Thus every $\vec{b} \in V(k : f)$ has the property that $b_i = b_k = b_j$. Let $\vec{a} \in V$. Then $\vec{a}(k : f(\vec{a})) \in V(k : f)$, so $a_i = a_j$. Therefore $V \in D_{ij}^+$.

Conversely, suppose $V \in D_{ij}^+$. Then every $\vec{a} \in V$ has the property that $a_i = a_j$. Since $i \notin J$, the projection $\mathrm{pr}_i \colon V \underset{J}{\to} A$ is independent of J, and every $\vec{b} \in V(k : \mathrm{pr}_i)$ has the property that $b_i = b_k = b_j$. Hence $V(k : \mathrm{pr}_i) \in D_{ik}^+ \cap D_{kj}^+ = (D_{ik} \cdot_\emptyset D_{kj})^+$. Thus $V \in C_{k,J}(D_{ik} \cdot_\emptyset D_{kj})^+$.

Now suppose $W \in C_{k,J}(D_{ik} \cdot_\emptyset D_{kj})^-$. Then $W(k : A) \in (D_{ik} \cdot_\emptyset D_{kj})^-$, which means $W(k : A) = W_1 \cup_\emptyset W_2$ for some $W_1 \in D_{ik}^-$ and $W_2 \in D_{kj}^-$. To show that $W \in D_{ij}^-$, suppose to the contrary that $W \notin D_{ij}^-$. Then there is an $\vec{a} \in W$ such that $a_i = a_j$. Let $\vec{b} = \vec{a}(k : a_i)$. Then $\vec{b} \in W(k : A)$, but $\vec{b} \notin W_1$ and $\vec{b} \notin W_2$, which contradicts the fact that $W(k : A) = W_1 \cup W_2$ is a disjoint cover. Therefore $W \in D_{ij}^-$.

Conversely, suppose $W \in D_{ij}^-$. Then every $\vec{a} \in W$ has the property that $a_i \neq a_j$. Let $W_1 = \{\, \vec{a} \in W(k : A) \mid a_i \neq a_k \,\}$ and $W_2 = \{\, \vec{a} \in W(k : A) \mid a_i = a_k \,\}$. Then we have $W(k : A) = W_1 \cup_\emptyset W_2$, $W_1 \in D_{ik}^-$, and $W_2 \in D_{kj}^-$. Hence $W(k : A) \in (D_{ik} \cdot_\emptyset D_{kj})^-$. Thus $W \in C_{k,J}(D_{ik} \cdot_\emptyset D_{kj})^-$. $\qquad\square$

2.4.7 Axiom C7

Axiom C7 is a complementation axiom, so it is not surprising that it fails in independence-friendly cylindric set algebras. Nevertheless, like the Boolean complementation axioms, axiom C7 fails in a nice way.

Proposition 2.55. *If X is a double suit and $i \neq j$, then*

$$C_{i,K}(D_{ij} \cdot_J X) \cdot_L C_{i,K}(D_{ij} \cdot_J X^\cup) \leq \Omega.$$

Proof. Suppose $V \in (C_{i,K}(D_{ij} \cdot_J X) \cdot_L C_{i,K}(D_{ij} \cdot_J X^\cup))^+$. Then $V \in C_{i,K}(D_{ij} \cdot_J X)^+$ and

$V \in C_{i,K}(D_{ij} \cdot_J X^\cup)^+$, which means that $V(i : f) \in (D_{ij} \cdot_J X)^+ = D_{ij}^+ \cap X^+$ and also

$V(i : g) \in (D_{ij} \cdot_J X^\cup)^+ = D_{ij}^+ \cap X^-$ for some $f \colon V \underset{K}{\to} A$ and $g \colon V \underset{K}{\to} A$. Hence, for every $\vec{a} \in V$,

$f(\vec{a}) = a_j = g(\vec{a})$, so $f = g$ and $V(i : f) = V(i : g)$. Thus $V(i : f) \in X^+ \cap X^- = \{\emptyset\}$, which

implies $V = \emptyset$. $\qquad\qquad\square$

Thus an IFG-formula of the form

$$\exists v_{i/K}(v_i = v_j \wedge_{/J} \phi) \wedge_{/L} \exists v_{i/K}(v_i = v_j \wedge_{/J} \sim\phi)$$

is never true. However, it might not be false. Let $\mathfrak{A}$ be the structure whose universe is $A = \{0, 1\}$

and in which every element is named by a constant, and consider the 2-dimensional IFG-cylindric

set algebra over $\mathfrak{A}$. Let $X = \|v_0 = 0\| +_N \|v_0 = 1\|$. Then

$$(C_{0,\emptyset}(D_{01} \cdot_\emptyset X) \cdot_\emptyset C_{0,\emptyset}(D_{01} \cdot_\emptyset X^\cup))^- \neq 0^-.$$

To see why, suppose to the contrary that $^N A \in (C_{0,\emptyset}(D_{01} \cdot_\emptyset X) \cdot_\emptyset C_{0,\emptyset}(D_{01} \cdot_\emptyset X^\cup))^-$. Then

by definition $^N A = W_1 \cup W_2$ for some $W_1 \in C_{0,\emptyset}(D_{01} \cdot_\emptyset X)^-$ and $W_2 \in C_{0,\emptyset}(D_{01} \cdot_\emptyset X^\cup)^-$.

Consider the valuations 00 and 11. Suppose for the sake of a contradiction that $00 \in W_1$. Then

$00 \in W_1(0 : A) \in (D_{01} \cdot_\emptyset X)^-$, which means $W_1(0 : A) = W_3 \cup W_4$ for some $W_3 \in D_{01}^-$ and

$W_4 \in X^- = \{\emptyset\}$. Hence $00 \in W_3 \in D_{01}^-$, which contradicts the definition of D_{01}^-. Thus $00 \notin W_1$.

Similarly, $11 \notin W_1$. Therefore, $\{00, 11\} \subseteq W_2$, which means $\{00, 01, 10, 11\} \subseteq W_2(0 : A) \in$

$(D_{01} \cdot_J X^\cup)^-$. Hence $\{00, 01, 10, 11\} = W_5 \cup W_6$ for some $W_5 \in D_{01}^-$ and $W_6 \in X^+$. Without

loss of generality we may assume $W_5 = \{01, 10\}$ and $W_6 = \{00, 11\}$, which contradicts the fact

that $\{00, 11\} \notin X^+$.

2.4.8 More on cylindrifications

Lemma 2.56. *Suppose X is rooted.*

(a) $C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X)^+ = \{\emptyset\}$ *if and only if $X^+ = \{\emptyset\}$.*

(b) $C_{0,N} \ldots C_{N-1,N}(X)^+ = \mathscr{P}(^NA)$ if and only if there is an $\vec{a} \in {}^NA$ such that $\{\vec{a}\} \in X^+$; otherwise $C_{0,N} \ldots C_{N-1,N}(X)^+ = \{\emptyset\}$.

(c) $C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X)^- = \mathscr{P}(^NA)$ if and only if $^NA \in X^-$.

(d) $C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X)^- = \{\emptyset\}$ if and only if $^NA \notin X^-$.

Proof. (a) If $X^+ = \{\emptyset\}$, then for any $V \subseteq {}^NA$ we have $V \in C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X)^+$ if and only if there exist $f_0, \ldots, f_{N-1}$ such that $V(0 : f_0) \ldots (N-1 : f_{n-1}) = \emptyset$ if and only if $V = \emptyset$. Hence $C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X)^+ = \{\emptyset\}$. Conversely, if there is a nonempty $V \in X^+$, then $V(0 : \mathrm{pr}_0) \ldots (N-1 : \mathrm{pr}_{N-1}) = V \in X^+$, so $V \in C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X)^+$.

(b) Suppose $\{\vec{a}\} \in X^+$, and let f_n be the function that takes the constant value a_n. Then for any $V \subseteq {}^NA$, $V(0 : f_0) \ldots (N-1, f_{N-1}) = \{\vec{a}\} \in X^+$. Hence $V \in C_{0,N} \ldots C_{N-1,N}(X)^+$. Thus $C_{0,N} \ldots C_{N-1,N}(X)^+ = \mathscr{P}(^NA)$. Conversely, suppose $C_{0,N} \ldots C_{N-1,N}(X)^+ = \mathscr{P}(^NA)$. Then there exist constant functions f_n such that $^NA(0 : f_0) \cdots (N-1 : f_{N-1}) \in X^+$. Let a_n be the constant value taken by f_n. Then $^NA(0 : f_0) \cdots (N-1 : f_{N-1}) = \{\vec{a}\}$.

If there is no $\vec{a} \in {}^NA$ such that $\{\vec{a}\} \in X^+$, then for any nonempty $V \subseteq {}^NA$ and constant functions $f_0, \ldots, f_{N-1}$, $V(0 : f_0) \ldots (N-1 : f_{N-1}) \notin X^+$. Hence $V \notin C_{0,N} \ldots C_{N-1,N}(X)^+$. Thus $C_{0,N} \ldots C_{N-1,N}(X)^+ = \{\emptyset\}$.

(c) and (d) Let $W \subseteq {}^NA$. By definition $W \in C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X)^-$ if and only if $W(0 : A) \ldots (N-1, A) \in X^-$. If $W = \emptyset$, then $W(0 : A) \ldots (N-1, A) = \emptyset$, otherwise $W(0 : A) \ldots (N-1, A) = {}^NA$. $\qquad\square$

Corollary 2.57. *If X is a pair of suits, then*

$$
C_{0,N} \ldots C_{N-1,N}(X) =
\begin{cases}
1 & \text{if } X \not\leq \Omega \text{ and } X \not\leq \mho, \\
\Omega & \text{if } 0 < X \leq \Omega, \\
\mho & \text{if } 0 < X \leq \mho, \\
0 & \text{if } X = 0.
\end{cases}
$$

Corollary 2.58. *If X is a double suit, then*

$$C_{0,N} \dots C_{N-1,N}(X) = \begin{cases} 1 & \text{if } X \not\leq \Omega, \\ \Omega & \text{if } 0 < X \leq \Omega, \\ 0 & \text{if } X = 0. \end{cases}$$

Proposition 2.59. *If X is a double suit, then*

$$C_{0,J_0} \dots C_{N-1,J_{N-1}}(X) = \begin{cases} 1 & \text{if } X \not\leq \Omega, \\ \Omega & \text{if } 0 < X \leq \Omega, \\ 0 & \text{if } X = 0. \end{cases}$$

Proof. By Proposition 2.48 and Proposition 2.49,

$$C_{0,N} \dots C_{N-1,N}(X) \leq C_{0,J_0} \dots C_{N-1,J_{N-1}}(X).$$

Thus $C_{0,J_0} \dots C_{N-1,J_{N-1}}(X) = 1$ if $X \not\leq \Omega$. If $0 < X \leq \Omega$, then $X^+ = \{\emptyset\}$ and ${}^N A \notin X^-$. Hence $C_{0,J_0} \dots C_{N-1,J_{N-1}}(X) = \Omega$. Finally, $C_{0,J_0} \dots C_{N-1,J_{N-1}}(0) = 0$ by Proposition 2.43. $\square$

In terms of IFG logic, what Proposition 2.59 says is that the semantic game associated with $\exists v_{0/J_0} \dots \exists v_{N-1/J_{N-1}} \phi$ is like the semantic game associated with ϕ except that Eloïse is allowed to choose the initial valuation. Since Eloïse can use constant functions to specify any initial valuation she wishes, the amount of information available to her is irrelevant.

Theorem 2.60. *If $\mathfrak{C}$ is a double-suited IFG_N-cylindric set algebra and $|\mathfrak{C}| > 2$, then $\Omega \in \mathfrak{C}$.*

Proof. Suppose $X \in \mathfrak{C} \setminus \{0,1\}$. Then ${}^N A \notin X^+ \cup X^-$, and there exists a nonempty team $V \in X^+ \cup X^-$. Hence $(X \cdot_N X^\cup)^+ = X^+ \cap X^- = \{\emptyset\}$, and $V \in X^+ \cup X^- = (X \cdot_N X^\cup)^-$, so $C_{0,N} \cdots C_{N-1,N}(X \cdot_N X^\cup) = \Omega$. $\square$

Chapter 3

Properties of Independence-Friendly Cylindric Set Algebras

3.1 The trivial algebra and IFG_0-cylindric set algebras

Thus far we have neglected the case when the base set $A = \emptyset$. If $A = \emptyset$ and $N > 0$, then $^NA = \emptyset$. Hence $0 = 1 = D_{ij} = \Omega = \mho$. Thus, by Proposition 2.23 and Proposition 2.45, $\{\Omega\}$ is an IFG_N-cylindric set algebra. We will refer to $\{\Omega\}$ as the trivial IFG_N-cylindric set algebra.

Proposition 3.1. *The trivial algebra $\{\Omega\}$ is the only double-suited IFG_N-cylindric set algebra with an empty base set. In fact, it is the only rooted IFG_N-cylindric set algebra with an empty base set. It is also the only IFG_N-cylindric set algebra with only one element.*

Proof. If $A = \emptyset$, then $\mathscr{P}(\mathscr{P}(^NA)) \times \mathscr{P}(\mathscr{P}(^NA)) = \{\langle \emptyset, \emptyset \rangle, \langle \emptyset, \{\emptyset\} \rangle, \langle \{\emptyset\}, \emptyset \rangle, \langle \{\emptyset\}, \{\emptyset\} \rangle\}$. The only rooted element is $\langle \{\emptyset\}, \{\emptyset\} \rangle$, which is a double suit.

Suppose $\mathfrak{C}$ is an IFG_N-cylindric set algebra such that $|\mathfrak{C}| = 1$. Then the base set A must be empty because otherwise $0 = \langle \{\emptyset\}, \mathscr{P}(^NA) \rangle \neq \langle \mathscr{P}(^NA), \{\emptyset\} \rangle = 1$. $\qquad\square$

If $N = 0$, then $^NA = \{\emptyset\}$. It is important to distinguish between the empty team $\emptyset$ and the team that includes only the empty valuation $\{\emptyset\}$. To avoid confusion let $\vec{a} = \emptyset$ be the empty valuation. Then $\mathscr{P}(\mathscr{P}(^NA)) \times \mathscr{P}(\mathscr{P}(^NA)) = \mathscr{P}(\mathscr{P}(\{\vec{a}\})) \times \mathscr{P}(\mathscr{P}(\{\vec{a}\}))$ has sixteen elements. The only suits are $\{\emptyset\}$ and $\mathscr{P}(\{\vec{a}\})$, so the only pairs of suits are $0 = \langle \{\emptyset\}, \mathscr{P}(\{\vec{a}\}) \rangle$, $\Omega = \langle \{\emptyset\}, \{\emptyset\} \rangle$, $\mho = \langle \mathscr{P}(\{\vec{a}\}), \mathscr{P}(\{\vec{a}\}) \rangle$, and $1 = \langle \mathscr{P}(\{\vec{a}\}), \{\emptyset\} \rangle$. Thus there are three suited

IFG$_0$-cylindric set algebras:
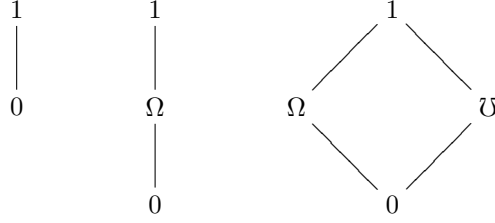
3.2   Changing dimension

In ordinary first-order logic there is only one formula $v_1 = v_3$. In IFG logic there are infinitely many versions of the formula $v_1 = v_3$, one with N variables for every $N > 3$. The meanings of these formulas are different because they are composed sets of teams with valuations of different lengths. It would be nice to know that all these different meanings of $v_1 = v_3$ are compatible.

Definition. If $M \leq N$ and $V \subseteq {}^N A$ define $\mathrm{pr}_M(V) = \{ \vec{a} \restriction M \mid \vec{a} \in V \}$.

Definition. Let $M \leq N$. If $X^* \subseteq \mathscr{P}({}^M A)$ is a set of teams, let the **extension** of X^* from M to N be

$$\mathrm{ext}_M^N(X^*) = \{ V \subseteq {}^N A \mid \mathrm{pr}_M(V) \in X^* \}.$$

If $X \in \mathscr{P}(\mathscr{P}({}^M A)) \times \mathscr{P}(\mathscr{P}({}^M A))$, let the **extension** of X from M to N be

$$\mathrm{ext}_M^N(X) = \langle \mathrm{ext}_M^N(X^+), \mathrm{ext}_M^N(X^-) \rangle.$$

If $\mathfrak{C}$ is a IFG$_M$-cylindric set algebra, let the **extension** of $\mathfrak{C}$ from M to N be

$$\mathrm{ext}_M^N(\mathfrak{C}) = \{ \mathrm{ext}_M^N(X) \mid X \in \mathfrak{C} \}.$$

Lemma 3.2. *If $X^* \subseteq \mathscr{P}({}^M A)$ is a suit, then $\mathrm{ext}_M^N(X^*)$ is a suit. If X is a double suit, then $\mathrm{ext}_M^N(X)$ is a double suit.*

Proof. Suppose X^* is a suit. Let $V' \subseteq V \in \mathrm{ext}_M^N(X^*)$. Then $\mathrm{pr}_M(V') \subseteq \mathrm{pr}_M(V) \in X^*$, so $\mathrm{pr}_M(V') \in X^*$. Hence $V' \in \mathrm{ext}_M^N(X^*)$.

Let X be a double suit. It suffices to show that $\text{ext}_M^N(X^+) \cap \text{ext}_M^N(X^-) = \{\emptyset\}$. First,

$\emptyset \in \text{ext}_M^N(X^+)$ because $\text{pr}_M(\emptyset) = \emptyset \in X^+$. Similarly, $\emptyset \in \text{ext}_M^N(X^-)$. Second, suppose $\emptyset \neq V \in$

$\text{ext}_M^N(X^+)$. Then $\emptyset \neq \text{pr}_M(V) \in X^+$. Hence $\text{pr}_M(V) \notin X^-$. Thus $V \notin \text{ext}_M^N(X^-)$. $\qquad\square$

Therefore the extension of a double-suited IFG_M-cylindric set algebra from M to N is a

double-suited IFG_N-cylindric set algebra, and the extension of a double-suited IFG_M-cylindric

set algebra is a double-suited IFG_N-cylindric set algebra.

Lemma 3.3. *If* $\text{ext}_M^N(X) = \text{ext}_M^N(Y)$, *then* $X = Y$.

Proof. Suppose $\text{ext}_M^N(X) = \text{ext}_M^N(Y)$. If $V \in X^+$, then $V' = \{\, \vec{a} \cup \vec{b} \in {}^N A \mid \vec{a} \in V,\ \vec{b} \in {}^{N \setminus M} A \,\} \in$

$\text{ext}_M^N(X^+)$, so $V' \in \text{ext}_M^N(Y^+)$. Hence $V = \text{pr}_M(V') \in Y^+$. Thus $X^+ \subseteq Y^+$, and by symmetry

$X^+ \supseteq Y^+$. Thus $X^+ = Y^+$. A similar argument shows that $X^- = Y^-$. $\qquad\square$

Lemma 3.4. *Let* $i, j, m < M \leq N$. *Let*

$$0 = \langle \mathscr{P}(\emptyset), \mathscr{P}({}^M A) \rangle, \qquad\qquad 0' = \langle \mathscr{P}(\emptyset), \mathscr{P}({}^N A) \rangle,$$

$$1 = \langle \mathscr{P}({}^M A), \mathscr{P}(\emptyset) \rangle, \qquad\qquad 1' = \langle \mathscr{P}({}^N A), \mathscr{P}(\emptyset) \rangle,$$

$$D_{ij} = \mathscr{P}(\{\, \vec{a} \in {}^M A \mid a_i = a_j \,\}), \qquad D'_{ij} = \mathscr{P}(\{\, \vec{a} \in {}^N A \mid a_i = a_j \,\}),$$

and let $X, Y \in \mathscr{P}(\mathscr{P}({}^M A)) \times \mathscr{P}(\mathscr{P}({}^M A))$.

(a) $\text{ext}_M^N(0) = 0'$.

(b) $\text{ext}_M^N(1) = 1'$.

(c) $\text{ext}_M^N(D_{ij}) = D'_{ij}$.

