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Cylindric Set Algebras and IF Logic*

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Independence-friendly logic (IF logic) [12, 13] is a conservative extension of first-order logic that can be viewed as a generalization of Henkin's branching quantifiers [8]. For example, in the branching quantifier sentence

$$\begin{pmatrix} \forall v_0 \, \exists v_1 \\ \forall v_2 \, \exists v_3 \end{pmatrix} \varphi(v_0, v_1, v_2, v_3), \tag{2.0.1}$$

the rows of the branching quantifier indicate that v_1 depends (only) on v_0, while v_3 depends (only) on v_2. It is a result due to Ehrenfeucht that in general such sentences cannot be expressed in ordinary first-order logic [8]. However, sentence (2.0.1) can be expressed in IF logic as

$$\forall v_0 \, \exists v_1 \, \forall v_{2/\{v_0,v_1\}} \, \exists v_{3/\{v_0,v_1\}} \varphi(v_0, v_1, v_2, v_3), \tag{2.0.2}$$

where the slashes indicate that v_2 and v_3 do not depend on v_0 or v_1. For our purposes, it will be more convenient to work with dependence-friendly logic (DF logic), in which we indicate on which variables each quantifier is dependent rather than independent. For example, in DF logic we would write sentence (2.0.1) as

$$\forall v_{0\backslash\emptyset} \, \exists v_{1\backslash\{v_0\}} \, \forall v_{2\backslash\emptyset} \, \exists v_{3\backslash\{v_2\}} \varphi(v_0, v_1, v_2, v_3), \tag{2.0.3}$$

where the backslashes indicate that v_0 and v_2 do not depend on any other variables, v_1 depends on v_0, and v_3 depends on v_2.

Although Henkin mentions semantic games, he chooses not to use them to interpret branching quantifiers. Instead, the meaning of a branching

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quantifier sentence is determined by its Skolemization. Sentence (2.0.1) is true if and only if there exist functions f and g satisfying

$$\forall v_0 \forall v_2 \varphi\big(v_0, f(v_0), v_2, g(v_2)\big)\,,$$

that is, if and only if the Σ^1_1 sentence

$$\exists f \exists g \forall v_0 \forall v_2 \varphi\big(v_0, f(v_0), v_2, g(v_2)\big) \tag{2.0.4}$$

is true. In contrast, the meaning of a DF sentence is defined via a game between two players, Eloïse ($\exists$) and Abélard ($\forall$). Given a structure $\mathfrak{A}$ and a DF sentence φ, Eloïse attempts to verify φ by picking elements of the universe to be the values of existentially quantified variables. For disjunctions, she chooses which disjunct to verify. Dually, Abélard attempts to falsify φ by picking the values of universally quantified variables and the conjuncts he wishes to falsify. The game ends when the players reach an atomic formula. Eloïse wins if the final assignment satisfies the atomic formula, and Abélard wins if it does not. The sentence is true if Eloïse has a winning strategy, and it is false if Abélard has a winning strategy.

Unlike the game for ordinary first-order logic, the semantic game for DF logic is a game of imperfect information. Just as the different rows of a branching quantifier indicate on which variables the Skolem functions should depend, the backslashes on the quantifiers (and connectives) of a DF sentence indicate the values of which variables the players are aware as they play. For example, in the semantic game for the sentence

$$\forall v_{0\backslash\emptyset} \exists v_{1\backslash\{v_0\}} R v_0 v_1 \tag{2.0.5}$$

Eloïse is allowed to see the value of v_0 when choosing v_1. Hence she has a winning strategy in any structure $\mathfrak{A}$ whose interpretation of R contains a function; otherwise Abélard has a winning strategy. In contrast, in the semantic game for

$$\forall v_{0\backslash\emptyset} \exists v_{1\backslash\emptyset} R v_0 v_1 \tag{2.0.6}$$

Eloïse must choose v_1 without knowing the value of v_0. Hence (2.0.6) is true if $R^{\mathfrak{A}}$ is the total relation, false if $R^{\mathfrak{A}}$ does not contain a function, and undetermined otherwise.