(d) $\text{ext}_M^N(X^\cup) = \text{ext}_M^N(X)^\cup$.

(e) $\text{ext}_M^N(X +_J Y) = \text{ext}_M^N(X) +_{J \cup (N \setminus M)} \text{ext}_M^N(Y)$.

(f) $\text{ext}_M^N(C_{m,J}(X)) = C_{m, J \cup (N \setminus M)}(\text{ext}_M^N(X))$.

Proof. (a) Observe that

$$V' \in \text{ext}_M^N(0)^+ \quad \text{iff} \quad \text{pr}_M(V') \in 0^+$$

$$\text{iff} \quad \text{pr}_M(V') = \emptyset$$

$$\text{iff} \quad V' = \emptyset$$

$$\text{iff} \quad V' \in (0')^+,$$

and

$$W' \in \text{ext}_M^N(0)^- \quad \text{iff} \quad \text{pr}_M(W') \in 0^-$$

$$\text{iff} \quad \text{pr}_M(W') \subseteq {}^M A$$

$$\text{iff} \quad W' \subseteq {}^N A$$

$$\text{iff} \quad W' \in (0')^-.$$

The proof of (b) is similar.

(c) Observe that

$$V' \in \text{ext}_M^N(D_{ij})^+ \quad \text{iff} \quad \text{pr}_M(V') \in D_{ij}^+$$

$$\text{iff} \quad a_i = a_j \text{ for all } \vec{a} \in \text{pr}_M(V')$$

$$\text{iff} \quad a_i = a_j \text{ for all } \vec{a} \in V'$$

$$\text{iff} \quad V' \in (D'_{ij})^+,$$

and

$$W' \in \text{ext}_M^N(D_{ij})^- \quad \text{iff} \quad \text{pr}_M(W') \in (D_{ij})^-$$

$$\text{iff} \quad a_i \neq a_j \text{ for all } \vec{a} \in \text{pr}_M(W')$$

$$\text{iff} \quad a_i \neq a_j \text{ for all } \vec{a} \in W'$$

$$\text{iff} \quad W' \in (D'_{ij})^-.$$

(d) $\text{ext}_M^N(X^\cup) = \langle \text{ext}_M^N(X^-), \text{ext}_M^N(X^+) \rangle = \text{ext}_M^N(X)^\cup.$

(e) Suppose $V' \in \text{ext}_M^N(X +_J Y)^+$. Then $\text{pr}_M(V') \in (X +_J Y)^+$. Let $V = \text{pr}_M(V')$.

Then $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$. Let $V_1' = \{ \vec{a} \in V' \mid \vec{a} \restriction M \in V_1 \}$ and

$V_2' = \{ \vec{b} \in V' \mid \vec{b} \restriction M \in V_2 \}$. To show that $V' = V_1' \cup_{J \cup (N \setminus M)} V_2'$, first note that if $\vec{a} \in V'$,

then $\vec{a} \restriction M \in V = V_1 \cup V_2$, so either $\vec{a} \restriction M \in V_1$ or $\vec{a} \restriction M \in V_2$. In the first case $\vec{a} \in V_1'$,

and in the second case $\vec{a} \in V_2'$. Hence $V' = V_1' \cup V_2'$. Second, note that $V_1' \cap V_2' = \emptyset$ because

$V_1 \cap V_2 = \emptyset$. Third, suppose $\vec{a} \in V_k'$ and $\vec{a} \approx_{J \cup (N \setminus M)} \vec{b}$. Then $\vec{a} \restriction M \in V_k$ and $\vec{a} \restriction M \approx_J \vec{b} \restriction M$, so

$\vec{b} \restriction M \in V_k$. Hence $\vec{b} \in V_k'$. Thus $V' = V_1' \cup_{J \cup (N \setminus M)} V_2'$. Finally, note that $V_1' \in \text{ext}_M^N(X)^+$ and

$V_2' \in \text{ext}_M^N(X)^-$. Therefore $V' \in (\text{ext}_M^N(X) +_{J \cup (N \setminus M)} \text{ext}_M^N(Y))^+$.

Conversely, suppose $V' \in (\text{ext}_M^N(X) +_{J \cup (N \setminus M)} \text{ext}_M^N(Y))^+$. Then $V' = V_1' \cup_{J \cup (N \setminus M)} V_2'$

for some $V_1' \in \text{ext}_M^N(X)^+$ and $V_2' \in \text{ext}_M^N(Y)^+$. Let $V = \text{pr}_M(V')$, $V_1 = \text{pr}_M(V_1')$, and $V_2 = \text{pr}_M(V_2')$. We wish to show that $V = V_1 \cup_J V_2$. First suppose $\vec{a} \in V$. Then $\vec{a} = \vec{c} \restriction M$ for some

$\vec{c} \in V' = V_1' \cup V_2'$. If $\vec{c} \in V_k'$, then $\vec{a} = \vec{c} \restriction M \in V_k$. Hence $V = V_1 \cup V_2$. Second, suppose $\vec{a} \in V_1$.

Then $\vec{a} = \vec{c} \restriction M$ for some $\vec{c} \in V_1'$. Suppose $\vec{a} = \vec{d} \restriction M$ for some $\vec{d} \in V'$. Then $\vec{c} \approx_{J \cup (N \setminus M)} \vec{d}$, so

$\vec{d} \notin V_2'$. Hence $\vec{a} \notin V_2$. Thus $V_1 \cap V_2 = \emptyset$. Third, suppose $\vec{a} \in V_k$ and $\vec{a} \approx_J \vec{b}$. Then $\vec{a} = \vec{c} \restriction M$

for some $\vec{c} \in V_k'$, $\vec{b} = \vec{d} \restriction M$ for some $\vec{d} \in V'$, and $\vec{c} \approx_{J \cup (N \setminus M)} \vec{d}$, so $\vec{d} \in V_k'$. Hence $\vec{b} \in V_k$.

Thus $V = V_1 \cup_J V_2$. Finally, note that $V_1 \in X^+$ and $V_2 \in Y^+$. Therefore $V \in (X +_J Y)^+$ and

$V' \in \text{ext}_M^N(X +_J Y)^+$.

On the falsity axis $W' \in \text{ext}_M^N(X +_J Y)^-$ if and only if $\text{pr}_M(W') \in (X +_J Y)^- = X^- \cap Y^-$

if and only if $W' \in \text{ext}_M^N(X)^- \cap \text{ext}_M^N(Y)^- = (\text{ext}_M^N(X) +_{J \cup (N \setminus M)} \text{ext}_M^N(Y))^-$.

(f) Suppose $V' \in \text{ext}_M^N(C_{m,J}(X))^+$. Then $V = \text{pr}_N(V') \in C_{m,J}(X)^+$, so there is a

function $f \colon V \underset{J}{\to} A$ such that $V(m : f) \in X^+$. Define a function $f' \colon V' \underset{J \cup (N \setminus M)}{\longrightarrow} A$ by $f'(\vec{a}) = f(\vec{a} \restriction M)$. Then $V(m : f) = \text{pr}_M(V'(m : f'))$ because for every $\vec{a} \in V'$ we have

$$(\vec{a} \restriction M)(m : f(\vec{a} \restriction M)) = \vec{a}(m : f'(\vec{a})) \restriction M.$$

Hence $V' \in \text{ext}_M^N(C_{m, J \cup (N \setminus M)}(X))^+$.

Conversely, suppose $V' \in C_{m, J \cup (N \setminus M)}(\text{ext}_M^N(X))^+$. Then there is an $f' \colon V' \underset{J \cup (N \setminus M)}{\longrightarrow} A$

such that $V'(m : f') \in \text{ext}_M^N(X)^+$. Let $V = \text{pr}_M(V')$, and define a function $f \colon V \underset{J}{\to} A$ by

$f(\vec{a} \upharpoonright M) = f'(\vec{a})$. The function f is well defined and independent of J because if $\vec{a} \upharpoonright M \approx_J \vec{b} \upharpoonright M$, then $\vec{a} \approx_{J \cup (N \setminus M)} \vec{b}$, so $f(\vec{a} \upharpoonright M) = f'(\vec{a}) = f'(\vec{b}) = f(\vec{b} \upharpoonright M)$. Again $V(m : f) = \mathrm{pr}_M(V'(m : f'))$, so $V(m : f) \in X^+$. Hence $V \in C_{m,J}(X)^+$. Thus $V' \in \mathrm{ext}_M^N(C_{m,J}(X))^+$.

On the falsity axis, $W' \in \mathrm{ext}_M^N(C_{m,J}(X))^-$ if and only if $W = \mathrm{pr}_M(W') \in C_{m,J}(X)^-$ if and only if $W(m : A) \in X^-$. Observe that $W(m : A) = \mathrm{pr}_M(W'(m : A))$ because for any $\vec{a} \in W'$ we have

$$(\vec{a} \upharpoonright M)(m : b) = \vec{a}(m : b) \upharpoonright M.$$

Hence $W(m : A) \in X^-$ if and only if $\mathrm{pr}_M(W'(m : A)) \in X^-$ if and only if $W'(m : A) \in \mathrm{ext}_M^N(X)^-$ if and only if $W' \in C_{m,J \cup (N \setminus M)}(X)^-$. $\square$

Definition. Let $M \leq N$. If $\mathfrak{C}$ is an IFG$_N$-cylindric set algebra, define the **reduct** of $\mathfrak{C}$ from N to M, denoted $\mathfrak{Rd}_M(\mathfrak{C})$, to be the reduct of $\mathfrak{C}$ to the signature

$$\langle 0, 1, D_{ij}, \cup, +_{J \cup (N \setminus M)}, \cdot_{J \cup (N \setminus M)}, C_{m,J \cup (N \setminus M)} \rangle$$

for $i, j, m < M$ and $J \subseteq M$.

Theorem 3.5. *If $\mathfrak{C}$ is an IFG$_M$-cylindric set algebra and $M \leq N$, then $\mathfrak{C} \cong \mathfrak{Rd}_M(\mathrm{ext}_M^N(\mathfrak{C}))$.*

Proof. Consider the function $\mathrm{ext}_M^N : \mathfrak{C} \to \mathfrak{Rd}_M(\mathrm{ext}_M^N(\mathfrak{C}))$. It is surjective by definition and injective by Lemma 3.3. It is a homomorphism by Lemma 3.4. $\square$

Definition. If $M \leq N$, and ϕ is an IFG$_M$-formula, the **extension** of ϕ from M to N is the IFG$_N$-formula $\mathrm{ext}_M^N(\phi)$ defined as follows.

- If ϕ is atomic, then $\mathrm{ext}_M^N(\phi)$ is the same formula as ϕ except that $\mathrm{ext}_M^N(\phi)$ has N variables instead of M variables.

- If ϕ is $\sim \psi$, then $\mathrm{ext}_M^N(\sim \psi)$ is $\sim \mathrm{ext}_M^N(\psi)$.

- If ϕ is $\psi_1 \vee_{/J} \psi_2$, then $\mathrm{ext}_M^N(\psi_1 \vee_{/J} \psi_2)$ is $\mathrm{ext}_M^N(\psi_1) \vee_{/J \cup (N \setminus M)} \mathrm{ext}_M^N(\psi_2)$.

- If ϕ is $\exists v_{n/J} \psi$, then $\mathrm{ext}_M^N(\exists v_{n/J} \psi)$ is $\exists v_{n/J \cup (N \setminus M)} \mathrm{ext}_M^N(\psi)$.

Lemma 3.6. *If ϕ is an IFG$_M$-formula, and $\mathfrak{A}$ is a suitable structure, then*

$$\mathrm{ext}_M^N(\|\phi\|_{\mathfrak{A}}) = \left\|\mathrm{ext}_M^N(\phi)\right\|_{\mathfrak{A}}.$$

Note that $\|\phi\|_{\mathfrak{A}} \in \mathscr{P}(\mathscr{P}(^MA)) \times \mathscr{P}(\mathscr{P}(^MA))$, while $\left\|\mathrm{ext}_M^N(\phi)\right\|_{\mathfrak{A}} \in \mathscr{P}(\mathscr{P}(^NA)) \times \mathscr{P}(\mathscr{P}(^NA))$.

Proof. Suppose ϕ is atomic. Then $V \in \mathrm{ext}_M^N(\|\phi\|)^+$ if and only if $\mathrm{pr}_M(V) \in \|\phi\|^+$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \models \phi[\vec{a} \upharpoonright M]$ if and only if for every $\vec{a} \in V$, $\mathfrak{A} \models \mathrm{ext}_M^N(\phi)[\vec{a}]$ if and only if $V \in \left\|\mathrm{ext}_M^N(\phi)\right\|^+$. Also $W \in \mathrm{ext}_M^N(\|\phi\|)^-$ if and only if $\mathrm{pr}_M(W) \in \|\phi\|^-$ if and only if for every $\vec{b} \in W$, $\mathfrak{A} \not\models \phi[\vec{b} \upharpoonright M]$ if and only if for every $\vec{b} \in W$, $\mathfrak{A} \not\models \mathrm{ext}_M^N(\phi)[\vec{b}]$ if and only if $W \in \left\|\mathrm{ext}_M^N(\phi)\right\|^-$.

Suppose ϕ is $\sim\psi$. Then $\mathrm{ext}_M^N(\|\sim\psi\|) = \mathrm{ext}_M^N(\|\psi\|^{\cup}) = \mathrm{ext}_M^N(\|\psi\|)^{\cup} = \left\|\mathrm{ext}_M^N(\psi)\right\|^{\cup} = \left\|\sim\mathrm{ext}_M^N(\psi)\right\| = \left\|\mathrm{ext}_M^N(\sim\psi)\right\|$.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. Then

$$\mathrm{ext}_M^N(\|\psi_1 \vee_{/J} \psi_2\|) = \mathrm{ext}_M^N(\|\psi_1\| +_J \|\psi_2\|)$$

$$= \mathrm{ext}_M^N(\|\psi_1\|) +_{J\cup(N\setminus M)} \mathrm{ext}_M^N(\|\psi_2\|)$$

$$= \left\|\mathrm{ext}_M^N(\psi_1)\right\| +_{J\cup(N\setminus M)} \left\|\mathrm{ext}_M^N(\psi_2)\right\|$$

$$= \left\|\mathrm{ext}_M^N(\psi_1) \vee_{/J\cup(N\setminus M)} \mathrm{ext}_M^N(\psi_2)\right\|$$

$$= \left\|\mathrm{ext}_M^N(\psi_1 \vee_{/J} \psi_2)\right\|.$$

Suppose ϕ is $\exists v_{n/J}\psi$. Then

$$\mathrm{ext}_M^N(\|\exists v_{n/J}\psi\|) = \mathrm{ext}_M^N(C_{n,J}(\|\psi\|))$$

$$= C_{n,J\cup(N\setminus M)}(\mathrm{ext}_M^N(\|\psi\|))$$

$$= C_{n,J\cup(N\setminus M)}(\left\|\mathrm{ext}_M^N(\psi)\right\|)$$

$$= \left\|\exists v_{n/J\cup(N\setminus M)} \mathrm{ext}_M^N(\psi)\right\|$$

$$= \left\|\mathrm{ext}_M^N(\exists v_{n/J}\psi)\right\|. \qquad \square$$

Theorem 3.7. *If $M \leq N$ then $\mathrm{ext}_M^N \colon \mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A}) \hookrightarrow \mathfrak{Rd}_M(\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A}))$.*

Proof. For every $\|\phi\| \in \mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A})$ we have $\mathrm{ext}_M^N(\|\phi\|) = \left\|\mathrm{ext}_M^N(\phi)\right\| \in \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$, so the function $\mathrm{ext}_M^N \colon \mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A}) \to \mathfrak{Rd}_M(\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A}))$ is well defined. It is injective by Lemma 3.3. It is a homomorphism by Lemma 3.4. $\qquad\square$

Corollary 3.8. $\mathrm{ext}_M^N(\mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A}))$ *is the subalgebra of* $\mathfrak{Rd}_M(\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A}))$ *generated by the meanings of atomic IFG$_N$-formulas in which only the variables* $v_0, \ldots, v_{M-1}$ *appear.*

Proof. Let $\mathrm{ext}_M^N(\|\phi\|) \in \mathrm{ext}_M^N(\mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A}))$. If ϕ is an atomic IFG$_M$-formula, then $\mathrm{ext}_M^N(\phi)$ is an atomic IFG$_N$-formula in which only the variables $v_0, \ldots, v_{M-1}$ appear.

Suppose ϕ is $\sim\psi$. Then $\mathrm{ext}_M^N(\sim\psi)$ is $\sim\mathrm{ext}_M^N(\psi)$, where $\mathrm{ext}_M^N(\psi)$ is generated by atomic IFG$_N$-formulas in which only the variables $v_0, \ldots, v_{M-1}$ appear.

Suppose ϕ is $\psi_1 \vee_{/J} \psi_2$. Then $\mathrm{ext}_M^N(\psi_1 \vee_{/J} \psi_2)$ is $\mathrm{ext}_M^N(\psi_1) \vee_{/J \cup (N \setminus M)} \mathrm{ext}_M^N(\psi_2)$, where $\mathrm{ext}_M^N(\psi_1)$ and $\mathrm{ext}_M^N(\psi_2)$ are both generated by atomic IFG$_N$-formulas in which only the variables $v_0, \ldots, v_{M-1}$ appear.

Suppose ϕ is $\exists v_{n/J}\psi$. Then $\mathrm{ext}_M^N(\exists v_{n/J}\psi)$ is $\exists v_{n/J \cup (N \setminus M)} \mathrm{ext}_M^N(\psi)$, where $\mathrm{ext}_M^N(\psi)$ is genereated by atomic IFG$_N$-formulas in which only the variables $v_0, \ldots, v_{M-1}$ appear. $\qquad\square$

3.3 Definitional and elementary equivalence

Definition. Let σ and τ be relational signatures. A σ-structure $\mathfrak{A}$ and a τ-structure $\mathfrak{B}$ are **definitionally equivalent** if $A = B$ and their fundamental relations are mutually definable:

(a) If R is an M-ary relation symbol in σ, then there is a formula $\psi_R \in \mathscr{L}_{\mathrm{IFG}}^\tau$ whose free variables are among $v_0, \ldots, v_{M-1}$ such that $\|R v_0 \ldots v_{M-1}\|_{\mathfrak{A}} = \|\psi_R\|_{\mathfrak{B}}$.

(b) If S is an M-ary relation symbol in τ, then there is a formula $\phi_S \in \mathscr{L}_{\mathrm{IFG}}^\sigma$ whose free variables are among $v_0, \ldots, v_{M-1}$ such that $\|S v_0 \ldots v_{M-1}\|_{\mathfrak{B}} = \|\phi_S\|_{\mathfrak{A}}$.

Theorem 3.9. *Let σ and τ be finite relational signatures, let $\mathfrak{A}$ be a σ-structure, and let $\mathfrak{B}$ be a τ-structure. Then $\mathfrak{A}$ and $\mathfrak{B}$ are definitionally equivalent if and only if $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A}) = \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{B})$ for all N sufficiently large.*

Proof. For every M-ary relation symbol R in σ, let ψ_R be a formula in the language of $\mathfrak{B}$ whose free variables are among $v_0, \dots, v_{M-1}$ such that $\|Rv_0 \dots v_{M-1}\|_{\mathfrak{A}} = \|\psi_R\|_{\mathfrak{B}}$. Fix N_1 greater than all the arities of the relation symbols in σ and greater than the number of variables that occur in any of the ψ_R. For any $N' \geq N_1$ define a function $\psi' \colon \mathscr{L}^{\sigma}_{\mathrm{IFG}_{N'}} \to \mathscr{L}^{\tau}_{\mathrm{IFG}_{N'}}$ by setting

$$\psi'(v_i = v_j) = (v_i = v_j),$$

$$\psi'(Rv_{i_0} \dots v_{i_{M-1}}) = \psi_R(v_{i_0}, \dots, v_{i_{M-1}}),$$

$$\psi'(\sim\chi) = \sim(\psi'(\chi)),$$

$$\psi'(\chi_1 \vee_{/J} \chi_2) = \psi'(\chi_1) \vee_{/J} \psi'(\chi_2),$$

$$\psi'(\exists v_{n/J}\chi) = \exists v_{n/J}\psi'(\chi).$$

We wish to show that $\|\phi\|_{\mathfrak{A}} = \|\psi'(\phi)\|_{\mathfrak{B}}$ for every $\phi \in \mathscr{L}^{\sigma}_{\mathrm{IFG}_N}$.

Suppose ϕ is $v_i = v_j$. Then $\|v_i = v_j\|_{\mathfrak{A}} = D_{ij} = \|v_i = v_j\|_{\mathfrak{B}}$.

Suppose ϕ is $Rv_{i_0} \dots v_{i_{M-1}}$. Then $\|Rv_{i_0} \dots v_{i_{M-1}}\| = \|\psi_R(v_{i_0}, \dots, v_{i_{M-1}})\|$ by hypothesis.

Suppose ϕ is $\sim\chi$. Then $\|\sim\chi\|_{\mathfrak{A}} = \|\chi\|^{\cup}_{\mathfrak{A}} = \|\psi'(\chi)\|^{\cup}_{\mathfrak{B}} = \|\sim\psi'(\chi)\|_{\mathfrak{B}} = \|\psi'(\sim\chi)\|$.

Suppose ϕ is $\chi_1 \vee_{/J} \chi_2$. Then

$$\|\chi_1 \vee_{/J} \chi_2\|_{\mathfrak{A}} = \|\chi_1\|_{\mathfrak{A}} +_J \|\chi_2\|_{\mathfrak{A}}$$

$$= \|\psi'(\chi_1)\|_{\mathfrak{B}} +_J \|\psi'(\chi_2)\|_{\mathfrak{B}}$$

$$= \|\psi'(\chi_1) \vee_{/J} \psi'(\chi_2)\|_{\mathfrak{B}}$$

$$= \|\psi'(\chi_1 \vee_{/J} \chi_2)\|_{\mathfrak{B}}.$$

Suppose ϕ is $\exists v_{n/J}\chi$. Then

$$\|\exists v_{n/J}\chi\|_{\mathfrak{A}} = C_{n,J}(\|\chi\|_{\mathfrak{A}})$$

$$= C_{n,J}(\|\psi'(\chi)\|_{\mathfrak{B}})$$

$$= \|\exists v_{n/J}\psi'(\chi)\|_{\mathfrak{B}}$$

$$= \|\psi'(\exists v_{n/J}\chi)\|_{\mathfrak{B}}.$$

Thus $\mathfrak{Cs}_{\text{IFG}_{N'}}(\mathfrak{A}) \subseteq \mathfrak{Cs}_{\text{IFG}_{N'}}(\mathfrak{B})$. Similarly fix N_2 greater than all the arities of the relation symbols in τ and greater than the number of variables that occur in any of the ϕ_S. Then for any $N' \geq N_2$, $\mathfrak{Cs}_{\text{IFG}_{N'}}(\mathfrak{A}) \supseteq \mathfrak{Cs}_{\text{IFG}_{N'}}(\mathfrak{B})$. Therefore for all $N \geq \max(N_1, N_2)$, $\mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{A}) = \mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{B})$.

Conversely, suppose $\mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{A}) = \mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{B})$ for all N sufficiently large. Choose an N greater than all the arities of the relation symbols in either σ or τ such that $\mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{A}) = \mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{B})$. Then for any M-ary relation symbol $R \in \sigma$ we have

$$\|Rv_0 \ldots v_{M-1}\|_{\mathfrak{A}} \in \mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{A}) = \mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{B}).$$

Let $\|Rv_0 \ldots v_{M-1}\|_{\mathfrak{A}} = \|\psi_R\|_{\mathfrak{B}}$. Observe that if $M \leq n < N$, then

$$\left\|\exists v_{n/\emptyset}\psi_R\right\|_{\mathfrak{B}} = C_{0,N} \ldots C_{N-1,N}(\|\psi_R\|_{\mathfrak{B}})$$
$$= C_{0,N} \ldots C_{N-1,N}(\|Rv_0 \ldots v_{M-1}\|_{\mathfrak{A}})$$
$$= \|Rv_0 \ldots v_{M-1}\|_{\mathfrak{A}} = \|\psi_R\|_{\mathfrak{B}}.$$

Hence we may assume that the free variables of ψ_R are among $v_0, \ldots, v_{M-1}$. By symmetry, $\mathfrak{A}$ is definitionally equivalent to $\mathfrak{B}$. $\square$

Definition. Similar structures $\mathfrak{A}$ and $\mathfrak{B}$ are **IFG$_N$-elementarily equivalent** if for every IFG$_N$-sentence ϕ, $\mathfrak{A} \models^+ \phi$ if and only if $\mathfrak{B} \models^+ \phi$. $\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent** if they are IFG$_N$-elementarily equivalent for all N.

Observe that if $\mathfrak{A}$ and $\mathfrak{B}$ are IFG-elementarily equivalent, then for every IFG-sentence ϕ, $\mathfrak{A} \models^- \phi$ if and only if $\mathfrak{A} \models^+ \sim\phi$ if and only if $\mathfrak{B} \models^+ \sim\phi$ if and only if $\mathfrak{B} \models^- \phi$.

Lemma 3.10. *If $\mathfrak{A}$ and $\mathfrak{B}$ are IFG$_N$-elementarily equivalent and $M < N$, then $\mathfrak{A}$ and $\mathfrak{B}$ are IFG$_M$-elementarily equivalent.*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are IFG$_N$-elementarily equivalent. Let ϕ be and IFG$_M$-sentence. Then $\mathfrak{A} \models^+ \phi[^M A]$ if and only if $\mathfrak{A} \models^+ \phi[^N A]$ if and only if $\mathfrak{B} \models^+ \phi[^N A]$ if and only if $\mathfrak{B} \models^+ \phi[^M A]$. $\square$

Theorem 3.11. *If $\mathfrak{A}$ and $\mathfrak{B}$ are σ-structures, and the relation $\{\,\langle\|\phi\|_{\mathfrak{A}},\|\phi\|_{\mathfrak{B}}\rangle\mid\phi\in\mathscr{L}^{\sigma}_{\mathrm{IFG}_N}\,\}$ is an isomorphism, then $\mathfrak{A}$ and $\mathfrak{B}$ are IFG_N-elementarily equivalent.*

Proof. Suppose $\{\,\langle\|\phi\|_{\mathfrak{A}},\|\phi\|_{\mathfrak{B}}\rangle\mid\phi\in\mathscr{L}^{\sigma}_{\mathrm{IFG}_N}\,\}$ is an isomorphism. Then for every IFG_N-sentence ϕ we have $\mathfrak{A}\models^{+}\phi$ if and only if $\|\phi\|_{\mathfrak{A}}=1$ if and only if $\|\phi\|_{\mathfrak{B}}=1$ if and only if $\mathfrak{B}\models^{+}\phi$. $\square$

Corollary 3.12. *If $\mathfrak{A}$ and $\mathfrak{B}$ are σ-structures, and for all N the relation $\{\,\langle\|\phi\|_{\mathfrak{A}},\|\phi\|_{\mathfrak{B}}\rangle\mid\phi\in\mathscr{L}^{\sigma}_{\mathrm{IFG}_N}\,\}$ is an isomorphism, then $\mathfrak{A}$ and $\mathfrak{B}$ are IFG-elementarily equivalent.*

Definition. Given two structures $\mathfrak{A}$ and $\mathfrak{B}$, define $\mathfrak{Cs}_{\mathrm{IFG}}(\mathfrak{A})\cong\mathfrak{Cs}_{\mathrm{IFG}}(\mathfrak{B})$ to mean that for each N there is an isomorphism $f_N\colon\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})\to\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{B})$, and that whenever $M\le N$ and $X\in\mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A})$ we have $f_N(\mathrm{ext}^N_M(X))=\mathrm{ext}^N_M(f_M(X))$.

Theorem 3.13. *Let σ and τ be finite relational signatures, let $\mathfrak{A}$ be a σ-structure, and let $\mathfrak{B}$ be a τ-structure. If $\mathfrak{Cs}_{\mathrm{IFG}}(\mathfrak{A})\cong\mathfrak{Cs}_{\mathrm{IFG}}(\mathfrak{B})$, then $\mathfrak{A}$ is IFG-elementarily equivalent to a σ-structure $\mathfrak{B}'$ that is definitionally equivalent to $\mathfrak{B}$.*

Proof. Suppose that for each N, $f_N\colon\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})\to\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{B})$ is an isomorphism, and that whenever $M\le N$ and $X\in\mathfrak{Cs}_{\mathrm{IFG}_M}(\mathfrak{A})$ we have $f_N(\mathrm{ext}^N_M(X))=\mathrm{ext}^N_M(f_M(X))$. For every M-ary relation symbol R in σ, let

$$R^{\mathfrak{B}'}=\{\,\vec{b}\in{}^{M}B\mid\{\vec{b}\}\in f_M(\|Rv_0\ldots v_{M-1}\|_{\mathfrak{A}})\,\}.$$

Then $f_M(\|Rv_0\ldots v_{M-1}\|_{\mathfrak{A}})=\|Rv_0\ldots v_{M-1}\|_{\mathfrak{B}'}$ because f_M sends perfect elements to perfect elements.

Let N be greater than the maximum arity of all the relation symbols in σ. We will prove by induction that $f_N(\|\phi\|_{\mathfrak{A}})=\|\phi\|_{\mathfrak{B}'}$ for all $\phi\in\mathscr{L}^{\sigma}_{\mathrm{IFG}_N}$. For any M-ary relation symbol R in σ we have

$$\begin{aligned}
f_N\big(\big\|\mathrm{ext}^N_M(Rv_0\ldots v_{M-1})\big\|_{\mathfrak{A}}\big)&=f_N\big(\mathrm{ext}^N_M(\|Rv_0\ldots v_{M-1}\|_{\mathfrak{A}})\big)\\
&=\mathrm{ext}^N_M\big(f_M(\|Rv_0\ldots v_{M-1}\|_{\mathfrak{A}})\big)\\
&=\mathrm{ext}^N_M\big(\|Rv_0\ldots v_{M-1}\|_{\mathfrak{B}'}\big).
\end{aligned}$$

Also, $f_N(\|v_i = v_j\|_\mathfrak{A}) = f_N(D_{ij}) = D_{ij} = \|v_i = v_j\|_{\mathfrak{B}'}$. If ϕ is $\sim\psi$, then $f_N(\|\sim\psi\|_\mathfrak{A}) = f_N(\|\psi\|_\mathfrak{A}^{\cup}) = f_N(\|\psi\|_\mathfrak{A})^{\cup} = \|\psi\|_{\mathfrak{B}'}^{\cup} = \|\sim\psi\|_{\mathfrak{B}'}$. If ϕ is $\psi_1 \vee_{/J} \psi_2$, then $f_N(\|\psi_1 \vee_{/J} \psi_2\|_\mathfrak{A}) = f_N(\|\psi_1\|_\mathfrak{A} +_J \|\psi_2\|_\mathfrak{A}) = f_N(\|\psi_1\|_\mathfrak{A}) +_J f_N(\|\psi_2\|_\mathfrak{A}) = \|\psi\|_{\mathfrak{B}'} +_J \|\psi\|_{\mathfrak{B}'} = \|\psi_1 \vee_{/J} \psi_2\|_{\mathfrak{B}'}$. Finally, if ϕ is $\exists v_{n/J}\psi$, then $f_N(\|\exists v_{n/J}\psi\|_\mathfrak{A}) = f_N(C_{n,J}(\|\psi\|_\mathfrak{A})) = C_{n,J}(f_N(\|\psi\|_\mathfrak{A})) = C_{n,J}(\|\psi\|_{\mathfrak{B}'}) = \|\exists v_{n/J}\psi\|_{\mathfrak{B}'}$.

In particular, if ϕ is an IFG$_N$-sentence $\mathfrak{A} \models^+ \phi$ if and only if $\|\phi\|_\mathfrak{A} = 1$ if and only if $\|\phi\|_{\mathfrak{B}'} = f_n(\|\phi\|_\mathfrak{A}) = 1$ if and only if $\mathfrak{B}' \models^+ \phi$. Therefore $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}'$.

To show that $\mathfrak{B}'$ definitionally equivalent to $\mathfrak{B}$, observe that for all N, $\mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{B}') = \mathfrak{Cs}_{\text{IFG}_N}(\mathfrak{B})$. $\square$

3.4   Counting suits and double suits

Definition. A set of teams $\{V_\alpha \mid \alpha < \kappa\} \subseteq \mathscr{P}(^N A)$ is called an **antichain** if for all $\alpha \neq \beta$ we have $V_\alpha \not\subseteq V_\beta$ and $V_\beta \not\subseteq V_\alpha$.

Lemma 3.14. *If $X^* \subseteq \mathscr{P}(^N A)$ is a suit with a finite base set A, then there is a finite antichain $\{V_0, \ldots, V_{k-1}\}$ such that $X^* = \mathscr{P}(V_0) \cup \cdots \cup \mathscr{P}(V_{k-1})$.*

Proof. Let X^* be partially ordered by $\subseteq$, and let $\{V_0, \ldots, V_{k-1}\}$ be the set of maximal teams in X^*. Since X^* is downward closed and every team in X^* is contained in a maximal team $X^* = \mathscr{P}(V_0) \cup \cdots \cup \mathscr{P}(V_{k-1})$. The set $\{V_0, \ldots, V_{k-1}\}$ is antichain because each of its elements is maximal. $\square$

Not every suit can be expressed as the collection of subsets of maximal teams. For example, if A is an infinite set, then the set of all finite subsets of A is a one-dimensional suit that has no maximal teams. We call a suit X^* **bounded** if every team in X^* is contained in some maximal team. A pair of suits X is **bounded** if both X^+ and X^- are.

Lemma 3.15. *If X^* is a bounded suit then there is a unique antichain of teams $\{U_\alpha \mid \alpha < \kappa\}$ such that $X^* = \bigcup\{\mathscr{P}(U_\alpha) \mid \alpha < \kappa\}$.*

Proof. Let $\{U_\alpha \mid \alpha < \kappa\}$ be the set of maximal teams in X^*. As in the finite case, $X^* = \bigcup\{\mathscr{P}(U_\alpha) \mid \alpha < \kappa\}$ because X^* is downward closed and every team in X^* is contained in some maximal team. Also, $\{U_\alpha \mid \alpha < \kappa\}$ is an antichain because each of its elements is maximal.

Suppose $\{V_\beta \mid \beta < \lambda\}$ is another antichain such that $X^* = \bigcup\{\mathscr{P}(V_\beta) \mid \beta < \lambda\}$. Then for every $\beta < \lambda$, $V_\beta \in \bigcup\{\mathscr{P}(U_\alpha) \mid \alpha < \kappa\}$, hence there is an $\alpha < \kappa$ such that $V_\beta \subseteq U_\alpha$. But V_β is maximal, so we must have $V_\beta = U_\alpha$. By symmetry, every U_α is equal to some V_β as well, so we have $\{U_\alpha \mid \alpha < \kappa\} = \{V_\beta \mid \beta < \lambda\}$. $\qquad\square$

We call a suit X^* **finitely bounded** if there is a finite antichain $\{U_0, \ldots, U_{k-1}\}$ such that $X^* = \mathscr{P}(U_0) \cup \ldots \cup \mathscr{P}(U_{k-1})$. A pair of suits X is **finitely bounded** if both X^+ and X^- are.

Lemma 3.16. *Let X and Y be bounded double suits, where*

$$X = \left\langle \bigcup\{\mathscr{P}(U_\alpha) \mid \alpha < \kappa\}, \ \bigcup\{\mathscr{P}(W_\gamma) \mid \gamma < \mu\}\right\rangle,$$
$$Y = \left\langle \bigcup\{\mathscr{P}(V_\beta) \mid \beta < \lambda\}, \ \bigcup\{\mathscr{P}(W'_\gamma) \mid \delta < \nu\}\right\rangle.$$

Then

$$(X +_\emptyset Y)^+ = \bigcup\{\mathscr{P}(U_\alpha \cup V_\beta) \mid \alpha < \kappa, \ \beta < \lambda\},$$
$$(X +_\emptyset Y)^- = \bigcup\{\mathscr{P}(W_\gamma \cap W'_\delta) \mid \gamma < \mu, \ \delta < \nu\}.$$

Furthermore, if X and Y are both finitely bounded then so is $X +_\emptyset Y$.

Proof. Suppose $U \in (X +_\emptyset Y)^+$. Then there exist $U' \in X^+$ and $V' \in Y^+$ such that $U = U' \cup V'$. Then for some $\alpha < \kappa$ and $\beta < \lambda$, $U' \subseteq U_\alpha$ and $V' \subseteq V_\beta$. Hence $U \in \mathscr{P}(U_\alpha \cup V_\beta)$. Conversely, suppose $U \in \mathscr{P}(U_\alpha \cup V_\beta)$ for some $\alpha < \kappa$ and $\beta < \lambda$. Then $U \subseteq U_\alpha \cup V_\beta$, which implies $U = (U \cap U_\alpha) \cup (U \cap V_\beta)$, where $U \cap U_\alpha \in X^+$ and $U \cap V_\beta \in Y^+$. Hence $U \in (X +_\emptyset Y)^+$.

Observe that

$$(X +_\emptyset Y)^- = X^- \cap Y^-$$

$$= \bigcup \{\, \mathscr{P}(W_\gamma) \mid \gamma < \mu \,\} \cap \bigcup \{\, \mathscr{P}(W'_\delta) \mid \delta < \nu \,\}$$

$$= \bigcup \{\, \mathscr{P}(W_\gamma) \cap \mathscr{P}(W'_\delta) \mid \gamma < \mu,\ \delta < \nu \,\}$$

$$= \bigcup \{\, \mathscr{P}(W_\gamma \cap W'_\delta) \mid \gamma < \mu,\ \delta < \nu \,\}.$$

The sets $\{\, U_\alpha \cup V_\beta \mid \alpha < \kappa,\ \beta < \lambda \,\}$ and $\{\, W_\gamma \cap W'_\delta \mid \gamma < \mu,\ \delta < \nu \,\}$ are most likely not antichains. However, if κ, λ, μ, and ν are all finite, then $\{\, U_\alpha \cup V_\beta \mid \alpha < \kappa,\ \beta < \lambda \,\}$ and $\{\, W_\gamma \cap W'_\delta \mid \gamma < \mu,\ \delta < \nu \,\}$ are finite sets partially ordered by $\subseteq$. In both cases, the subset of maximal teams is a finite antichain, and the collection of subsets of those maximal teams is equal to $\bigcup \{\, \mathscr{P}(U_\alpha \cup V_\beta) \mid \alpha < \kappa,\ \beta < \lambda \,\}$ and $\bigcup \{\, \mathscr{P}(W_\gamma \cap W'_\delta) \mid \gamma < \mu,\ \delta < \nu \,\}$, respectively. To see why, let $\{\, U_{\alpha_0} \cup V_{\beta_0}, \ldots, U_{\alpha_{k-1}} \cup V_{\beta_{k-1}} \,\}$ be the set of maximal teams in $\{\, U_\alpha \cup V_\beta \mid \alpha < \kappa,\ \beta < \lambda \,\}$. It is an antichain because each of its elements is maximal. Also $\mathscr{P}(U_{\alpha_0} \cup V_{\beta_0}) \cup \cdots \cup \mathscr{P}(U_{\alpha_{k-1}} \cup V_{\beta_{k-1}}) = \bigcup \{\, \mathscr{P}(U_\alpha \cup V_\beta) \mid \alpha < \kappa,\ \beta < \lambda \,\}$. Similarly for $\{\, W_\gamma \cap W'_\delta \mid \gamma < \mu,\ \delta < \nu \,\}$. $\qquad\square$

Lemma 3.17. *If X and Y are finitely-bounded, N-dimensional double suits, then so is $X +_N Y$.*

Proof. Let $\{U_0, \ldots, U_{k-1}\}$, $\{V_0, \ldots, V_{\ell-1}\}$, $\{W_0, \ldots, W_{m-1}\}$, and $\{W'_0, \ldots, W'_{n-1}\}$ be finite antichains of teams such that

$$X = \langle \mathscr{P}(U_0) \cup \cdots \cup \mathscr{P}(U_{k-1}),\ \mathscr{P}(W_0) \cup \cdots \cup \mathscr{P}(W_{m-1}) \rangle,$$

$$Y = \langle \mathscr{P}(V_0) \cup \cdots \cup \mathscr{P}(V_{\ell-1}),\ \mathscr{P}(W'_0) \cup \cdots \cup \mathscr{P}(W'_{n-1}) \rangle.$$