The major difference between first-order logic with branching quantifiers and DF logic is their treatment of negation. Enderton [6] and Walkoe [25] proved independently that every branching quantifier sentence is equivalent to a Σ^1_1 sentence, and vice versa. Since the contradictory negation of a Σ^1_1 sentence is in general a Π^1_1 sentence, the contradictory negation of a branching quantifier sentence is not always expressible as a branching quantifier sentence. The same is true for DF sentences. However, there is a natural way to interpret negation as the role-reversal of the players. Eloïse attempts to verify $\sim \varphi$ by falsifying φ, while Abélard attempts to falsify $\sim \varphi$ by verifying φ. Thus $\sim \varphi$ is true if and only if φ is false, $\sim \varphi$ is false if and only if φ is true, and $\sim \varphi$ is undetermined if and only if φ is. This suggests that the propositional logic underlying DF logic is Kleene's strong three-valued logic, which we will prove below.

The fact that Eloïse has a winning strategy for $G(\mathfrak{A}, \varphi)$ is expressible by a Σ^1_1 sentence. For example, in (2.0.4) the functions f and g encode Eloïse's winning strategy for (2.0.3). Likewise, the fact that Abélard has a winning strategy. Thus we can treat one Σ^1_1 sentence as the truth-condition for φ, and another as its falsity-condition. Hence a DF sentence has the same expressive power as a pair of contrary Σ^1_1 sentences. Moreover, working with branching quantifiers, Burgess [2] proved that every pair of contrary Σ^1_1 sentences can be viewed as the truth- and falsity-conditions for some DF sentence.

1. Dependence-Friendly Logic

To simplify our notation, we assume that all variables come from a fixed set $\{v_n : n < \alpha\}$ for some ordinal α.

Definition 2.1.1. Given a first-order signature σ, $\mathscr{L}^{\mathrm{DF}_\alpha}(\sigma)$ is the language generated by the following grammar:

$$\varphi := \Phi \mid \sim \varphi \mid \{\varphi \vee_J \psi\}_{J \subseteq \alpha} \mid \{\exists\, v_{n \backslash J} \varphi\}_{J \subseteq \alpha}$$

where Φ consists of all atomic σ-formulas. An element of $\mathscr{L}^{\mathrm{DF}_\alpha}(\sigma)$ is called a DF_α *formula*. When the signature is implicit, we will simply write $\mathscr{L}^{\mathrm{DF}_\alpha}$.

We adopt the standard abbreviations of $\varphi \wedge_J \psi$ for $\sim (\sim \varphi \vee_J \sim \psi)$ and $\forall\, v_{n \backslash J} \varphi$ for $\sim \exists\, v_{n \backslash J} \sim \varphi$. We suppress the backslash symbol on connectives for aesthetic reasons, and we are content to keep track of which variables are in each dependence set by their indices.

We must extend the game-theoretical semantics for DF sentences presented in the introduction to all DF formulas. To do so, we need a way to keep track of the information available to the players at each position of the semantic game. In ordinary first-order logic, it is sufficient to use a single assignment to encode the values of the variables. In DF logic, we must use sets of assignments, called *teams*, to encode the information available to the players. Given a structure $\mathfrak{A}$, a DF_α formula φ, and a team $V \subseteq {}^\alpha A$, the semantic game $G(\mathfrak{A}, \varphi, V)$ is played as usual except that at the beginning of the game a third player, whom we will call Nature, randomly chooses an assignment $\vec{a} \in V$ which is used to assign values to the free variables in φ, the bound variables in φ, and even the variables that do not appear in φ.

Definition 2.1.2. Let $\mathfrak{A}$ be a structure, φ a DF_α formula, and $V, W \subseteq {}^\alpha A$.

(+) $\mathfrak{A} \models^+_V \varphi$ if and only if Eloïse has a winning strategy for $G(\mathfrak{A}, \varphi, V)$.

(−) $\mathfrak{A} \models^-_W \varphi$ if and only if Abélard has a winning strategy for $G(\mathfrak{A}, \varphi, W)$.

In the first case we say that V is a *winning team*[1] for φ in $\mathfrak{A}$, and in the second case we say that W is a *losing team* for φ in $\mathfrak{A}$. We write $\mathfrak{A} \models^+ \varphi$ when $\mathfrak{A} \models^+_{{}^\alpha A} \varphi$, and write $\mathfrak{A} \models^- \varphi$ when $\mathfrak{A} \models^-_{{}^\alpha A} \varphi$. The previous statement can be summarized by saying $\mathfrak{A} \models^\pm \varphi$ if and only if $\mathfrak{A} \models^\pm_{{}^\alpha A} \varphi$.

For a fuller treatment of game-theoretical semantics in the context of IF logic, see [20, Section 1.2].