Then

$$(X +_N Y)^+ = \mathscr{P}(U_0) \cup \cdots \cup \mathscr{P}(U_{k-1}) \cup \mathscr{P}(V_0) \cup \cdots \cup \mathscr{P}(V_{\ell-1}).$$

The set $\{U_0, \ldots, U_{k-1}, V_0, \ldots, V_{\ell-1}\}$ may not be an antichain, be we can always throw out the

non-maximal teams without affecting the collection of subsets. Similarly,

$$(X +_N Y)^- = (\mathscr{P}(W_0) \cup \cdots \cup \mathscr{P}(W_{m-1})) \cap (\mathscr{P}(W'_0) \cup \cdots \cup \mathscr{P}(W'_{n-1}))$$

$$= \bigcup \{ \mathscr{P}(W_i) \cap \mathscr{P}(W'_j) \mid i < k, \ j < \ell \}$$

$$= \bigcup \{ \mathscr{P}(W_i \cap W'_j) \mid i < k, \ j < \ell \}.$$

The set $\{ W_i \cap W'_j \mid i < k, \ j < \ell \}$ may not be an antichain, but again we can throw out the

non-maximal teams without affecting $\bigcup \{ \mathscr{P}(W_i \cap W'_j) \mid i < k, \ j < \ell \}$. $\qquad\square$

Lemma 3.18. *Every element of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})$ is finitely bounded.*

Proof. $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})$ is generated by perfect elements which are all finitely bounded. Suppose

$X, Y \in \mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})$ are finitely bounded. Then $X^{\cup}$ is finitely bounded, and $X +_\emptyset Y$ is finitely

bounded by Lemma 3.16. Also, $X +_{\{0\}} Y$ is finitely bounded by the fact that $(X +_{\{0\}} Y)^+ =$

$X^+ \cup Y^+$ and $(X +_{\{0\}} Y)^- = X^- \cap Y^-$. Finally, $C_{0,J}(X)$ is finitely bounded because $C_{0,J}(X) \in$

$\{0, \Omega, 1\}$. $\qquad\square$

If $\mathfrak{A}$ is a finite structure, then the number of double suits in $\mathscr{P}(\mathscr{P}(^N A)) \times \mathscr{P}(\mathscr{P}(^N A))$

gives an upper bound for the size of $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$. Let $|A| = m$, let $f(m)$ be the number of suits

in $\mathscr{P}(\mathscr{P}(A))$, and let $g(m)$ be the number of double suits in $\mathscr{P}(\mathscr{P}(A)) \times \mathscr{P}(\mathscr{P}(A))$. Cameron

and Hodges [5] calculate $f(m)$ and $g(m)$ for small values of m (Table 3.1). They remark that

"one can think of the ratio of $g(m)$ to 2^m as measuring the expressive strength of [IFG logic]

compared with ordinary first-order logic—always bearing in mind that [IFG logic] may have a

rather unorthodox notion of what is worth expressing." Cameron and Hodges also prove that

given any finite set A, there is a structure with universe A such that the universe of $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$

is exactly the set of all double suits in $\mathscr{P}(\mathscr{P}(^N A)) \times \mathscr{P}(\mathscr{P}(^N A))$.

Proposition 3.19. *Let $\mathfrak{A}$ be a finite structure in which every element is named by a constant.*

Then $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A}) = \mathrm{DSuit}_N(A)$.

Proof. For every $\vec{a} \in {}^N A$, let $\phi_{\vec{a}}$ be the formula $v_0 = a_0 \wedge_{/\emptyset} \cdots \wedge_{/\emptyset} v_{N-1} = a_{N-1}$. For every

m	2^m	$f(m)$	$g(m)$
0	1	1	1
1	2	2	3
2	4	5	11
3	8	19	55
4	16	167	489
5	32	7,580	17,279
6	64	7,828,353	15,758,603
7	128	2,414,682,040,997	4,829,474,397,415
8	256	56,130,437,228,687,557,907,787	112,260,874,496,010,913,723,317

Table 3.1: Counting suits and double suits

$V \subseteq {}^N A$, let ϕ_V be the formula $\bigvee_{/\emptyset} \{ \phi_{\vec{a}} \mid \vec{a} \in V \}$. We claim that

$$\|\phi_{\vec{a}}\| = \langle \mathscr{P}(\{\vec{a}\}), \mathscr{P}({}^N A \setminus \{\vec{a}\}) \rangle,$$

$$\|\phi_V\| = \langle \mathscr{P}(V), \mathscr{P}({}^N A \setminus V) \rangle.$$

First, observe that $\{\vec{a}\} \in \|v_0 = a_0\|^+ \cap \cdots \cap \|v_{N-1} = a_{N-1}\|^+ = \|\phi_{\vec{a}}\|^+$. Next, suppose $\vec{a} \neq \vec{b} \in V$.

Then for some $i < N$, $a_i \neq b_i$, so $V \notin \|v_i = a_i\|^+ \subseteq \|\phi_{\vec{a}}\|^+$. Thus $\|\phi_{\vec{a}}\|^+ = \mathscr{P}(\{\vec{a}\})$. Therefore,

since $\phi_{\vec{a}}$ is a perfect formula, we know that $\|\phi_{\vec{a}}\| = \langle \mathscr{P}(\{\vec{a}\}), \mathscr{P}({}^N A \setminus \{\vec{a}\}) \rangle$. Now let $V \subseteq {}^N A$.

By Lemma 3.16,

$$\|\phi_V\|^+ = \sum_{\emptyset} \{ \|\phi_{\vec{a}}\|^+ \mid \vec{a} \in V \} = \sum_{\emptyset} \{ \mathscr{P}(\{\vec{a}\}) \mid \vec{a} \in V \} = \mathscr{P}(V).$$

Since ϕ_V is a perfect formula, $\|\phi_V\| = \langle \mathscr{P}(V), \mathscr{P}({}^N A \setminus V) \rangle$.

Let X be a double suit, and let $X^+ = \mathscr{P}(V_0) \cup \cdots \cup \mathscr{P}(V_{k-1})$. Let ϕ be the formula

$$\phi_{V_1} \bigvee_{/N} \cdots \bigvee_{/N} \phi_{V_{k-1}}.$$

Then $\|\phi\|^+ = X^+$ and $\|\phi\|^- = \mathscr{P}({}^N A \setminus V_0) \cap \cdots \cap \mathscr{P}({}^N A \setminus V_{k-1})$. Similarly, there is a formula

ψ such that $\|\psi\|^+ = (X^\cup)^+ = X^-$. Let

$$X' = \|\phi\| +_N \Omega = \langle X^+, \{\emptyset\} \rangle,$$

$$Y = \|\psi\| +_N \Omega = \langle X^-, \{\emptyset\} \rangle,$$

$$Z = \|\phi_V\| = \langle \mathscr{P}(V), \mathscr{P}({}^N A \setminus V) \rangle,$$

where $V = V_0 \cup \cdots \cup V_{k-1}$. Then

$$X' \cdot_N (Y^\cup +_N Z) = \langle X^+ \cap \mathscr{P}(V),\ X^- \cap \mathscr{P}(^N A \setminus V) \rangle = \langle X^+, X^- \rangle = X.$$

Therefore $X \in \mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$. $\qquad\square$

3.5   Simplicity

Every finite-dimensional cylindric set algebra is simple (Theorem 3.1.70(iv) in [12]). Also, given any structure $\mathfrak{A}$, the ω-dimensional cylindric set algebra $\mathfrak{Cs}(\mathfrak{A})$ is simple (Theorem 8.1 in [20]). The following theorem, although easy to prove, does not appear in the literature.

Theorem 3.20. *If $\mathfrak{C}$ is an N-dimensional cylindric set algebra, then there is a structure $\mathfrak{A}$ such that $\mathfrak{C} = \mathfrak{Cs}_N(\mathfrak{A})$.*

Proof. Let $\mathfrak{C}$ be an N-dimensional cylindric set algebra. For every $x \in \mathfrak{C}$, let R_x be an N-ary relation symbol, and let $\mathfrak{A}$ be the structure whose universe A is the base set of $\mathfrak{C}$ and in which $R_x^{\mathfrak{A}} = x$. $\qquad\square$

It is natural to ask which of these results carry over to independence-friendly cylindric set algebras. It is unknown at this time whether every finite-dimensional independence-friendly cylindric set algebra over a structure is simple, but we will give an example of an IFG_1-cylindric set algebra that is neither simple, nor the IFG_1-cylindric set algebra over a structure.

Let **2** be the structure with universe $\{0, 1\}$ in which both elements are named by constant symbols. Then the universe of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2}) = \mathrm{DSuit}_1(\{0, 1\})$. The distributive lattice structure of

$\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ is shown below.

$$\langle \mathscr{P}(\{0,1\}),\ \mathscr{P}(\emptyset)\rangle$$

$$\langle \mathscr{P}(\{0\}) \cup \mathscr{P}(\{1\}),\ \mathscr{P}(\emptyset)\rangle$$

$$\langle \mathscr{P}(\{0\}),\ \mathscr{P}(\emptyset)\rangle \qquad\qquad \langle \mathscr{P}(\{1\}),\ \mathscr{P}(\emptyset)\rangle$$

$$\langle \mathscr{P}(\{0\}),\ \mathscr{P}(\{1\})\rangle \qquad \langle \mathscr{P}(\emptyset),\ \mathscr{P}(\emptyset)\rangle \qquad \langle \mathscr{P}(\{1\}),\ \mathscr{P}(\{0\})\rangle$$

$$\langle \mathscr{P}(\emptyset),\ \mathscr{P}(\{1\})\rangle \qquad\qquad \langle \mathscr{P}(\emptyset),\ \mathscr{P}(\{0\})\rangle$$

$$\langle \mathscr{P}(\emptyset),\ \mathscr{P}(\{0\}) \cup \mathscr{P}(\{1\})\rangle$$

$$\langle \mathscr{P}(\emptyset),\ \mathscr{P}(\{0,1\})\rangle$$
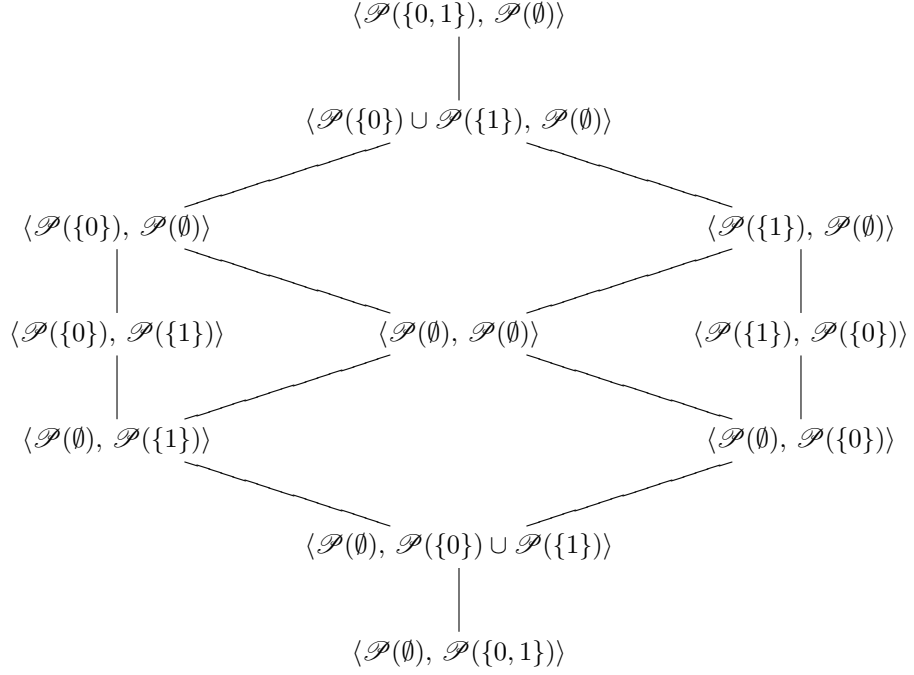
If we let $A = \langle \mathscr{P}(\{0\}) \cup \mathscr{P}(\{1\}),\ \mathscr{P}(\emptyset)\rangle$, $B = \langle \mathscr{P}(\{0\}),\ \mathscr{P}(\emptyset)\rangle$, and $C = \langle \mathscr{P}(\{1\}),\ \mathscr{P}(\emptyset)\rangle$, we

can express the same lattice by:

$$1$$

$$A$$

$$B \qquad\qquad C$$

$$\|v_0 = 0\| \qquad \Omega \qquad \|v_0 = 1\|$$

$$C^{\cup} \qquad\qquad B^{\cup}$$

$$A^{\cup}$$

$$0$$
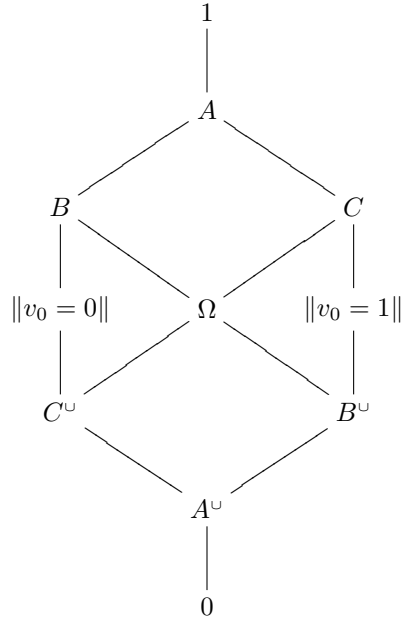
Lemma 3.21. *Let $\equiv$ be a congruence on any double-suited IFG_N-cylindric set algebra. If 0, Ω*

(if present), or 1 are congruent to any other element, then $\equiv$ is the total congruence.

Proof. First we will show that if $0 \equiv 1$, then $\equiv$ is the total congruence. If $0 \equiv 1$, then for every X we have $X = X +_\emptyset 0 \equiv X +_\emptyset 1 = 1$. Hence $\equiv$ is the total congruence. Next we will show that if $0 \equiv \Omega$ or $\Omega \equiv 1$, then $\equiv$ is the total congruence. If $0 \equiv \Omega$, then $1 = 0^\cup \equiv \Omega^\cup = \Omega$. Hence $0 \equiv 1$. Similarly if $\Omega \equiv 1$.

Now suppose $X \neq 0$ and $X \equiv 0$. Then $C_{0,N} \ldots C_{N-1,N}(X) = 1$ or $C_{0,N} \ldots C_{N-1,N}(X) = \Omega$. In the first case, $0 = C_{0,N} \ldots C_{N-1,N}(0) \equiv C_{0,N} \ldots C_{N-1,N}(X) = 1$. In the second case, $0 = C_{0,N} \ldots C_{N-1,N}(0) \equiv C_{0,N} \ldots C_{N-1,N}(X) = \Omega$.

If $X \neq 1$ and $X \equiv 1$, then $X^\cup \neq 0$ and $X^\cup \equiv 0$.

Finally, if $X \neq \Omega$ and $X \equiv \Omega$, then either $X \not\leq \Omega$ or $X^\cup \not\leq \Omega$. Hence either

$$1 = C_{0,N} \ldots C_{N-1,N}(X) \equiv C_{0,N} \ldots C_{N-1,N}(\Omega) = \Omega.$$

or

$$1 = C_{0,N} \ldots C_{N-1,N}(X^\cup) \equiv C_{0,N} \ldots C_{N-1,N}(\Omega) = \Omega. \qquad \square$$

Lemma 3.22. *Let $\equiv$ be a congruence on any double-suited IFG$_N$-cylindric set algebra that includes Ω. If $X < \Omega < Y$ and $X \equiv Y$, then $\equiv$ is the total congruence.*

Proof. If $X < \Omega < Y$ and $X \equiv Y$, then $\Omega = X +_N \Omega \equiv Y +_N \Omega = Y$, so by the previous lemma $\equiv$ is the total congruence. $\qquad \square$

Lemma 3.23. *Let $\equiv$ be a nontrivial congruence on any double-suited IFG$_N$-cylindric set algebra that includes Ω. Then there exist elements X and Y such that $\Omega \leq X < Y$ and $X \equiv Y$.*

Proof. Since $\equiv$ is nontrivial, there exist distinct elements X'' and Y'' such that $X'' \equiv Y''$. Either $(X'')^+ \neq (Y'')^+$ or $(X'')^- \neq (Y'')^-$. In the first case let $X' = X'' +_N \Omega$ and $Y' = Y'' +_N \Omega$; otherwise let $X' = (X'')^\cup +_N \Omega$ and $Y' = (Y'')^\cup +_N \Omega$. Either way $(X')^+ \neq (Y')^+$, $\Omega \leq X', Y'$, and $X' \equiv Y'$. Now let $X = X' = X' +_N X'$ and $Y = X' +_N Y'$. Then $\Omega \leq X < Y$ and $X \equiv Y$, as desired. $\qquad \square$

Theorem 3.24. $\mathfrak{Cs}_{\mathrm{IFG}_1}(2)$ *is simple.*

Proof. By the previous lemma it suffices to consider the congruences generated by pairs of elements from the interval above Ω. Using the technique of perspective edges, we can see that if $A \equiv B$, then $C \equiv \Omega$ because $A \cdot_N C = C$ and $B \cdot_N C = \Omega$. Similarly, if $A \equiv C$ then $B \equiv \Omega$. Finally, if $B \equiv C$ then $B \equiv A$ because $B = B +_N \|v_0 = 0\|$ and $A = C +_N \|v_0 = 0\|$. $\qquad\square$

Proposition 3.25. $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ *has the following proper subalgebras:*
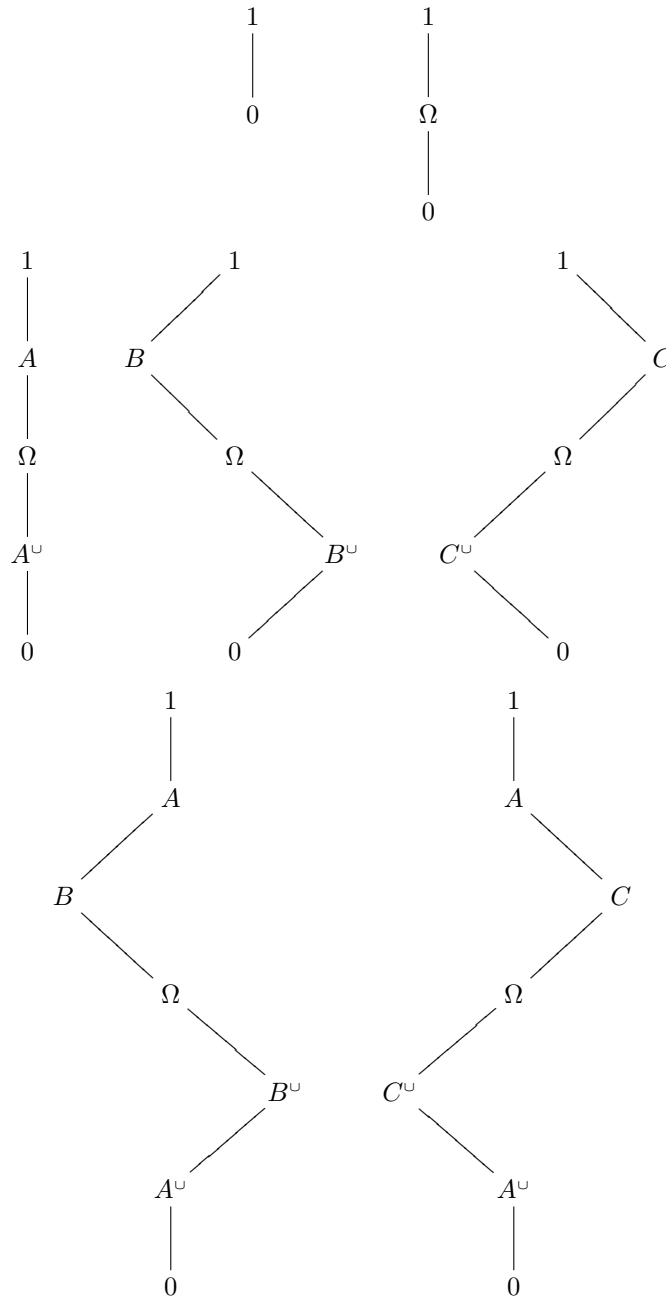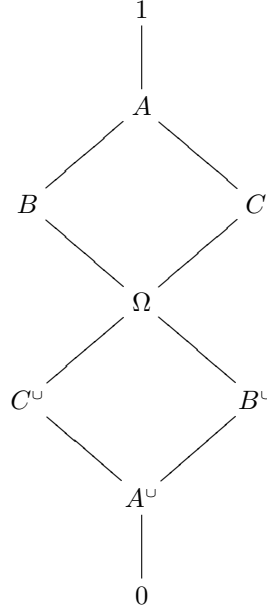
Proof. By the results of Chapter 2, we know that $\{0, 1\}$ and $\{0, \Omega, 1\}$ are subalgebras of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$.