Game-theoretical semantics gives us an intuitive definition of truth for DF formulas. However, it is not compositional, which makes it unsuitable for proving theorems by induction. Luckily, Wilfrid Hodges discovered a compositional semantics for IF logic called *trump semantics* that we can adapt to DF logic [15, 16].

Definition 2.1.3. Two assignments $\vec{a}, \vec{b} \in {}^\alpha A$ are *indistinguishable on J*, denoted $\vec{a} \approx_J \vec{b}$, if $\vec{a} \upharpoonright J = \vec{b} \upharpoonright J$.

Definition 2.1.4. Define a partial operation $\cup_J$ on pairs of teams by declaring $V_1 \cup_J V_2 = V_1 \cup V_2$ whenever

- $V_1 \cap V_2 = \emptyset$,
- if $\vec{a} \in V_1$ and $\vec{b} \in V_2$, then $\vec{a} \not\approx_J \vec{b}$,

[1] Hodges [15, 16] calls a winning team a *trump*, and a losing team a *cotrump*. The term *team* was introduced by Väänänen [24].

and letting $V_1 \cup_J V_2$ be undefined otherwise. Thus the formula $V = V_1 \cup_J V_2$ asserts that $\{V_1, V_2\}$ is a pair of disjoint teams that cover V in such a way that assignments that are indistinguishable on J are always in the same cell.

Definition 2.1.5. A function $f : V \to A$ is *determined by* J, denoted $f : V \underset{J}{\to} A$, if $f(\vec{a}) = f(\vec{b})$ whenever $\vec{a} \approx_J \vec{b}$.

Definition 2.1.6. The assignment $\vec{a}(n : b)$ is defined by

$$\vec{a}(n : b)_i = \begin{cases} b & \text{if } \ i = n, \\ a_i & \text{otherwise.} \end{cases}$$

Furthermore, if $V \subseteq {}^{\alpha}A$ and $f : V \to A$,

$$V(n : f) = \left\{ \vec{a}\big(n : f(\vec{a})\big) : \vec{a} \in V \right\},$$

$$V(n : A) = \left\{ \vec{a}(n : b) : \vec{a} \in V, \ b \in A \right\}.$$

Theorem 2.1.7 ([15, Theorem 7.5], see also [5, Theorem 5.3.5])**.** *Let $\mathfrak{A}$ be a structure, let φ be a* DF_α *formula, and let $V, W \subseteq {}^{\alpha}A$.*

- *If φ is atomic, then*
  - (+) $\mathfrak{A} \models^+_V \varphi$ *if and only if* $\mathfrak{A} \models_{\vec{a}} \varphi$ *for all* $\vec{a} \in V$,
  - (−) $\mathfrak{A} \models^-_W \varphi$ *if and only if* $\mathfrak{A} \not\models_{\vec{b}} \varphi$ *for all* $\vec{b} \in W$.
- *If φ is $\sim \psi$, then* $\mathfrak{A} \models^\pm_V \sim \psi$ *if and only if* $\mathfrak{A} \models^\mp_V \psi$.
- *If φ is $\psi_1 \vee_J \psi_2$, then*
  - (+) $\mathfrak{A} \models^+_V \psi_1 \vee_J \psi_2$ *if and only if* $\mathfrak{A} \models^+_{V_1} \psi_1$ *and* $\mathfrak{A} \models^+_{V_2} \psi_2$ *for some* $V = V_1 \cup_J V_2$,
  - (−) $\mathfrak{A} \models^-_W \psi_1 \vee_J \psi_2$ *if and only if* $\mathfrak{A} \models^-_W \psi_1$ *and* $\mathfrak{A} \models^-_W \psi_2$.
- *If φ is $\exists v_{n \setminus J} \psi$, then*
  - (+) $\mathfrak{A} \models^+_V \exists v_{n \setminus J} \psi$ *if and only if* $\mathfrak{A} \models^+_{V(n:f)} \psi$ *for some* $f : V \underset{J}{\to} A$,
  - (−) $\mathfrak{A} \models^-_W \exists v_{n \setminus J} \psi$ *if and only if* $\mathfrak{A} \models^-_{W(n:A)} \psi$.

Dependence-friendly formulas have two important properties that reflect the fact that it is impossible for both players to have winning strategies for the same game. Also, if one player has a winning strategy given a certain amount of information, then the same player has a winning strategy given more information.

Proposition 2.1.8 ([16, page 57], see also [5, Lemma 6.2.1]). *Let* $V \subseteq {}^{\alpha}A$.