Consider the subalgebra of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ generated by A. Recall that since $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ is a double-suited algebra, $X \leq \Omega \leq Y$ implies $X +_J Y = Y$. It is easily checked that

$$A +_\emptyset A = 1, \qquad A +_{\{0\}} A = A,$$

$$A^\cup +_\emptyset A^\cup = A^\cup, \qquad A^\cup +_{\{0\}} A^\cup = A^\cup.$$

Finally, recall that for any $X \in \mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$, $C_{0,J}(X) \in \{0, \Omega, 1\}$. Thus $\langle A \rangle = \{0, A^\cup, \Omega, A, 1\}$.

Now consider $\langle B \rangle \subseteq \mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$. Since B is flat,

$$B +_\emptyset B = B, \qquad B +_{\{0\}} B = B,$$

$$B^\cup +_\emptyset B^\cup = B^\cup, \qquad B^\cup +_{\{0\}} B^\cup = B^\cup.$$

Thus $\langle B \rangle = \{0, B^\cup, \Omega, B, 1\}$. Similarly $\langle C \rangle = \{0, C^\cup, \Omega, C, 1\}$.

To show that $\langle A, B \rangle = \{0, A^\cup, B^\cup, \Omega, B, A, 1\}$ observe that

$$A +_\emptyset B = 1, \qquad A +_{\{0\}} B = A,$$

$$A^\cup +_\emptyset B^\cup = B^\cup, \qquad A^\cup +_{\{0\}} B^\cup = B^\cup.$$

Similarly $\langle A, C \rangle = \{0, A^\cup, C^\cup, \Omega, C, A, 1\}$.

To show that $\langle B, C \rangle = \{0, A^\cup, B^\cup, C^\cup, \Omega, C, B, A, 1\}$, observe that

$$B +_\emptyset C = 1, \qquad B +_{\{0\}} C = A,$$

$$B^\cup +_\emptyset C^\cup = \Omega, \qquad B^\cup +_{\{0\}} C^\cup = \Omega.$$

Finally, note that if $\mathfrak{D}$ be a subalgebra of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$, and $\|v_0 = 0\| \in \mathfrak{D}$, then $\|v_0 = 1\| = \|v_0 = 0\|^\cup \in \mathfrak{D}$. Thus $\mathfrak{D}$ includes all of the perfect elements in $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$. Hence $\mathfrak{D} = \mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$. Similarly, if $\|v_0 = 1\| \in \mathfrak{D}$, then $\mathfrak{D} = \mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$. $\qquad\square$

Theorem 3.26. $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ *is hereditarily simple.*

Proof. It follows from Lemma 3.21 and Lemma 3.22 that the subalgebras $\{0, \Omega, 1\}$, $\langle A \rangle$, $\langle B \rangle$, and $\langle C \rangle$ are all simple. To show the subalgebra $\langle A, B \rangle$ is simple, by Lemma 3.23 it suffices to show that the congruence $\mathrm{Cg}(A, B)$ generated by A and B is the total congruence. Observe that if $A \equiv B$, then $1 = A +_\emptyset A \equiv B +_\emptyset B = B$, so $\mathrm{Cg}(A, B)$ is the total congruence. A similar argument shows that the subalgebra $\langle A, C \rangle$ is simple. Finally, to prove the subalgebra $\langle A, B, C \rangle$ and $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ are simple it suffices to show that the congruences $\mathrm{Cg}(A, B)$ and $\mathrm{Cg}(A, C)$ are both the total congruence. But the calculations are the same as before, so we are done. $\qquad\square$

Proposition 3.27. $\mathrm{DSuit}_N(A)$ *is simple.*

Proof. Suppose X and Y are distinct elements of $\mathrm{DSuit}_N(A)$ such that $X \equiv Y$. Without loss of generality we may assume that there exists a $V \in Y^+ \setminus X^+$. Let $Z = \langle \mathscr{P}(^N A \setminus V), \mathscr{P}(V) \rangle$. Since $V \notin X^+$ we know that for every $U \in X^+$ there is an $\vec{a} \in V \setminus U$. Hence $U \cup (^N A \setminus V) \neq {}^N A$. Thus $1 \neq X +_\emptyset Z \equiv Y +_\emptyset Z = 1$. Therefore $\equiv$ is the total congruence. $\qquad\square$

Recall that in the proof that $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{2})$ is hereditarily simple, we used the fact that $A +_\emptyset A = 1$ but $B +_\emptyset B \neq 1$. For any element X of an IFG-cylindric set algebra, let nX be an abbreviation for $\underbrace{X +_\emptyset \cdots +_\emptyset X}_{n}$.

Definition. The **order** of an element X is the least positive integer n such that $nX = 1$. If no such positive integer exists then the order of X is infinite.

Lemma 3.28. *Let $\equiv$ be a congruence on a double-suited IFG_N-cylindric set algebra. If any two elements of different order are congruent, then $\equiv$ is the total congruence.*

Proof. Let $X \equiv Y$. If X has finite order and Y has infinite order, then for some positive integer n we have $1 = nX \equiv nY \neq 1$. If X has order m, Y has order n, and $m < n$, then $1 = mX \equiv mY \neq 1$. $\qquad\qquad\square$

Let **3** be the structure with universe $\{0, 1, 2\}$ in which all three elements are named by constant symbols. Then $|\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{3})| = 55$. Part of the lattice structure of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{3})$ is shown below. For simplicity, we only show the interval above Ω. Furthermore, we omit the falsity coordinate and denote each truth coordinate by its generating antichain. For example, the vertex labeled $\{0, 1\}, \{2\}$ denotes the element $\langle \mathscr{P}(\{0, 1\}) \cup \mathscr{P}(\{2\}), \mathscr{P}(\emptyset) \rangle$, and the vertex labeled $\emptyset$ denotes $\langle \mathscr{P}(\emptyset), \mathscr{P}(\emptyset) \rangle = \Omega$.
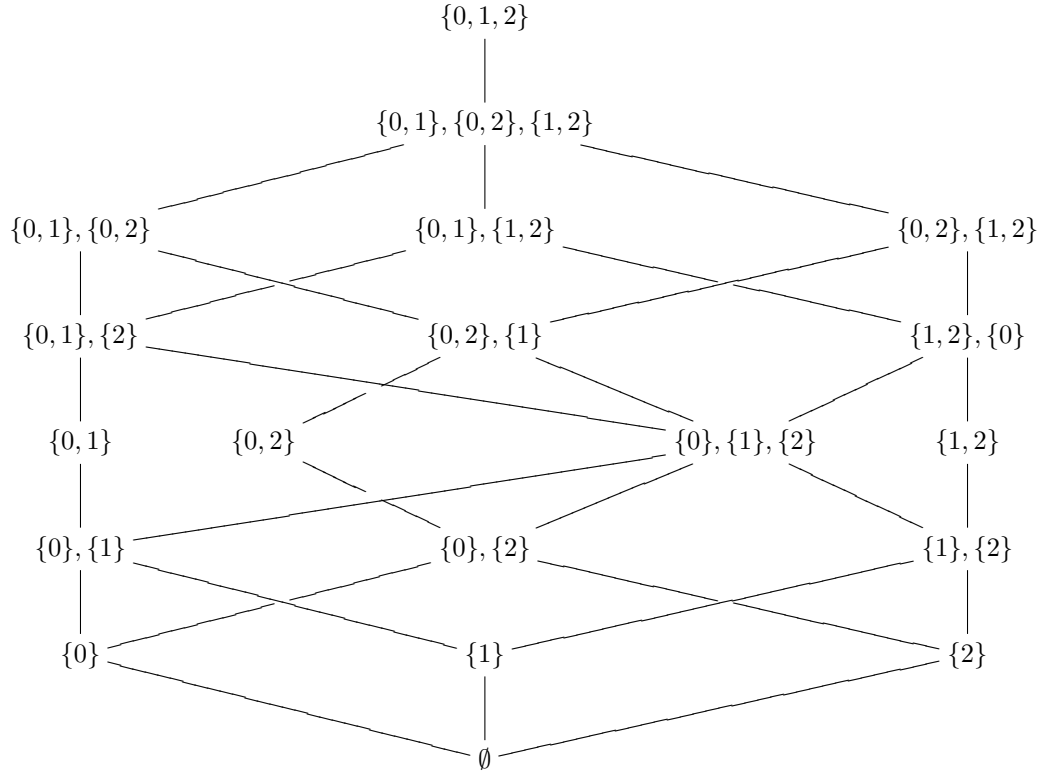
We know by Proposition 3.19 and Proposition 3.27 that $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{3})$ is simple. We can verify this directly by using the previous lemmas and the technique of perspective edges. For

example, if $\{0\}, \{1\} \equiv \{0, 1\}$, then $\{0\}, \{1\}, \{2\} \equiv \{0, 1\}, \{2\}$. But $\{0\}, \{1\}, \{2\}$ has order 2, while $\{0, 1\}, \{2\}$ has order 1, so by Lemma 3.28 we have that $\equiv$ is the total congruence.

Theorem 3.29. $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{3})$ *is not hereditarily simple.*

Proof. Let $A = \langle \mathscr{P}(\{0, 1\}), \mathscr{P}(\emptyset) \rangle$ and $B = \langle \mathscr{P}(\{0\}) \cup \mathscr{P}(\{1\}), \mathscr{P}(\emptyset) \rangle$. Consider the subalgebra of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{3})$ generated by B. The following calculations show that

$$\langle B \rangle = \{0, A^{\cup}, B^{\cup}, \Omega, B, A, 1\}.$$

First of all, $B +_{\emptyset} B = A$ because $(B +_{\emptyset} B)^{+} = \mathscr{P}(\{0\} \cup \{1\}) = \mathscr{P}(\{0, 1\})$ and $(B +_{\emptyset} B)^{-} = \mathscr{P}(\emptyset) \cap \mathscr{P}(\emptyset) = \mathscr{P}(\emptyset)$. Similarly, $A +_{\emptyset} A = A$ because $(A +_{\emptyset} A)^{+} = \mathscr{P}(\{0, 1\} \cup \{0, 1\}) = \mathscr{P}(\{0, 1\})$. Also $A +_{\emptyset} B = A$ because $(A +_{\emptyset} B)^{+} = \mathscr{P}(\{0, 1\} \cup \{0\}) \cup \mathscr{P}(\{0, 1\} \cup \{1\}) = \mathscr{P}(\{0, 1\})$.

The set $\{0, A^{\cup}, B^{\cup}, \Omega, B, A, 1\}$ is closed under $^{\cup}$ because $0^{\cup} = 1$, $\Omega^{\cup} = \Omega$, $1^{\cup} = 0$, $(A^{\cup})^{\cup} = A$ and $(B^{\cup})^{\cup} = B$. It is closed under $+_{\emptyset}$ and $+_{\{0\}}$ because

$$
\begin{array}{ll}
A +_{\emptyset} A = A, & A +_{\{0\}} A = A, \\[2mm]
A +_{\emptyset} B = A, & A +_{\{0\}} B = A, \\[2mm]
B +_{\emptyset} B = A, & B +_{\{0\}} B = B,
\end{array}
$$

$$
\begin{array}{ll}
A +_{\emptyset} A^{\cup} = A, & A +_{\{0\}} A^{\cup} = A, \\[2mm]
A +_{\emptyset} B^{\cup} = A, & A +_{\{0\}} B^{\cup} = A, \\[2mm]
B +_{\emptyset} A^{\cup} = B, & B +_{\{0\}} A^{\cup} = B, \\[2mm]
B +_{\emptyset} B^{\cup} = B, & B +_{\{0\}} B^{\cup} = B,
\end{array}
$$

$$
\begin{array}{ll}
A^{\cup} +_{\emptyset} A^{\cup} = A^{\cup}, & A^{\cup} +_{\{0\}} A^{\cup} = A^{\cup}, \\[2mm]
A^{\cup} +_{\emptyset} B^{\cup} = B^{\cup}, & A^{\cup} +_{\{0\}} B^{\cup} = B^{\cup}, \\[2mm]
B^{\cup} +_{\emptyset} B^{\cup} = B^{\cup}, & B^{\cup} +_{\{0\}} B^{\cup} = B^{\cup}.
\end{array}
$$

We perform a few representative calculations. All of the $+_{\{0\}}$ calculations are easy to check by looking at the lattice. For example, $A +_{\{0\}} B = A$ because $B \leq A$. The $+_\emptyset$ calculations require some computation. We have already checked the first three. Checking two more should suffice. $A +_\emptyset A^\cup = A$ because $(A +_\emptyset A^\cup)^+ = \mathscr{P}(\{0,1\} \cup \emptyset) = \mathscr{P}(\{0,1\})$ and $(A +_\emptyset A^\cup)^- = \mathscr{P}(\emptyset) \cap \mathscr{P}(\{0,1\}) = \mathscr{P}(\emptyset)$, while $A^\cup +_\emptyset B^\cup = B^\cup$ because $(A^\cup +_\emptyset B^\cup)^+ = \mathscr{P}(\emptyset \cup \emptyset) = \mathscr{P}(\emptyset)$ and $(A^\cup +_\emptyset B^\cup)^- = \mathscr{P}(\{0,1\}) \cap (\mathscr{P}(\{0\}) \cup \mathscr{P}(\{1\})) = \mathscr{P}(\{0\}) \cup \mathscr{P}(\{1\})$. The set is closed under $C_{0,J}$ because for any X we have $C_{0,J}(X) \in \{0, \Omega, 1\}$.

We claim that $\mathrm{Cg}(A, B)$ is a nontrivial, non-total congruence. Let $\equiv$ denote the equivalence relation that makes $A \equiv B$ and $A^\cup \equiv B^\cup$, but makes no other pair of distinct elements equivalent. To verify that $\equiv$ is a congruence, observe that $\equiv$ is preserved under $^\cup$ because $A^\cup \equiv B^\cup$ and $(A^\cup)^\cup = A \equiv B = (B^\cup)^\cup$. It is preserved under $C_{0,J}$ because $C_{0,J}(A) = 1 = C_{0,J}(B)$ and $C_{0,J}(A^\cup) = \Omega = C_{0,J}(B^\cup)$. Finally, the calculations above show that $\equiv$ is preserved under $+_\emptyset$ and $+_{\{0\}}$. Thus $\equiv$ is $\mathrm{Cg}(A, B)$. Therefore $\langle B \rangle$ is not simple. $\qquad\square$

Note that the only perfect elements in $\langle B \rangle$ are 0 and 1, which do not generate $\langle B \rangle$. Therefore $\langle B \rangle$ is not the IFG_1-cylindric set algebra over any structure.

Theorem 3.29 has an interesting consequence for the syntax of IFG logic. In ordinary first-order logic $\phi^\mathfrak{A} = \psi^\mathfrak{A}$ if and only if $\mathfrak{A} \models \phi \leftrightarrow \psi$. Let $\phi \underset{J}{\rightarrow} \psi$ be an abbreviation for $\sim\!\phi \vee_{/J} \psi$, and let $\phi \underset{J}{\leftrightarrow} \psi$ be an abbreviation for $(\phi \underset{J}{\rightarrow} \psi) \wedge_{/J} (\psi \underset{J}{\rightarrow} \phi)$. In IFG logic it is not the case that $\|\phi\|_\mathfrak{A} = \|\psi\|_\mathfrak{A}$ if and only if $\mathfrak{A} \models^+ \phi \underset{J}{\leftrightarrow} \psi$. In fact, there is no schema $\xi(\phi, \psi)$ involving two IFG-formulas such that $\|\phi\|_\mathfrak{A} = \|\psi\|_\mathfrak{A}$ if and only if $\mathfrak{A} \models^+ \xi(\phi, \psi)$.

It will be useful to know when $\mathfrak{A} \models^\pm \phi \underset{J}{\rightarrow} \psi[V]$ and $\mathfrak{A} \models^\pm \phi \underset{J}{\leftrightarrow} \psi[V]$. It follows immediately from the definitions that

(+) $\mathfrak{A} \models^+ \phi \underset{J}{\rightarrow} \psi[V]$ if and only if there is a J-saturated disjoint cover $V = V_1 \cup_J V_2$ such that $\mathfrak{A} \models^- \phi[V_1]$ and $\mathfrak{A} \models^+ \psi[V_2]$,

(−) $\mathfrak{A} \models^- \phi \underset{J}{\rightarrow} \psi[W]$ if and only if $\mathfrak{A} \models^+ \phi[W]$ and $\mathfrak{A} \models^- \psi[W]$.

In particular,

(+) $\mathfrak{A} \models^+ \phi \underset{N}{\rightarrow} \psi[V]$ if and only if $\mathfrak{A} \models^- \phi[V]$ or $\mathfrak{A} \models^+ \psi[V]$,

(−) $\mathfrak{A} \models^- \phi \underset{N}{\rightarrow} \psi[W]$ if and only if $\mathfrak{A} \models^+ \phi[W]$ and $\mathfrak{A} \models^- \psi[W]$.

Also,

(+) $\mathfrak{A} \models^+ \phi \underset{J}{\leftrightarrow} \psi[V]$ if and only if there are two J-saturated disjoint covers $V = V_1 \cup_J V_2$

and $V = V_3 \cup_J V_4$ such that $\mathfrak{A} \models^+ \phi[V_1]$, $\mathfrak{A} \models^- \psi[V_2]$, $\mathfrak{A} \models^- \phi[V_3]$, and $\mathfrak{A} \models^+ \psi[V_4]$,

(−) $\mathfrak{A} \models^- \phi \underset{J}{\leftrightarrow} \psi[W]$ if and only if there is a J-saturated disjoint cover $W = W_1 \cup_J W_2$

such that $\mathfrak{A} \models^+ \phi[W_1]$, $\mathfrak{A} \models^- \psi[W_1]$, $\mathfrak{A} \models^- \phi[W_2]$, and $\mathfrak{A} \models^+ \psi[W_2]$.

In particular,

(+) $\mathfrak{A} \models^+ \phi \underset{N}{\leftrightarrow} \psi[V]$ if and only if $\mathfrak{A} \models^+ \phi[V]$ and $\mathfrak{A} \models^+ \psi[V]$, or $\mathfrak{A} \models^- \phi[V]$ and

$\mathfrak{A} \models^- \psi[V]$,

(−) $\mathfrak{A} \models^- \phi \underset{N}{\leftrightarrow} \psi[W]$ if and only if $\mathfrak{A} \models^+ \phi[W]$ and $\mathfrak{A} \models^- \psi[W]$, or $\mathfrak{A} \models^- \phi[W]$ and

$\mathfrak{A} \models^+ \psi[W]$.

For example, $\mathfrak{A} \models^+ \phi \underset{N}{\leftrightarrow} \psi[V]$ if and only if $\mathfrak{A} \models^+ (\phi \vee_{/N} \sim \psi) \wedge_{/N} (\sim \phi \vee_{/N} \psi)[V]$ if and only if $\mathfrak{A} \models^+ \phi \vee_{/N} \sim \psi[V]$ and $\mathfrak{A} \models^+ \sim \phi \vee_{/N} \psi$ if and only if $\mathfrak{A} \models^+ \phi[V]$ or $\mathfrak{A} \models^- \psi[V]$, and $\mathfrak{A} \models^- \phi[V]$ or $\mathfrak{A} \models^+ \psi[V]$ if and only if $\mathfrak{A} \models^+ \phi[V]$ and $\mathfrak{A} \models^- \phi[V]$, or $\mathfrak{A} \models^+ \phi[V]$ and $\mathfrak{A} \models^+ \psi[V]$, or $\mathfrak{A} \models^- \psi[V]$ and $\mathfrak{A} \models^- \phi[V]$, or $\mathfrak{A} \models^- \psi[V]$ and $\mathfrak{A} \models^+ \psi[V]$ if and only if $\mathfrak{A} \models^+ \phi[V]$ and $\mathfrak{A} \models^+ \psi[V]$, or $\mathfrak{A} \models^- \psi[V]$ and $\mathfrak{A} \models^- \phi[V]$.