(i) $\mathfrak{A} \models^{+}_{V} \varphi$ *and* $\mathfrak{A} \models^{-}_{V} \varphi$ *if and only if* $V = \emptyset$.

(ii) *If* $V' \subseteq V$, *then* $\mathfrak{A} \models^{\pm}_{V} \varphi$ *implies* $\mathfrak{A} \models^{\pm}_{V'} \varphi$.

2. Dependence-Friendly Cylindric Set Algebras

The *meaning* of a DF_α formula φ in a structure $\mathfrak{A}$ is a pair whose first coordinate is the set of winning teams for the formula, and whose second coordinate is the set of losing teams. That is, $\|\varphi\|_{\mathfrak{A}} = \left\langle \|\varphi\|^{+}_{\mathfrak{A}}, \|\varphi\|^{-}_{\mathfrak{A}} \right\rangle$, where

$$\|\varphi\|^{+}_{\mathfrak{A}} = \{V \subseteq {}^{\alpha}A : \mathfrak{A} \models^{+}_{V} \varphi\},$$

$$\|\varphi\|^{-}_{\mathfrak{A}} = \{W \subseteq {}^{\alpha}A : \mathfrak{A} \models^{-}_{W} \varphi\}.$$

Analogous to Definition 4.3.4 in [10], the universe of the α-dimensional dependence-friendly cylindric set algebra over $\mathfrak{A}$ consists of the meanings of all the DF_α formulas expressible in the language of $\mathfrak{A}$. In symbols,

$$\mathsf{Cs}_{\mathrm{DF}_\alpha}(\mathfrak{A}) = \left\{ \|\varphi\|_{\mathfrak{A}} : \varphi \in \mathscr{L}^{\mathrm{DF}_\alpha} \right\}.$$

We then define constants 1, 0, and diagonal elements D_{ij}, as well as operations ${}^{\cup}$, $+_J$, $\cdot_J$, and $C_{n,J}$ corresponding to $\top$, $\bot$, $v_i = v_j$, $\sim$, $\vee_J$, $\wedge_J$, and $\exists\, v_{n\setminus J}$, respectively (see Definition 2.2.1). We denote the *α-dimensional dependence-friendly cylindric set algebra over* $\mathfrak{A}$ by $\mathfrak{Cs}_{\mathrm{DF}_\alpha}(\mathfrak{A})$. As in the case of ordinary cylindric set algebras, we can define dependence-friendly cylindric set algebras without reference to a base structure $\mathfrak{A}$.

Definition 2.2.1. A *dependence-friendly cylindric power set algebra* is an algebra whose universe is $\mathscr{P}\big(\mathscr{P}({}^{\alpha}U)\big) \times \mathscr{P}\big(\mathscr{P}({}^{\alpha}U)\big)$, where U is a set called the *base,* and α is an ordinal called the *dimension* of the algebra. Every element X of a dependence-friendly cylindric power set algebra is an ordered pair of sets of teams. We will use the notation X^+ to refer to the first coordinate of the pair, and X^- to refer to the second coordinate. There are a finite number of operations:

- the constant $1 = \left\langle \mathscr{P}({}^\alpha U), \{\emptyset\} \right\rangle$;
- the constant $0 = \left\langle \{\emptyset\}, \mathscr{P}({}^\alpha U) \right\rangle$;
- for all $i, j < \alpha$, the constant D_{ij} is defined by
  (+) $D_{ij}^+ = \mathscr{P}\big(\{\vec{a} \in {}^\alpha U : a_i = a_j\}\big)$,
  (−) $D_{ij}^- = \mathscr{P}\big(\{\vec{a} \in {}^\alpha U : a_i \neq a_j\}\big)$;
- if $X = \langle X^+, X^- \rangle$, then $X^\cup = \langle X^-, X^+ \rangle$;
- for every $J \subseteq \alpha$, the binary operation $+_J$ is defined by
  (+) $V \in (X +_J Y)^+$ if and only if $V = V_1 \cup_J V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$,
  (−) $(X +_J Y)^- = X^- \cap Y^-$;
- for every $J \subseteq \alpha$, the binary operation $\cdot_J$ is defined by
  (+) $(X \cdot_J Y)^+ = X^+ \cap Y^+$,
  (−) $W \in (X \cdot_J Y)^-$ if and only if $W = W_1 \cup_J W_2$ for some $W_1 \in X^-$ and $W_2 \in Y^-$;
- for every $n < \alpha$ and $J \subseteq \alpha$, the unary operation $C_{n,J}$ is defined by
  (+) $V \in C_{n,J}(X)^+$ if and only if $V(n : f) \in X^+$ for some $f : V \underset{J}{\rightarrow} U$,
  (−) $W \in C_{n,J}(X)^-$ if and only if $W(n : U) \in X^-$.

Definition 2.2.2. A *dependence-friendly cylindric set algebra* (or DF *algebra*) is any subalgebra of a dependence-friendly cylindric power set algebra. A DF_α *cylindric set algebra* (or DF_α *algebra*) is a DF cylindric set algebra of dimension α.

Independence-friendly cylindric set algebras (IF algebras) were defined in [20, Definition 2.1].

In view of Proposition 2.1.8, not every element of a dependence-friendly cylindric power set algebra can be the meaning of a DF formula. Only those elements X with the property that $X^+ \cap X^- = \{\emptyset\}$, and $V' \subseteq V \in X^\pm$

implies $V' \in X^{\pm}$ can be meanings. Such elements are called *double suits* [3, Section 3]. A *double-suited* DF *algebra* is one in which every element is a double suit.

For the rest of the paper we will focus on double-suited algebras. The reader should verify that the DF algebra generated by a set of double suits is a double-suited algebra. A similar phenomenon occurs with ordinary cylindric set algebras. The meaning $\varphi^{\mathfrak{A}}$ of an ordinary first-order formula has the property that for all $\vec{a} \in \varphi^{\mathfrak{A}}$ and $\vec{b} \in {}^{\alpha}A$, if $\vec{b}$ agrees with $\vec{a}$ on the free variables of φ, then $\vec{b} \in \varphi^{\mathfrak{A}}$. An element of a cylindric set algebra with the corresponding property is called *regular*, where the set of free variables is replaced by the dimension set of the element. A cylindric set algebra is *regular* if all its elements are regular [10, Definition 3.1.1(viii)], and the cylindric set algebra generated by a set of regular elements with finite dimension sets is regular [10, Corollary 3.1.64]. See also [11, pages 145–149].

3. The Perfect Subreduct

Naturally, one wonders about the relationship between ordinary cylindric set algebras and their dependence-friendly brethren. Every element of an ordinary cylindric set algebra has the form $V \subseteq {}^{\alpha}U$, while every double suit has the form

$$\left\langle \bigcup_{i<\beta} \mathscr{P}(V_i), \bigcup_{j<\gamma} \mathscr{P}(W_j) \right\rangle,$$

for some $V_i, W_j \subseteq {}^{\alpha}U$ such that $V_i \cap W_j = \emptyset$ for all i, j. Furthermore, we may assume that $\{V_i : i < \beta\}$ and $\{W_j : j < \gamma\}$ are antichains when partially ordered by $\subseteq$. Double suits of the form $\langle \mathscr{P}(V), \mathscr{P}({}^{\alpha}U \setminus V) \rangle$ are called *perfect* because they are the meanings of those DF formulas whose semantic game is one of perfect information. There is a natural mapping

$$V \mapsto \langle \mathscr{P}(V), \mathscr{P}({}^{\alpha}U \setminus V) \rangle$$

that sends meanings of ordinary first-order formulas to the meanings of their dependence-friendly analogues. In fact, this mapping embeds any α-dimensional cylindric set algebra $\mathfrak{C}$ into the reduct of the DF_{α} algebra $\mathfrak{D}$ generated by

$$\{ \langle \mathscr{P}(V), \mathscr{P}({}^{\alpha}U \setminus V) \rangle : V \in \mathfrak{C} \}$$

to the signature $\langle 1, 0, D_{ij}, +_\alpha, \cdot_\alpha, C_{n,\alpha} \rangle$. The image of $\mathfrak{C}$ under this mapping is called the *perfect subreduct* of $\mathfrak{D}$. In particular, $\mathfrak{Cs}_\alpha(\mathfrak{A})$ is isomorphic to the perfect subreduct of $\mathfrak{Cs}_{\mathrm{DF}_\alpha}(\mathfrak{A})$, which captures algebraically the fact that DF logic is a conservative extension of first-order logic [19].