Proposition 3.30. *For any two IFG$_N$-formulas ϕ and ψ, $\mathfrak{A} \models^+ \phi \underset{\emptyset}{\leftrightarrow} \psi$ if and only if $\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}}$ and both are perfect.*

Proof. Suppose $\mathfrak{A} \models^+ (\phi \underset{\emptyset}{\leftrightarrow} \psi)[^N A]$. Then there exist $V, V' \subseteq {}^N A$ such that $\mathfrak{A} \models^+ \phi[V]$, $\mathfrak{A} \models^- \psi[^N A \backslash V]$, $\mathfrak{A} \models^- \phi[V']$, and $\mathfrak{A} \models^+ \psi[^N A \backslash V']$. Thus $V \cap V' = \emptyset$ and $(^N A \backslash V) \cap (^N A \backslash V') = \emptyset$. Therefore $V' = {}^N A \setminus V$, and $\|\phi\|_{\mathfrak{A}} = \langle \mathscr{P}(V), \mathscr{P}(^N A \setminus V) \rangle = \|\psi\|_{\mathfrak{A}}$. $\qquad \square$

Proposition 3.31. *For any IFG$_N$-formulas ϕ and ψ, $\mathfrak{A} \models^+ \phi \underset{N}{\leftrightarrow} \psi$ if and only if $\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}} = 0$ or $\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}} = 1$.*

Proof. Suppose $\mathfrak{A} \models^+ (\phi \underset{N}{\leftrightarrow} \psi)[^N A]$. Then $\mathfrak{A} \models^+ \phi[^N A]$ and $\mathfrak{A} \models^+ \psi[^N A]$, or $\mathfrak{A} \models^- \phi[^N A]$ and $\mathfrak{A} \models^- \psi[^N A]$. $\qquad \square$

Definition. An **IFG$_N$-schema** involving k formula variables is an element of the smallest set Ξ satisfying the following conditions.

  (a) The formula variables $\alpha_0, \ldots, \alpha_{k-1}$ belong to Ξ.

  (b) For all $i, j < N$, the formula $v_i = v_j$ belongs to Ξ.

  (c) If ξ belongs to Ξ, then $\sim\xi$ belongs to Ξ.

  (d) If ξ_1 and ξ_2 belong to Ξ, and $J \subseteq N$, then $\xi_1 \vee_{/J} \xi_2$ belongs to Ξ.

  (e) If ξ belongs to Ξ, $n < N$, and $J \subseteq N$, then $\exists v_{n/J}\xi$ belongs to Ξ.

Note that the symbols $\alpha_0, \ldots, \alpha_{k-1}$ are distinct from the ordinary IFG-variables $v_0, \ldots, v_{N-1}$. If ξ is an IFG$_N$-schema involving k formula variables, and $\phi_0, \ldots, \phi_{k-1}$ are IFG$_N$-formulas, then the IFG$_N$-formula $\xi(\phi_0, \ldots, \phi_{k-1})$ is called an **instance** of ξ.

Every IFG$_N$-schema ξ has a corresponding term T_ξ in the language of IFG$_N$-cylindric set algebras. T_ξ is defined recursively as follows.

  (a) $T_{\alpha_i} = X_i$.

  (b) $T_{v_i=v_j} = D_{ij}$.

  (c) $T_{\sim\xi} = (T_\xi)^\cup$.

  (d) $T_{\xi_1 \vee_{/J} \xi_2} = T_{\xi_1} +_J T_{\xi_2}$.

  (e) $T_{\exists v_{n/J}\xi} = C_{n,J}(T_\xi)$.

Lemma 3.32. *Let ξ be an IFG$_N$-schema involving k formula variables, and let T_ξ be its corresponding term. Then for any IFG$_N$-formulas $\phi_0, \ldots, \phi_{k-1}$ and any suitable structure $\mathfrak{A}$,*

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

Proof. If ξ is a formula variable α_i, then $T_\xi = X_i$, so

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = \|\phi_i\| = T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

If ξ is $v_i = v_j$, then $T_\xi = D_{ij}$, so

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = \|v_i = v_j\| = D_{ij}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})} = T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

Now assume that

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),$$

$$\|\xi_1(\phi_0, \ldots, \phi_{k-1})\| = T_{\xi_1}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),$$

$$\|\xi_2(\phi_0, \ldots, \phi_{k-1})\| = T_{\xi_2}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

Then

$$\begin{aligned}
\|\sim\xi(\phi_0, \ldots, \phi_{k-1})\| &= \|\xi(\phi_0, \ldots, \phi_{k-1})\|^\cup \\
&= \left(T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|)\right)^\cup \\
&= T_{\sim\xi}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),
\end{aligned}$$

$$\begin{aligned}
\left\|\xi_1 \vee_{/J} \xi_2(\phi_0, \ldots, \phi_{k-1})\right\| &= \|\xi_1(\phi_0, \ldots, \phi_{k-1})\| +_J \|\xi_2(\phi_0, \ldots, \phi_{k-1})\| \\
&= T_{\xi_1}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|) +_J T_{\xi_2}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|) \\
&= T_{\xi_1 \vee_J \xi_2}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),
\end{aligned}$$

$$\begin{aligned}
\left\|\exists v_n {/}_J \xi(\phi_0, \ldots, \phi_{k-1})\right\| &= C_{n,J} \|\xi(\phi_0, \ldots, \phi_{k-1})\| \\
&= C_{n,J} \left(T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|)\right) \\
&= T_{\exists v_n {/}_J \xi}^{\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|). \qquad \square
\end{aligned}$$

Proposition 3.33. *Any independence-friendly cylindric set algebra that has a term operation* $T(X,Y)$ *such that* $T(X,Y) = 1$ *if and only if* $X = Y$ *is hereditarily simple.*

Proof. Suppose $\mathfrak{C}$ is an independence-friendly cylindric set algebra that has such a term operation. Then for any $X \neq Y$ we have $\langle 1, Z \rangle = \langle T(X,X), T(X,Y) \rangle \in \mathrm{Cg}(X,Y)$, where Z is some element different than 1. Hence $\mathrm{Cg}(X,Y)$ is the total congruence. Thus $\mathfrak{C}$ is simple. Furthermore, the sentence

$$\forall X \forall Y [T(X,Y) = 1 \leftrightarrow X = Y]$$

is universal, and so must hold in every subalgebra of $\mathfrak{C}$. Hence $\mathfrak{C}$ is hereditarily simple. $\qquad\square$

Theorem 3.34. *There is no IFG$_1$-schema ξ involving two formula variables such that for every pair of IFG$_1$-formulas ϕ and ψ, and every suitable structure $\mathfrak{A}$, we have*

$$\mathfrak{A} \models^+ \xi(\phi, \psi) \quad \text{iff} \quad \|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}}.$$

Proof. Suppose ξ were such a schema. Then the corresponding term T_ξ would have the property that for any $\mathfrak{A}$ and any $\|\phi\|_{\mathfrak{A}}, \|\psi\|_{\mathfrak{A}} \in \mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})$,

$$
\begin{aligned}
T_\xi^{\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})}(\|\phi\|_{\mathfrak{A}}, \|\psi\|_{\mathfrak{A}}) = 1 \quad &\text{iff} \quad \|\xi(\phi,\psi)\|_{\mathfrak{A}} = 1 \\
&\text{iff} \quad \mathfrak{A} \models^+ \xi(\phi,\psi) \\
&\text{iff} \quad \|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}}.
\end{aligned}
$$

Thus every $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})$ would be hereditarily simple. However $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathbf{3})$ is not hereditarily simple. $\qquad\square$

Chapter 4

Finite Axiomatizability

We would like to know whether the equational theory of IFG_N-cylindric set algebras is finitely axiomatizable. Unfortunately determining the answer is beyond our current abilities, even in the one-dimensional case. To make the problem more tractable, we consider certain reducts of IFG_1-cylindric set algebras. First we will examine reducts of IFG_1-cylindric set algebras to the signature $\langle 0, 1, \cup, +_{\{0\}}, \cdot_{\{0\}} \rangle$. Then will consider reducts to the signature $\langle 0, 1, \cup, +_{\{0\}}, \cdot_{\{0\}}, C_{0,\{0\}} \rangle$.

4.1 The De Morgan reduct

Definition. The reduct of an IFG_N-cylindric set algebra $\mathfrak{C}$ to the signature $\langle 0, 1, \cup, +_N, \cdot_N \rangle$ is called the **De Morgan reduct** of $\mathfrak{C}$.

Definition. An element x of a De Morgan algebra is a **fixed point** (or **center**) if $\sim x = x$. A **centered De Morgan algebra** is a De Morgan algebra with a center.

Proposition 4.1. *The center of a centered Kleene algebra is unique.*

Proof. Suppose a and b are both fixed points of a centered Kleene algebra. Then $a = a \wedge \sim a \leq b \vee \sim b = b$, and $b = b \wedge \sim b \leq a \vee \sim a = a$. $\qquad\square$

Let $\underline{\mathbf{B}}$ be the two-element De Morgan algebra, let $\underline{\mathbf{K}}$ be the three-element De Morgan algebra with fixed point a, and let $\underline{\mathbf{M}}$ be the four-element De Morgan algebra with two fixed

points a and b such that $a \wedge b = 0$ and $a \vee b = 1$:

$$\mathbf{\underline{B}} \qquad \mathbf{\underline{K}} \qquad \mathbf{\underline{M}}$$
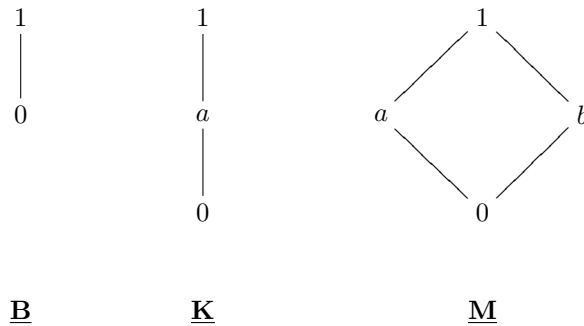
Theorem 4.2 (Kalman [18])**.** *The subdirectly irreducible De Morgan algebras are exactly* $\mathbf{\underline{B}}$, $\mathbf{\underline{K}}$, *and* $\mathbf{\underline{M}}$.

The following corollary is an immediate consequence of Theorem 4.2 and Birkhoff's subdirect representation theorem (II§8.6 in [3]).

Corollary 4.3. *Every De Morgan algebra is isomorphic to a subdirect product of copies of* $\mathbf{\underline{B}}$, $\mathbf{\underline{K}}$, *and* $\mathbf{\underline{M}}$.

Corollary 4.4.   *(a) Every Kleene algebra is isomorphic to a subdirect product of copies of* $\mathbf{\underline{B}}$ *and* $\mathbf{\underline{K}}$.

   (b) Every centered Kleene algebra is isomorphic to a subdirect product of copies of $\mathbf{\underline{K}}$.

   (c) Every De Morgan algebra that has a subalgebra isomorphic to $\mathbf{\underline{M}}$ *is isomorphic to a subdirect product of copies of* $\mathbf{\underline{M}}$.

Proof. (a) Let $\mathfrak{A}$ be a Kleene algebra. If $\mathbf{\underline{M}}$ were a factor in a subdirect representation of $\mathfrak{A}$, then $\mathbf{\underline{M}}$ would be a homomorphic image of $\mathfrak{A}$, and hence a Kleene algebra.

   (b) Let $\mathfrak{A}$ be a Kleene algebra, and let $\mathfrak{B}$ be a subdirect representation of $\mathfrak{A}$. If $\mathbf{\underline{B}}$ is a factor of $\mathfrak{B}$ (say the first), then every tuple in $\mathfrak{B}$ has the form $\langle 0, \ldots \rangle$ or $\langle 1, \ldots \rangle$, so no tuple in $\mathfrak{B}$ is a fixed point.

   (c) Let $\mathfrak{A}$ be a De Morgan algebra, and let $\mathfrak{B}$ be a subdirect representation of $\mathfrak{A}$. If $\mathbf{\underline{B}}$ is a factor of $\mathfrak{B}$, then $\mathfrak{B}$ would have no fixed points at all, and if $\mathbf{\underline{K}}$ is a factor of $\mathfrak{B}$ (say the first),

then every fixed point in $\mathfrak{B}$ would have the form $\langle a, \ldots \rangle$, so no two fixed points could meet to $\langle 0, 0, \ldots \rangle$ or join to $\langle 1, 1, \ldots \rangle$. □

Let $\mathcal{B}$ denote the variety of Boolean algebras, $\mathcal{K}$ denote the variety of Kleene algebras, and $\mathcal{M}$ denote the variety of De Morgan algebras.

Theorem 4.5 (Kalman [18])**.** *The nontrivial subvarieties of $\mathcal{M}$ are $\mathcal{B} \subset \mathcal{K} \subset \mathcal{M}$.*

Theorem 4.6. *The variety generated by the De Morgan reducts of all suited IFG_N-cylindric set algebras is $\mathcal{M}$. The variety generated by the De Morgan reducts of all double-suited IFG_N-cylindric set algebras is $\mathcal{K}$.*

Proof. The De Morgan reduct of every suited IFG_N-cylindric set algebra is a De Morgan algebra. Hence the variety generated by the De Morgan reducts of all suited IFG_N-cylindric set algebras is contained in $\mathcal{M}$. On the other hand, $\{0, 1\}$, $\{0, \Omega, 1\}$, and $\{0, \Omega, \mho, 1\}$ are all suited IFG_N-cylindric set algebras, and their De Morgan reducts are isomorphic to $\underline{\mathbf{B}}$, $\underline{\mathbf{K}}$, and $\underline{\mathbf{M}}$, respectively. Thus the De Morgan reducts of suited IFG_N-cylindric set algebras generate all the subdirectly irreducible De Morgan algebras, which in turn generate $\mathcal{M}$.

The De Morgan reduct of every double-suited IFG_N-cylindric set algebra is a Kleene algebra. Hence the variety generated by the De Morgan reducts of all double-suited IFG_N-cylindric set algebras is contained in $\mathcal{K}$. On the other hand, $\{0, 1\} \cong \underline{\mathbf{B}}$ and $\{0, \Omega, 1\} \cong \underline{\mathbf{K}}$ are both double-suited IFG_N-cylindric set algebras. Thus the De Morgan reducts of double-suited IFG_N-cylindric set algebras generate all the subdirectly irreducible Kleene algebras, which in turn generate $\mathcal{K}$. □

It follows from Theorem 4.6 that the equational theory of De Morgan reducts of suited IFG_N-cylindric set algebras is axiomatized by the axioms of De Morgan algebra, and the equational theory of De Morgan reducts of double-suited IFG_N-cylindric set algebras is axiomatized by the axioms of Kleene algebra.

4.2   Monadic De Morgan algebras

A monadic Boolean algebra is a Boolean algebra with an additional unary operation called a quantifier. Monadic Boolean algebras were first studied by Halmos [9] following the investigations of McKinsey and Tarski [19] into the algebraic properties of closure operators. Cignoli [6] generalized the notion of a monadic Boolean algebra by adding quantifiers to bounded distributive lattices. Cignoli calls a bounded distributive lattice with a quantifier a Q-distributive lattice. Petrovich [21, 22] extended the results of Cignoli to monadic De Morgan algebras. Our presentation follows [22].

Definition. An **existential quantifier** on a bounded distributive lattice is a unary operation ∇ such that:

(Q1) $\nabla 0 = 0$,

(Q2) $x \leq \nabla x$,

(Q3) $\nabla(x \vee y) = \nabla x \vee \nabla y$,

(Q4) $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$.

A **universal quantifier** is a unary operation Δ such that:

(Q1$'$) $\Delta 1 = 1$,

(Q2$'$) $\Delta x \leq x$,

(Q3$'$) $\Delta(x \wedge y) = \Delta x \wedge \Delta y$,

(Q4$'$) $\Delta(x \vee \Delta y) = \Delta x \vee \Delta y$.

The two kinds of quantifiers are dual to each other in the usual way. A **quantifier** without modification is assumed to be existential. A **Q-distributive lattice** $\langle L, \nabla \rangle$ is a bounded distributive lattice L equipped with a quantifier ∇.

Definition. An **quantifier** on a De Morgan algebra must satisfy the additional condition:

(Q5) $\nabla(\sim\nabla x) = \sim\nabla x$.

A De Morgan algebra equipped with a quantifier is called a **monadic De Morgan algebra**.

Proposition 4.7. *In any Q-distributive lattice,*

    (a) $\nabla 1 = 1$,

    (b) $\nabla\nabla x = \nabla x$.

Proof. (a) $1 \leq \nabla 1$ by axiom Q2, and $\nabla 1 \leq 1$ because for all x, $x \leq 1$.

    (b) $\nabla\nabla x = \nabla(1 \wedge \nabla x) = \nabla 1 \wedge \nabla x = 1 \wedge \nabla x = \nabla x$.   □

Lemma 4.8. *If $\mathfrak{A}$ is a De Morgan algebra, and ∇ is a quantifier on its underlying distributive lattice, then ∇ is a quantifier on $\mathfrak{A}$ if and only if the range of ∇ is a subalgebra of $\mathfrak{A}$.*

Proof. Suppose ∇ is a quantifier on $\mathfrak{A}$. Then $0 = \nabla 0$, $1 = \nabla 1$, and $\sim\nabla x = \nabla(\sim\nabla x)$. Also, $\nabla x \vee \nabla y = \nabla(x \vee y)$, and $\nabla x \wedge \nabla y = \nabla(x \wedge \nabla y)$. Thus the range of ∇ is a subalgebra of $\mathfrak{A}$. Conversely, suppose the range of ∇ is a subalgebra of $\mathfrak{A}$. Then $\sim\nabla x$ is in the range of ∇, so $\nabla(\sim\nabla x) = \sim\nabla x$.   □

Example. Let $\mathfrak{A}$ be a De Morgan algebra, and let ∇ be defined by

$$\nabla x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then ∇ is a quantifier on $\mathfrak{A}$. Such a quantifier will be called a **quantifier of type 0**. If $\mathfrak{A}$ is any De Morgan algebra, then $\langle\mathfrak{A}, \nabla_0\rangle$ will denote the monadic De Morgan algebra $\mathfrak{A}$ with the type 0 quantifier. In particular, $\langle\underline{\mathbf{B}}, \nabla_0\rangle$, $\langle\underline{\mathbf{K}}, \nabla_0\rangle$, and $\langle\underline{\mathbf{M}}, \nabla_0\rangle$ are all monadic De Morgan algebras.

Lemma 4.9. *If $\mathfrak{A}$ is a centered Kleene algebra with fixed point c, then*

$$\{\, x \in \mathfrak{A} \mid x \wedge c = 0 \,\} = \{0\}.$$

Proof. If $\mathfrak{A}$ is a centered Kleene algebra with fixed point c, then $\mathfrak{A}$ is isomorphic to a subdirect product $\mathfrak{B}$ of copies of $\underline{\mathbf{K}}$. If $x \neq 0$ in $\mathfrak{A}$, then the corresponding tuple $\vec{y} \in \mathfrak{B}$ has at least one

nonzero coordinate y_i. Let $\vec{a} = \langle a, a, \ldots \rangle$ be the fixed point of $\mathfrak{B}$. Then $y_i \in \{a, 1\}$, so $y_i \wedge a = a$ in $\underline{\mathbf{K}}$. Hence $\vec{y} \wedge \vec{a} \neq \vec{0}$ in $\mathfrak{B}$. Thus $x \wedge c \neq 0$ in $\mathfrak{A}$. $\qquad\square$

Example. Let $\mathfrak{A}$ be a centered De Morgan algebra, and let a be a fixed point of $\mathfrak{A}$. Define ∇ by

$$
\nabla x = \begin{cases} 1 & \text{if } x \not\leq a, \\ a & \text{if } 0 < x \leq a, \\ 0 & \text{if } x = 0. \end{cases}
$$

Then ∇ is a quantifier on $\mathfrak{A}$ if and only if $\{\, x \in \mathfrak{A} \mid x \wedge a = 0 \,\} = \{0\}$. Such a quantifier will be called a **quantifier of type 1**. If $\mathfrak{A}$ is a centered Kleene algebra with fixed point a, then a is unique and $\{\, x \in \mathfrak{A} \mid x \wedge a = 0 \,\} = \{0\}$. Hence ∇ is a quantifier on $\mathfrak{A}$. If $\mathfrak{A}$ is a centered Kleene algebra, let $\langle \mathfrak{A}, \nabla_1 \rangle$ denote the monadic Kleene algebra $\mathfrak{A}$ with the type 1 quantifier. In particular, $\langle \underline{\mathbf{K}}, \nabla_1 \rangle$ is a monadic Kleene algebra.