4. The De Morgan Reduct

When restricted to perfect double suits, the perfect-information operations $+_\alpha$ and $\cdot_\alpha$ correspond to the Boolean operations $\cup$ and $\cap$ of an ordinary cylindric set algebra. Unfortunately, when applied to all double suits, $+_\alpha$ and $\cdot_\alpha$ may not even be lattice operations. Consider the one-dimensional double suit $X = \big\langle \mathscr{P}\big(\{0\}\big) \cup \mathscr{P}\big(\{1\}\big), \{\emptyset\} \big\rangle$, and observe that

$$X +_1 X = \big\langle \mathscr{P}\big(\{0, 1\}\big), \{\emptyset\} \big\rangle \neq X.$$

At the other extreme, we consider the zero-information operations $+_\emptyset$ and $\cdot_\emptyset$, which correspond to positions in the semantic game where the players must move in complete ignorance of the current assignment. Unlike $+_\alpha$ and $\cdot_\alpha$, the operations $+_\emptyset$ and $\cdot_\emptyset$ are lattice operations on double suits. The present section and the next are adapted from [20].

Definition 2.4.1. A *De Morgan algebra* $\mathfrak{A} = \langle A; 1, 0, \sim, \vee, \wedge \rangle$ is a bounded distributive lattice with an additional unary operation $\sim$ that satisfies

$$\sim\sim x = x \quad \text{and} \quad \sim(x \vee y) = \sim x \wedge \sim y.$$

Unlike a Boolean algebra, it is possible for a De Morgan algebra to have an element such that $\sim a = a$. Such an element is called a *center* or *fixed point*. A *centered De Morgan algebra* is a De Morgan algebra with a center.

Definition 2.4.2. A *Kleene algebra* is a De Morgan algebra that satisfies the additional condition

$$x \wedge \sim x \leq y \vee \sim y.$$

Proposition 2.4.3. *A Kleene algebra has at most one center.*

Proof. Suppose a and b are both centers. Then $a = a \wedge \sim a \leq b \vee \sim b = b$ and $b = b \wedge \sim b \leq a \vee \sim a = a$. ■

For the elementary theory of De Morgan and Kleene algebras, we refer the reader to the excellent book [1].

Definition 2.4.4. The *De Morgan reduct* of a DF_α algebra is the reduct of the algebra to the signature $\langle 1, 0, {}^\cup, +_\emptyset, \cdot_\emptyset \rangle$.

Proposition 2.4.5. *Let X and Y be double suits.*

(i) $X +_\emptyset Y = \langle X^+ \cup Y^+, X^- \cap Y^- \rangle$.

(ii) $X \cdot_\emptyset Y = \langle X^+ \cap Y^+, X^- \cup Y^- \rangle$.

Proof. (i) Suppose $V \in (X +_\emptyset Y)^+$. Then $V = V_1 \cup_\emptyset V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$. However, since $\approx_\emptyset$ is the total relation, either $V_1 = V$ and $V_2 = \emptyset$ or vice versa. Thus $V \in X^+ \cup Y^+$. Conversely, suppose $V \in X^+ \cup Y^+$. Without loss of generality, assume $V \in X^+$. Then $V = V \cup_\emptyset \emptyset$, where $V \in X^+$ and $\emptyset \in Y^+$, so $V \in (X +_\emptyset Y)^+$. Thus $(X +_\emptyset Y)^+ = X^+ \cup Y^+$. Observe that $(X +_\emptyset Y)^- = X^- \cap Y^-$ by definition.

The proof of (ii) is similar. ∎

Definition 2.4.6. Let X and Y be double suits. Define $X \leq Y$ if $X^+ \subseteq Y^+$ and $Y^- \subseteq X^-$.

Theorem 2.4.7. *The class of De Morgan reducts of double-suited* DF_α *algebras generates the variety of all Kleene algebras.*

Proof. Let $\mathfrak{D}$ be the De Morgan reduct of a double-suited DF_α algebra. First, we will prove that $\mathfrak{D}$ is a Kleene algebra. Using Proposition 2.4.5, it is easy to show that $\mathfrak{D}$ is a bounded distributive lattice and that the lattice order agrees with $\leq$ as defined above. We verify the other three axioms. For all $X, Y \in \mathfrak{D}$,

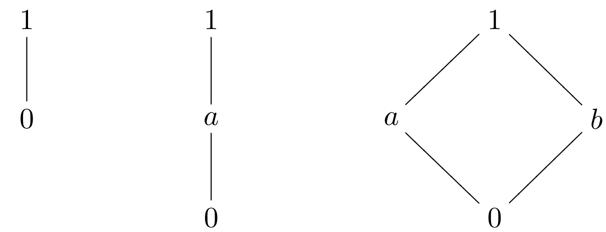
$$(X^\cup)^\cup = \langle X^-, X^+ \rangle^\cup = \langle X^+, X^- \rangle = X,$$

$$(X +_\emptyset Y)^\cup = \langle X^- \cap Y^-, X^+ \cup Y^+ \rangle = X^\cup \cdot_\emptyset Y^\cup,$$

$$X \cdot_\emptyset X^\cup = \big\langle \{\emptyset\}, X^- \cup X^+ \big\rangle \leq \big\langle Y^+ \cup Y^-, \{\emptyset\} \big\rangle = Y +_\emptyset Y^\cup.$$

Therefore $\mathfrak{D}$ is a Kleene algebra.

To prove that the class of De Morgan reducts of double-suited DF_α algebras generates the variety of all Kleene algebras, it suffices to show that every nontrivial, subdirectly irreducible Kleene algebra is isomorphic to the De Morgan reduct of some double-suited DF_α algebra. Kalman [17] proved that, up to isomorphism, the only nontrivial, subdirectly irreducible De Morgan algebras are:



$\underline{\mathbf{B}}$ $\underline{\mathbf{K}}$ $\underline{\mathbf{M}}$

where $\sim a = a$ and $\sim b = b$. Of these, only $\underline{\mathbf{B}}$ and $\underline{\mathbf{K}}$ are Kleene algebras. For any base U, let

$$\begin{aligned} 1 &= \left\langle \mathscr{P}({}^{\alpha}U), \{\emptyset\} \right\rangle, \\ \Omega &= \left\langle \{\emptyset\}, \{\emptyset\} \right\rangle, \\ 0 &= \left\langle \{\emptyset\}, \mathscr{P}({}^{\alpha}U) \right\rangle. \end{aligned}$$

Then $\{0, 1\}$ and $\{0, \Omega, 1\}$ are double-suited DF_α algebras whose De Morgan reducts are isomorphic to $\underline{\mathbf{B}}$ and $\underline{\mathbf{K}}$, respectively. ■

It can be shown that 0, Ω, and 1 are the only possible meanings of DF sentences. Furthermore, if $X, Y \in \{0, \Omega, 1\}$, then for any $J \subseteq \alpha$ we have $X +_J Y = X +_\emptyset Y$ and $X \cdot_J Y = X \cdot_\emptyset Y$. Therefore, the propositional logic underlying DF logic is Kleene's strong three-valued logic [18, Section 64] (see also [23]). The connection between IF logic and Kleene's strong three-valued logic was first observed by Hintikka and Sandu [14, Proposition 5.1].

5. The Monadic De Morgan Reduct

Generalizing Halmos' [7] work on monadic Boolean algebras, Cignoli [4] and Petrovich [21] equip bounded distributive lattices with an additional unary operation to obtain what they call Q-distributive lattices. Soon after, Petrovich [22] extended their investigations by reintroducing negation, defining monadic De Morgan algebras.

Definition 2.5.1. A *quantifier* on a De Morgan algebra is a unary operation ∇ such that:

(Q1) $\nabla 0 = 0$,

(Q2) $x \leq \nabla x$,

(Q3) $\nabla(x \vee y) = \nabla x \vee \nabla y$,

(Q4) $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$.

(Q5) $\nabla(\sim \nabla x) = \sim \nabla x$.

A De Morgan algebra equipped with a quantifier is called a *monadic De Morgan algebra.*

Definition 2.5.2. The *monadic De Morgan reduct* of a DF_1 algebra is the reduct of the algebra to the signature $\langle 1, 0, {}^{\cup}, +_{\emptyset}, \cdot_{\emptyset}, C_{0,\emptyset} \rangle$.

Lemma 2.5.3. *In any* DF_1 *algebra we have* $C_{0,J}(1) = 1$ *and* $C_{0,J}(0) = 0$. *In any double-suited* DF_1 *algebra with* Ω,

$$C_{0,J}(X) = \begin{cases} 1 & \text{if } X \not\leq \Omega, \\ \Omega & \text{if } 0 < X \leq \Omega, \\ 0 & \text{if } X = 0. \end{cases}$$

Proof. $C_{0,J}(1) = \left\langle \mathscr{P}(U), \{\emptyset\} \right\rangle$ because $V \in C_{0,J}(1)^+$ if and only if there exists an $f : V \underset{J}{\to} U$ such that $V(0 : f) \in \mathscr{P}(U)$, which is always true. Also, $W \in C_{0,J}(1)^-$ if and only if $W(0 : U) = \emptyset$ if and only if $W = \emptyset$.