Lemma 4.10. *Let L be a bounded distributive lattice. If $a \wedge b = 0$ and $a \vee b = 1$, then*

$$
\{\, x \in L \mid x \wedge a = 0 \,\} = [0, b] \quad \text{and} \quad \{\, x \in L \mid x \wedge b = 0 \,\} = [0, a].
$$

Proof. If $0 \leq x \leq b$, then $0 \leq a \wedge x \leq a \wedge b = 0$. Hence $x \wedge a = 0$. Conversely, if $x \wedge a = 0$, then

$$
x \vee b = (x \vee b) \wedge (a \vee b) = (x \wedge a) \vee b = b.
$$

Thus $0 \leq x \leq b$. $\qquad\square$

Example. Let $\mathfrak{A}$ be a centered De Morgan algebra with two fixed points a and b such that $a \wedge b = 0$ and $a \vee b = 1$, i.e., $\mathfrak{A}$ contains a subalgebra isomorphic to $\underline{\mathbf{M}}$. Define ∇ by

$$
\nabla x = \begin{cases} 1 & \text{if } x \not\leq a \text{ and } x \not\leq b, \\ a & \text{if } 0 < x \leq a, \\ b & \text{if } 0 < x \leq b, \\ 0 & \text{if } x = 0. \end{cases}
$$

It is easy to check that ∇ satisfies axioms Q1–Q3. To check Q4, suppose $x, y \in \mathfrak{A}$. If $x = 0$ or $y = 0$, then $\nabla(x \wedge \nabla y) = 0 = \nabla x \wedge \nabla y$. If $0 < x, y \leq a$, then $\nabla(x \wedge \nabla y) = a = \nabla x \wedge \nabla y$. Similarly, if $0 < x, y \leq b$, then $\nabla(x \wedge \nabla y) = b = \nabla x \wedge \nabla y$. If $x \nleq a, b$ and $0 < y \leq a$, then by Lemma 4.10, $0 < x \wedge a \leq a$, so

$$\nabla(x \wedge \nabla y) = \nabla(x \wedge a) = a = 1 \wedge a = \nabla x \wedge \nabla y.$$

Similarly, if $x \nleq a, b$ and $0 < y \leq b$, then by Lemma 4.10, $0 < x \wedge b \leq b$, so

$$\nabla(x \wedge \nabla y) = \nabla(x \wedge b) = b = 1 \wedge b = \nabla x \wedge \nabla y.$$

Finally, if $y \nleq a, b$, then $\nabla(x \wedge \nabla y) = \nabla(x \wedge 1) = \nabla x = \nabla x \wedge 1 = \nabla x \wedge \nabla y$.

Thus ∇ is a quantifier on the underlying bounded distributive lattice of $\mathfrak{A}$, and since the range of ∇ is a subalgebra of $\mathfrak{A}$, Lemma 4.8 tells us that ∇ is a quantifier on $\mathfrak{A}$. Such a quantifier will be called a **quantifier of type 2**. If $\mathfrak{A}$ is a De Morgan algebra with two specified fixed points a and b, then $\langle \mathfrak{A}, \nabla_2 \rangle$ will denote the monadic De Morgan algebra $\mathfrak{A}$ with the associated type 2 quantifier. In particular, $\langle \underline{\mathbf{M}}, \nabla_2 \rangle$ is a monadic De Morgan algebra.

Proposition 4.11 (Petrovich [22]). *Let $\langle \mathfrak{A}, \nabla \rangle$ be a subdirectly irreducible monadic De Morgan algebra. Then the range of ∇ is a subdirectly irreducible De Morgan algebra.*

Corollary 4.12 (Petrovich [22]). *Let $\langle \mathfrak{A}, \nabla \rangle$ be a subdirectly irreducible monadic De Morgan algebra. Then ∇ is a quantfier of type 0, a quantifer of type 1, or a quantifier of type 2.*

The varieties of monadic Boolean, Kleene, and De Morgan algebras will be denoted by $\mathcal{B}^{\nabla}$, $\mathcal{K}^{\nabla}$, and $\mathcal{M}^{\nabla}$, respectively. We can form subvarieties of $\mathcal{M}^{\nabla}$ by imposing additional conditions on ∇. For example, let $\mathcal{M}^{\nabla_{01}}$ denote the variety of monadic De Morgan algebras that satisfy the equation equivalent to $\nabla x \wedge \sim \nabla x \leq \nabla y \vee \sim \nabla y$, i.e., the range of ∇ is a Kleene algebra, and let $\mathcal{M}^{\nabla_0}$ denote the variety of monadic De Morgan algebras that satisfy the equation $\nabla x \wedge \sim \nabla x = 0$, i.e., the range of ∇ is a Boolean algebra. It follows that if $\langle \mathfrak{A}, \nabla \rangle$ is a subdirectly irreducible monadic De Morgan algebra in $\mathcal{M}^{\nabla_{01}}$, then ∇ must be a quantifier of type 0 or 1, and if it is subdirectly irreducible in $\mathcal{M}^{\nabla_0}$, then ∇ is a quantifier of type 0. Similarly, let $\mathcal{K}^{\nabla_0}$ denote the

variety of monadic Kleene algebras that satisfy $\nabla x \wedge \sim \nabla x = 0$. In particular, we have that $\langle \underline{\mathbf{M}}, \nabla_2 \rangle$ belongs to $\mathcal{M}^{\nabla}$ but not $\mathcal{M}^{\nabla_{01}}$, that $\langle \underline{\mathbf{M}}, \nabla_0 \rangle$ belongs to $\mathcal{M}^{\nabla_0}$ but not $\mathcal{K}^{\nabla}$, and that $\langle \underline{\mathbf{K}}, \nabla_1 \rangle$ belongs to $\mathcal{K}^{\nabla}$ but not $\mathcal{M}^{\nabla_0}$. Also, $\mathcal{M}^{\nabla_0} \cap \mathcal{K}^{\nabla} = \mathcal{K}^{\nabla_0}$, and $\langle \underline{\mathbf{K}}, \nabla_0 \rangle$ is in $\mathcal{K}^{\nabla_0}$ but not $\mathcal{B}^{\nabla}$.

Thus we have the following sublattice of the lattice of subvarieties of $\mathcal{M}^{\nabla}$:

$$\mathcal{M}^{\nabla}$$

$$\mathcal{M}^{\nabla_{01}}$$

$$\mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla}$$

$$\mathcal{M}^{\nabla_0} \qquad\qquad \mathcal{K}^{\nabla}$$

$$\mathcal{K}^{\nabla_0}$$

$$\mathcal{B}^{\nabla}$$
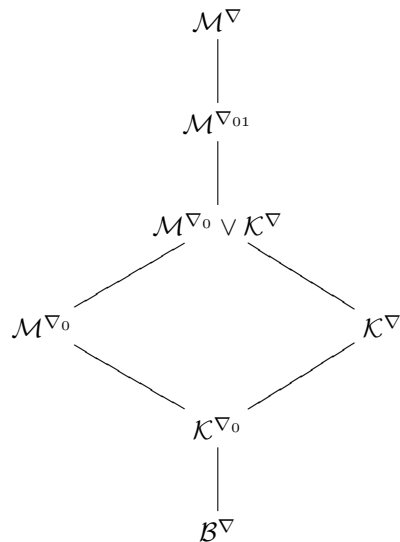
To show that $\mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla}$ is properly contained in $\mathcal{M}^{\nabla_{01}}$ will require some results from universal algebra.

Definition. A class of algebras is **congruence-distributive** if every algebra in the class has a distributive lattice of congruences.

Theorem 4.13 (Jónsson [17])**.** *A variety $\mathcal{V}$ is congruence-distributive if and only if for some integer $n \geq 2$ there exist terms $p_0(x, y, z), \ldots, p_n(x, y, z)$ such that $\mathcal{V}$ satisfies*

$$p_0(x, y, z) = x,$$

$$p_n(x, y, z) = z,$$

$$p_i(x, y, x) = x \qquad\qquad 0 \leq i \leq n,$$

$$p_i(x, x, y) = p_{i+1}(x, x, y) \qquad \text{for } i \text{ even,}$$

$$p_i(x, y, y) = p_{i+1}(x, y, y) \qquad \text{for } i \text{ odd.}$$

Example. The variety of lattices is congruence-distributive because it has Jónsson terms

$$p_0(x, y, z) = x,$$

$$p_1(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x),$$

$$p_2(x, y, z) = z.$$

It follows that any algebra that has an underlying lattice structure is congruence-distributive. In particular, the variety of De Morgan algebras and the variety of monadic De Morgan algebras are both congruence-distributive.

Lemma 4.14 (Jónsson [17])**.** *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are varieties such that $\mathcal{V}_1 \vee \mathcal{V}_2$ is congruence-distributive, then every member of $\mathcal{V}_1 \vee \mathcal{V}_2$ is isomorphic to a subdirect product of a member of $\mathcal{V}_1$ and a member of $\mathcal{V}_2$ and, in particular, every subdirectly irreducible member of $\mathcal{V}_1 \vee \mathcal{V}_2$ belongs either to $\mathcal{V}_1$ or $\mathcal{V}_2$.*

Corollary 4.15 (Jónsson [17])**.** *If $\mathcal{V}$ is a congruence-distributive variety, then then lattice of all subvarieties of $\mathcal{V}$ is distributive.*

Lemma 4.16. *There is a simple monadic De Morgan algebra $\langle \mathfrak{A}, \nabla \rangle$ with a quantifier of type 1 such that $\mathfrak{A}$ is not a Kleene algebra.*

Proof. Let $\mathfrak{A}$ be the following subalgebra of $\underline{\mathbf{K}} \times \underline{\mathbf{M}}$, where a is the fixed point of $\underline{\mathbf{K}}$, and b, c are the fixed points of $\underline{\mathbf{M}}$:
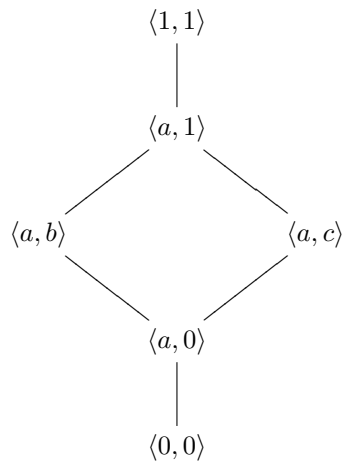
Consider the fixed point $\langle a, b \rangle$. The only $x \in \mathfrak{A}$ that satisfies $x \wedge \langle a, b \rangle = \langle 0, 0 \rangle$ is $\langle 0, 0 \rangle$. Therefore $\{\langle 0, 0 \rangle, \langle a, b \rangle, \langle 1, 1 \rangle\}$ is the range of a type 1 quantifier ∇ on $\mathfrak{A}$.

To show that $\langle \mathfrak{A}, \nabla \rangle$ is simple, suppose $\langle 0, 0 \rangle \equiv \langle a, 0 \rangle$. Then $\langle 0, 0 \rangle = \nabla \langle 0, 0 \rangle \equiv \nabla \langle a, 0 \rangle = \langle a, b \rangle$, which implies $\langle 1, 1 \rangle = \sim \langle 0, 0 \rangle \equiv \sim \langle a, b \rangle = \langle a, b \rangle$. Hence $\langle 0, 0 \rangle \equiv \langle 1, 1 \rangle$, and $\equiv$ is the total congruence. Now suppose $\langle a, 0 \rangle \equiv \langle a, b \rangle$. Then $\langle a, 1 \rangle = \sim \langle a, 0 \rangle \equiv \sim \langle a, b \rangle = \langle a, b \rangle$, which implies $\langle 1, 1 \rangle = \nabla \langle a, 1 \rangle \equiv \nabla \langle a, b \rangle = \langle a, b \rangle$. Hence $\langle 0, 0 \rangle \equiv \langle 1, 1 \rangle$, and $\equiv$ is the total congruence. Finally, suppose $\langle a, b \rangle \equiv \langle a, c \rangle$. Then $\langle a, b \rangle = \nabla \langle a, b \rangle \equiv \nabla \langle a, c \rangle = \langle 1, 1 \rangle$. Hence $\equiv$ is the total congruence. $\qquad \square$

Proposition 4.17. $\mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla} \subset \mathcal{M}^{\nabla_{01}}$.

Proof. Let $\langle \mathfrak{A}, \nabla \rangle$ be a simple monadic De Morgan algebra $\langle \mathfrak{A}, \nabla \rangle$ with a quantifier of type 1 such that $\mathfrak{A}$ is not a Kleene algebra. Then $\langle \mathfrak{A}, \nabla \rangle \notin \mathcal{M}^{\nabla_0} \cup \mathcal{K}^{\nabla}$, and by Lemma 4.14, $\langle \mathfrak{A}, \nabla \rangle \notin \mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla}$. $\qquad \square$

We have shown that we can separate the monadic De Morgan algebras with quantifiers of type 0 from those with quantifiers of type 1 or type 2 by means of an equation asserting that the range of the quantifier is a Boolean algebra. We can separate the monadic De Morgan algebras with quantifiers of type 0 or type 1 from those with quantifiers of type 2 by an equation asserting that the range of the quantifier is a Kleene algebra. However what we cannot yet do is separate those De Morgan algebras with type 1 quantifiers from those with type 0 or type 2 quantifiers. In particular, is there a variety that includes all monadic Kleene algebras with a quantifier of type 1 but not those with a quantifier of type 0? For starters, we should look for an equation that is true in $\langle \underline{\mathbf{K}}, \nabla_1 \rangle$ but false in $\langle \underline{\mathbf{K}}, \nabla_0 \rangle$. Let a be the fixed point of $\underline{\mathbf{K}}$. Notice that for all $x \in \underline{\mathbf{K}}$,

$$\nabla_1(x \wedge \sim x) \leq \sim \nabla_1(x \wedge \sim x),$$

while

$$\nabla_0(a \wedge \sim a) \nleq \sim \nabla_0(a \wedge \sim a).$$

Proposition 4.18. *Let* $\langle \mathfrak{A}, \nabla \rangle$ *be a monadic De Morgan algebra. If* $\langle \mathfrak{A}, \nabla \rangle$ *satisfies*

$$\nabla(x \wedge \sim x) \leq \sim \nabla(x \wedge \sim x),$$

then every fixed point of $\mathfrak{A}$ *is fixed by* ∇.

Proof. First, suppose $\langle \mathfrak{A}, \nabla \rangle$ is subdirectly irreducible and satisfies $\nabla(x \wedge \sim x) \leq \sim \nabla(x \wedge \sim x)$. Then ∇ is a quantifier of type 0, type 1, or type 2. If c is a fixed point that is not fixed by ∇, then $\nabla c = 1$ because any two distinct fixed points are incomparable. But then $\nabla(c \wedge \sim c) = \nabla c = 1$ and $\sim \nabla(c \wedge \sim c) = 0$, contrary to hypothesis.

Now let $\langle \mathfrak{A}, \nabla \rangle$ be any monadic De Morgan algebra that satisfies $\nabla(x \wedge \sim x) \leq \sim \nabla(x \wedge \sim x)$. Every fixed point c in $\mathfrak{A}$ corresponds to a tuple of fixed points in the subdirect representation of $\mathfrak{A}$ in which every coordinate is fixed by ∇. Hence c is fixed by ∇. □

Corollary 4.19. *Let* $\langle \mathfrak{A}, \nabla \rangle$ *be a subdirectly irreducible monadic De Morgan algebra that satisfies* $\nabla(x \wedge \sim x) \leq \sim \nabla(x \wedge \sim x)$. *Then either*

    (a) $\mathfrak{A}$ *has exactly two fixed points and* ∇ *is a quantifier of type 2,*

    (b) $\mathfrak{A}$ *is a Kleene algebra and* ∇ *is a quantifier of type 1, or*

    (c) $\mathfrak{A}$ *is a Boolean algebra and* ∇ *is a quantifier of type 0.*

Proof. Since ∇ is a quantifier of type 0, type 1, or type 2, and every fixed point in $\mathfrak{A}$ is fixed by ∇, $\mathfrak{A}$ can have at most two fixed points. Furthermore, if $\mathfrak{A}$ has two distinct fixed points, then ∇ must be a quantifier of type 2.

If $\mathfrak{A}$ has a unique fixed point c, then ∇ must be a quantifier of type 1. Suppose for the sake of a contradiction that there exists an $x \in \mathfrak{A}$ such that $x \wedge \sim x \not\leq c$. Then $\nabla(x \wedge \sim x) = 1$ and $\sim \nabla(x \wedge \sim x) = 0$, contrary to hypothesis. Therefore, for every $x, y \in \mathfrak{A}$ we have $x \wedge \sim x \leq c \leq y \vee \sim y$. Hence $\mathfrak{A}$ is a Kleene algebra.

If $\mathfrak{A}$ is not centered, then ∇ must be a quantifier of type 0. Suppose for the sake of a contradiction that there exists an $x \in \mathfrak{A}$ such that $x \wedge \sim x > 0$. Then $\nabla(x \wedge \sim x) = 1$ and

$\sim\nabla(x \wedge \sim x) = 0$, contrary to hypothesis. Therefore $\mathfrak{A}$ satisfies the equation $x \wedge \sim x = 0$. Hence $\mathfrak{A}$ is a Boolean algebra. $\qquad\square$

Let $\mathcal{M}^{\nabla_2}$ denote the variety of monadic De Morgan algebras that satisfy the equation equivalent to $\nabla(x \wedge \sim x) \leq \sim\nabla(x \wedge \sim x)$, and let $\mathcal{K}^{\nabla_1}$ denote the variety of monadic Kleene algebras that satisfy $\nabla(x \wedge \sim x) \leq \sim\nabla(x \wedge \sim x)$.

Proposition 4.20. *The following diagram is a sublattice of the lattice of subvarieties of $\mathcal{M}^{\nabla}$.*
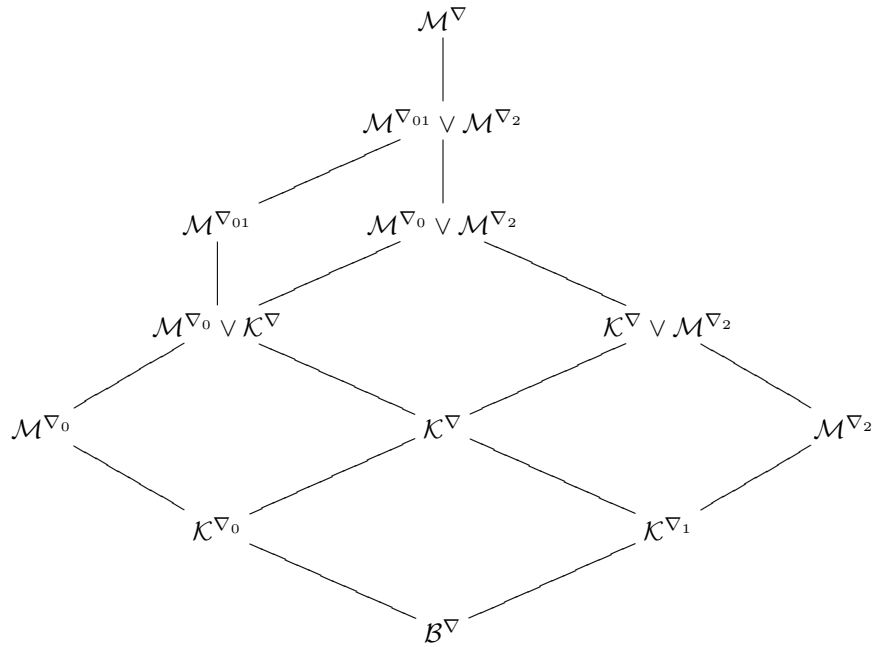
Proof. Starting at the bottom, $\mathcal{K}^{\nabla_0} \wedge \mathcal{K}^{\nabla_1} = \mathcal{B}^{\nabla}$ by Corollary 4.19, and $\mathcal{K}^{\nabla_0} \vee \mathcal{K}^{\nabla_1} = \mathcal{K}^{\nabla}$ because if $\langle \mathfrak{A}, \nabla \rangle$ is a subdirectly irreducible algebra in $\mathcal{K}^{\nabla}$, then either ∇ is of type 0 or 1. In the first case $\langle \mathfrak{A}, \nabla \rangle \in \mathcal{K}^{\nabla_0}$. In the second case, let a be the fixed point of $\mathfrak{A}$. Then for any $x \in \mathfrak{A}$, $x \wedge \sim x \leq a$, so $\nabla(x \wedge \sim x) \leq a \leq \sim\nabla(x \wedge \sim x)$. Hence $\langle \mathfrak{A}, \nabla \rangle \in \mathcal{K}^{\nabla_1}$.

Moving up and to the right, $\mathcal{K}^{\nabla} \wedge \mathcal{M}^{\nabla_2} = \mathcal{K}^{\nabla_1}$ by definition. The monadic Kleene algebra $\langle \mathbf{K}, \nabla_0 \rangle$ belongs to $\mathcal{K}^{\nabla} \setminus \mathcal{M}^{\nabla_2}$, while $\langle \mathbf{M}, \nabla_2 \rangle$ belongs to $\mathcal{M}^{\nabla_2} \setminus K^{\nabla}$, so neither $\mathcal{K}^{\nabla} \subseteq \mathcal{M}^{\nabla_2}$ nor $\mathcal{M}^{\nabla_2} \subseteq \mathcal{K}^{\nabla}$.

Moving left, $\mathcal{M}^{\nabla_0} \wedge \mathcal{K}^{\nabla} = \mathcal{K}^{\nabla_0}$ by definition. Also, $\langle \mathbf{M}, \nabla_0 \rangle \in \mathcal{M}^{\nabla_0} \setminus \mathcal{K}^{\nabla}$, and $\langle \mathbf{K}, \nabla_1 \rangle \in \mathcal{K}^{\nabla} \setminus \mathcal{M}^{\nabla_0}$. However, $\mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla} \subset \mathcal{M}^{\nabla_{01}}$ by Proposition 4.17.

Before moving right it will be useful to note that by Corollary 4.19, $\mathcal{M}^{\nabla_0} \wedge \mathcal{M}^{\nabla_2} = \mathcal{B}^{\nabla}$ and $\mathcal{M}^{\nabla_{01}} \wedge \mathcal{M}^{\nabla_2} = \mathcal{K}^{\nabla_1}$. Also, $\langle \underline{\mathbf{M}}, \nabla_0 \rangle \in \mathcal{M}^{\nabla_0} \setminus \mathcal{M}^{\nabla_2}$, while $\langle \underline{\mathbf{M}}, \nabla_2 \rangle \in \mathcal{M}^{\nabla_2} \setminus \mathcal{M}^{\nabla_{01}}$. Thus

$$(\mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla}) \vee (\mathcal{K}^{\nabla} \vee \mathcal{M}^{\nabla_2}) = \mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla} \vee \mathcal{M}^{\nabla_2}$$

$$= \mathcal{M}^{\nabla_0} \vee (\mathcal{K}^{\nabla_0} \vee \mathcal{K}^{\nabla_1}) \vee \mathcal{M}^{\nabla_2}$$

$$= (\mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla_0}) \vee (\mathcal{K}^{\nabla_1} \vee \mathcal{M}^{\nabla_2})$$

$$= \mathcal{M}^{\nabla_0} \vee \mathcal{M}^{\nabla_2}.$$

By Corollary 4.15,

$$\mathcal{M}^{\nabla_{01}} \wedge (\mathcal{M}^{\nabla_0} \vee \mathcal{M}^{\nabla_2}) = (\mathcal{M}^{\nabla_{01}} \wedge \mathcal{M}^{\nabla_0}) \vee (\mathcal{M}^{\nabla_{01}} \wedge \mathcal{M}^{\nabla_2})$$

$$= \mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla_1}$$

$$= \mathcal{M}^{\nabla_0} \vee \mathcal{K}^{\nabla}.$$

Also, $\mathcal{M}^{\nabla_{01}} \vee (\mathcal{M}^{\nabla_0} \vee \mathcal{M}^{\nabla_2}) = (\mathcal{M}^{\nabla_{01}} \vee \mathcal{M}^{\nabla_0}) \vee \mathcal{M}^{\nabla_2} = \mathcal{M}^{\nabla_{01}} \vee \mathcal{M}^{\nabla_2}$.

Finally to show $\mathcal{M}^{\nabla_{01}} \vee \mathcal{M}^{\nabla_2} \subset \mathcal{M}^{\nabla}$ let a and b be the fixed points of $\underline{\mathbf{M}}$, and let $\mathfrak{A}$ be the following subalgebra of $\underline{\mathbf{M}} \times \underline{\mathbf{M}}$:
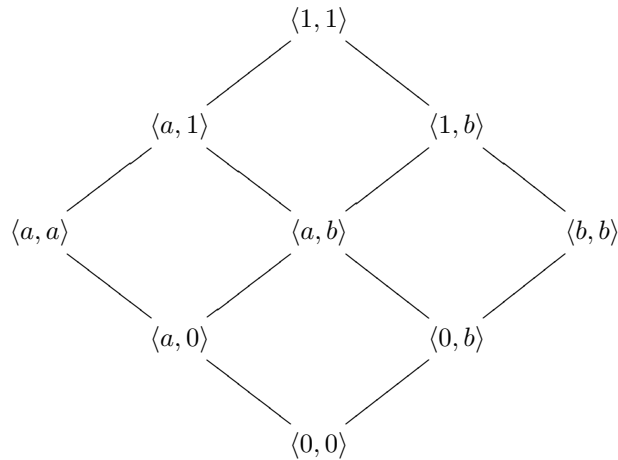
Observe that $\langle a, a \rangle$ and $\langle b, b \rangle$ are fixed points in $\mathfrak{A}$ such that $\langle a, a \rangle \wedge \langle b, b \rangle = \langle 0, 0 \rangle$ and $\langle a, a \rangle \vee \langle b, b \rangle = \langle 1, 1 \rangle$. Let ∇ be the type 2 quantifier whose range is $\{\langle 0, 0 \rangle, \langle a, a \rangle, \langle b, b \rangle, \langle 1, 1 \rangle\}$. Then $\langle \mathfrak{A}, \nabla \rangle$ is a monadic De Morgan algebra that does not belong to $\mathcal{M}^{\nabla_{01}} \cup \mathcal{M}^{\nabla_2}$, on the one hand because ∇ is a quantifier of type 2, and on the other hand because $\langle a, b \rangle$ is a fixed point that

is not fixed by ∇. In order to apply Lemma 4.14 we need to show that $\langle \mathfrak{A}, \nabla \rangle$ is subdirectly irreducible. In fact, $\langle \mathfrak{A}, \nabla \rangle$ is simple. We perform three representative calculations. Let $\equiv$ be a congruence on $\langle \mathfrak{A}, \nabla \rangle$. If $\langle 0, 0 \rangle \equiv \langle a, 0 \rangle$, then $\langle 0, 0 \rangle = \nabla \langle 0, 0 \rangle \equiv \nabla \langle a, 0 \rangle = \nabla \langle a, a \rangle = \langle a, a \rangle$, in which case $\langle 0, 0 \rangle \equiv \langle a, a \rangle = \sim \langle a, a \rangle \equiv \langle 1, 1 \rangle$. Hence $\equiv$ is the total congruence. Similarly, if $\langle 0, 0 \rangle \equiv \langle 0, b \rangle$, then $\equiv$ is the total congruence. If $\langle a, 0 \rangle \equiv \langle a, a \rangle$, then $\langle a, b \rangle \equiv \langle a, 1 \rangle$, and $\langle 1, b \rangle \equiv \langle 1, 1 \rangle$. Hence $\langle 0, 0 \rangle = \sim \langle 1, 1 \rangle \equiv \sim \langle 1, b \rangle = \langle 0, b \rangle$, so $\equiv$ is the total congruence. If $\langle a, a \rangle \equiv \langle a, b \rangle$, then $\langle a, a \rangle = \nabla \langle a, a \rangle \equiv \nabla \langle a, b \rangle = \langle 1, 1 \rangle$, so $\equiv$ is the total congruence. Thus $\langle \mathfrak{A}, \nabla \rangle$ is simple. Therefore $\langle \mathfrak{A}, \nabla \rangle$ does not belong to $\mathcal{M}^{\nabla_{01}} \vee \mathcal{M}^{\nabla_2}$. $\qquad \square$

4.3 The monadic De Morgan reduct

Let X be a one-dimensional pair of suits. It follows from Corollary 2.57 that

$$
C_{0,\{0\}}(X) = \begin{cases} 1 & \text{if } X \not\leq \Omega \text{ and } X \not\leq \mho, \\[2mm] \Omega & \text{if } 0 < X \leq \Omega, \\[2mm] \mho & \text{if } 0 < X \leq \mho, \\[2mm] 0 & \text{if } X = 0. \end{cases}
$$

If X is a double suit, then

$$
C_{0,\{0\}}(X) = \begin{cases} 1 & \text{if } X \not\leq \Omega, \\[2mm] \Omega & \text{if } 0 < X \leq \Omega, \\[2mm] 0 & \text{if } X = 0. \end{cases}
$$

Thus the reduct of a suited IFG$_1$-cylindric set algebra that includes Ω and $\mho$ to the signature $\langle 0, 1, {}^{\cup}, +_{\{0\}}, \cdot_{\{0\}}, C_{0,\{0\}} \rangle$ is a monadic De Morgan algebra with a quantifier of type 2, while the same reduct of a double-suited IFG$_1$-cylindric set algebra that includes Ω is a monadic Kleene algebra with a quantifier of type 1. We will refer to the reduct of a suited IFG$_1$-cylindric set algebra $\mathfrak{C}$ to the signature

$$
\langle 0, 1, {}^{\cup}, +_{\{0\}}, \cdot_{\{0\}}, C_{0,\{0\}} \rangle
$$

as the **monadic De Morgan reduct** of $\mathfrak{C}$. It follows that the variety generated by the monadic

De Morgan reducts of all suited IFG$_1$-cylindric set algebras is contained in $\mathcal{M}^\nabla$.

Proposition 4.21. *If X is a double suit, then*

$$C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X \cdot_N X^\cup) \le C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X \cdot_N X^\cup)^\cup.$$

Proof. If X is a double suit, then $0 \le X \cdot_N X^\cup \le \Omega$. If $X \cdot_N X^\cup = 0$, then

$$C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X \cdot_N X^\cup) = 0 \quad \text{and} \quad C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X \cdot_N X^\cup)^\cup = 1.$$

If $0 < X \cdot_N X^\cup \le \Omega$, then

$$C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X \cdot_N X^\cup) = \Omega = C_{0,J_0} \ldots C_{N-1,J_{N-1}}(X \cdot_N X^\cup)^\cup. \qquad \square$$

In particular, in the one-dimensional case $C_{0,\{0\}}(X \cdot_{\{0\}} X^\cup) \le C_{0,\{0\}}(X \cdot_{\{0\}} X^\cup)^\cup$. Thus

the variety generated by the monadic De Morgan reducts of all double-suited IFG$_1$-cylindric set

algebras is contained in $\mathcal{K}^{\nabla_1}$.

Conjecture. The variety generated by the monadic De Morgan reducts of all double-suited

IFG$_1$-cylindric set algebras is $\mathcal{K}^{\nabla_1}$.

To prove the conjecture it would be sufficient to show that every monadic Kleene algebra

in $\mathcal{K}^{\nabla_1}$ is a homomorphic image of a subalgebra of a product of De Morgan reducts of double-

suited IFG$_1$-cylindric set algebras. By Corollary 4.19 it suffices to consider monadic Kleene

algebras with type 1 quantifiers and monadic Boolean algebras with type 0 quantifiers.

Definition. Let L be a lattice. An element $a \in L$ is **join irreducible** if $a = x \vee y$ implies $a = x$

or $a = y$. Dually, $b \in L$ is **meet irreducible** if $b = x \wedge y$ implies $b = x$ or $b = y$.

Proposition 4.22. *In any De Morgan reduct of a suited IFG$_N$-cylindric set algebra 0 is meet*

irreducible and 1 is join irreducible.

Proof. Let $\mathfrak{C}$ be a the De Morgan reduct of a double-suited IFG$_N$-cylindric set algebra, and let

$X, Y \in \mathfrak{C}$. If $X +_N Y = 1$, then $^N A \in (X +_N Y)^+ = X^+ \cup Y^+$. If $^N A \in X^+$ then $X = 1$, and if

$^N A \in Y^+$ then $Y = 1$. $\qquad \square$

It follows that not every Kleene algebra is isomorphic to a double-suited IFG_1-cylindric set algebra. For example, let K be the following centered Kleene algebra with fixed point c:
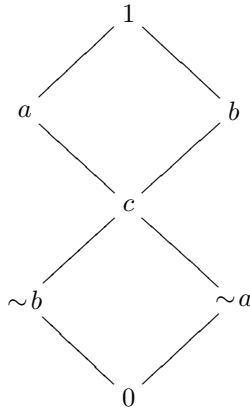
Observe that 0 is not meet irreducible, and 1 is not join irreducible, so K cannot be embedded into any suited IFG_N-cylindric set algebra. Perhaps every monadic Kleene algebra with a type 1 quantifier in which 0 is meet irreducible and 1 is join irreducible is isomorphic to the De Morgan reduct of a double-suited IFG_1-cylindric set algebra. However, my advisor and I have only been able to prove the following partial result.

Theorem 4.23. *Every monadic Kleene algebra with a quantifier of type 1 in which 0 is meet irreducible and 1 is join irreducible is isomorphic to the De Morgan reduct of a rooted IFG_1-cylindric set algebra.*

Proof. Let $\langle K, \nabla \rangle$ be a monadic Kleene algebra with a quantifier of type 1 in which 0 is meet irreducible and 1 is join irreducible. Let c be the fixed point of K, and consider the interval $L = [c, 1]$ as a bounded distributive lattice with minimum c and maximum 1. Let A be the set of prime filters on L, and let σ be the Priestley isomorphism $x \mapsto \{ F \in A \mid x \in F \}$. Since 1 is join irreducible we have that $\{1\}$ is a prime filter, and for all $x \in L$, $\{1\} \in \sigma(x)$ if and only if $x = 1$.

There is a partition P of $\mathscr{P}(A)$ such that $|P| = |A|$, every cell of the partition includes a singleton, but no two singletons belong to the same cell. Let $f \colon A \to P$ be a bijection such that

for all $a \in A$, $\{a\} \in f(a)$, and $A \in f(\{1\})$. Define $g \colon \mathscr{P}(A) \to \mathscr{P}(\mathscr{P}(A)) \setminus \{\emptyset\}$ by

$$g(U) = \{\emptyset\} \cup \bigcup_{a \in U} f(a).$$

If $U \neq V$, then without loss of generality there is a $b \in U \setminus V$. Consequently $f(b) \subseteq g(U) \setminus g(V)$. Since $f(b)$ is nonempty, $g(U) \neq g(V)$. Hence g is injective. Note that there is a singleton $\{a\} \in g(U)$ if and only if U is nonempty, and $A \in g(U)$ if and only if $\{1\} \in U$.

Let $G = g \circ \sigma$. Note that

$$G(c) = g(\sigma(c)) = g(\emptyset) = \{\emptyset\},$$

$$G(1) = g(\sigma(1)) = g(A) = \mathscr{P}(A),$$

$$G(x \vee y) = g(\sigma(x \vee y)) = g(\sigma(x) \cup \sigma(y)) = g(\sigma(x)) \cup g(\sigma(y)) = G(x) \cup G(y).$$

To prove $G(x \wedge y) = G(x) \cap G(y)$ we need to show that

$$\{\emptyset\} \cup \bigcup_{a \in \sigma(x) \cap \sigma(y)} f(a) = \left(\{\emptyset\} \cup \bigcup_{a \in \sigma(x)} f(a) \right) \cap \left(\{\emptyset\} \cup \bigcup_{b \in \sigma(y)} f(b) \right).$$

Suppose u is a nonempty member of the left-hand side. Then for some $a \in \sigma(x) \cap \sigma(y)$, $u \in f(a)$. That is, there is a prime filter a such that $x, y \in a$ and $u \in f(a)$, which is enough to show that u belongs to the right-hand side. Conversely, suppose v is a nonempty member of the right-hand side. Then for some $a \in \sigma(x)$ and some $b \in \sigma(y)$, $v \in f(a)$ and $v \in f(b)$. However, if a and b were distinct, $f(a)$ and $f(b)$ would be disjoint. Thus $a = b$, and v belongs to the left-hand side.

The function G is injective because g and σ are. Also note that there is a singleton $\{a\} \in G(x)$ whenever $x > c$, and $A \in G(x)$ if and only if $\{1\} \in \sigma(x)$ if and only if $x = 1$.

Define a function h from $\langle K, \nabla \rangle$ to the monadic De Morgan reduct of $\mathrm{Root}_1(A)$ by $h(x) = \langle G(x \vee c), G(\sim x \vee c) \rangle$. If $x \neq y$, then either $x \vee c \neq y \vee c$ or $\sim x \vee c \neq \sim y \vee c$ because in a distributive lattice $x \vee c = y \vee c$ and $x \wedge c = y \wedge c$ imply $x = y$. Thus h is injective.

To show that h is a homomorphism, observe that

$$h(0) = \langle G(0 \vee c), G(1 \vee c) \rangle = \langle G(c), G(1) \rangle = \langle \{\emptyset\}, \mathscr{P}(A) \rangle = 0,$$

$$h(1) = \langle G(1 \vee c), G(0 \vee c) \rangle = \langle G(1), G(c) \rangle = \langle \mathscr{P}(A), \{\emptyset\} \rangle = 1,$$

$$h(\sim x) = \langle G(\sim x \vee c), G(x \vee c)\rangle = h(x)^{\cup},$$

$$h(x \vee y) = \langle G((x \vee y) \vee c), G(\sim(x \vee y) \vee c)\rangle$$

$$= \langle G((x \vee c) \vee (y \vee c)), G((\sim x \vee c) \wedge (\sim y \vee c))\rangle$$

$$= \langle G(x \vee c) \cup G(y \vee c), G(\sim x \vee c) \cap G(\sim y \vee c)\rangle$$

$$= h(x) +_{\{0\}} h(y).$$

Now we check the quantifier:

$$h(\nabla 0) = h(0) = 0 = C_{0,\{0\}}(0) = C_{0,\{0\}}(h(0)).$$

If $0 < x \leq c$, then $x \vee c = c$ and $\sim x \vee c < 1$. Hence $G(x \vee c) = \{\emptyset\}$ and $A \notin G(\sim x \vee c)$. Thus

$$h(\nabla x) = h(c) = \langle G(c), G(c)\rangle = \Omega = C_{0,\{0\}}(\langle G(x \vee c), G(\sim x \vee c)\rangle) = C_{0,\{0\}}(h(x)).$$

If $x \not\leq c$, then $x \vee c > c$ and $\sim x \vee c < 1$. Hence there is a singleton $\{a\} \in G(x \vee c)$ and $A \notin G(\sim x \vee c)$. Thus

$$h(\nabla x) = h(1) = 1 = C_{0,\{0\}}(\langle G(x \vee c), G(\sim x \vee c)\rangle) = C_{0,\{0\}}(h(x)).$$

by Lemma 2.56. Therefore h is an embedding. $\qquad\square$

Note that Theorem 4.23 does not resolve the conjecture because the elements in the range of h are not double suits (or even pairs of suits).

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Appendix A

Index of Symbols

Appendix B

Index of Words