$C_{0,J}(0) = \left\langle \{\emptyset\}, \mathscr{P}(U) \right\rangle$ because $V \in C_{0,J}(0)^+$ if and only if $V(0 : f) = \emptyset$ for some $f : V \underset{J}{\to} U$ if and only if $V = \emptyset$, while $C_{0,\emptyset}(0)^- = \mathscr{P}(U)$ because $W \in C_{0,J}(0)^-$ if and only if $W(0 : U) \in \mathscr{P}(U)$, which is always true.

For the rest of the proof, assume X and $C_{0,J}(X)$ are double suits. Suppose $X \not\leq \Omega$. Then there is a $\emptyset \neq V \in X^+$. Let $b \in V$, and let $f : U \underset{J}{\to} U$ be the constant function that sends every element to b. Then $U(0 : f) = \{b\} \in X^+$, so $U \in C_{0,J}(X)^+$. Hence $C_{0,J}(X) = \left\langle \mathscr{P}(U), \{\emptyset\} \right\rangle$.

Suppose $0 < X \leq \Omega$. Then $X^+ = \emptyset$ and $U \notin X^-$. It follows that $C_{0,J}(X) = \left\langle \{\emptyset\}, \{\emptyset\} \right\rangle$ because $V \in C_{0,J}(X)^+$ if and only if there exists a function $f : V \underset{J}{\to} U$ such that $V(0 : f) = \emptyset$ if and only if $V = \emptyset$. Also, $W \in C_{0,J}(X)^-$ if and only if $W(0 : U) \in X^-$ if and only if $W = \emptyset$. ■

Theorem 2.5.4. *The class of monadic De Morgan reducts of double-suited* DF_1 *algebras generates a proper subvariety of the variety of all monadic Kleene algebras.*

Proof. Let $\mathfrak{D}$ be the monadic De Morgan reduct of a double-suited DF_1 algebra with base U. First we verify that $\mathfrak{D}$ satisfies the axioms. We have already checked (Q1).

(Q2) Suppose $V \in X^+$. If $V = \emptyset$, then $V \in C_{0,\emptyset}(X)^+$ because $C_{0,\emptyset}(X)$ is a double suit. If $b \in V$, let $f : V \underset{\emptyset}{\to} U$ be the constant function that sends every element to b. Then $V(0 : f) = \{b\} \in X^+$. Thus $V \in C_{0,\emptyset}(X)^+$.

Now suppose $W \in C_{0,\emptyset}(X)^-$. If $W = \emptyset$, then $W = \emptyset = W(0 : U) \in X^-$. If $W \neq \emptyset$, then $U = W(0 : U) \in X^-$. Hence $W \in \mathscr{P}(U) = X^-$.

(Q3) For the rest of the proof we will assume that V and W are nonempty. Suppose $V \in C_{0,\emptyset}(X +_\emptyset Y)^+$. Then there is a function $f : V \underset{\emptyset}{\to} U$ such that

$$V(0 : f) \in (X +_\emptyset Y)^+ = X^+ \cup Y^+.$$

Without loss of generality, assume $V(0 : f) \in X^+$. It follows that

$$V \in C_{0,\emptyset}(X)^+ \cup C_{0,\emptyset}(Y)^+ = \big(C_{0,\emptyset}(X) +_\emptyset C_{0,\emptyset}(Y)\big)^+.$$

Conversely, suppose $V \in \big(C_{0,\emptyset}(X) +_\emptyset C_{0,\emptyset}(Y)\big)^+$. Then without loss of generality $V \in C_{0,\emptyset}(X)^+$, so there is an $f : V \underset{\emptyset}{\to} U$ such that

$$V(0 : f) \in X^+ \subseteq (X +_\emptyset Y)^+.$$

Hence $V \in C_{0,\emptyset}(X +_\emptyset Y)^+$.

Also, $W \in C_{0,\emptyset}(X +_\emptyset Y)^-$ if and only if

$$W(0 : U) \in X^- \cap Y^- = (X +_\emptyset Y)^-$$

if and only if

$$W \in C_{0,\emptyset}(X)^- \cap C_{0,\emptyset}(Y)^- = \big(C_{0,\emptyset}(X) +_\emptyset C_{0,\emptyset}(Y)\big)^-.$$

(Q4) Suppose we have $V \in C_{0,\emptyset}\big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^+$. Then there is a constant function $f : V \underset{\emptyset}{\to} U$ and an element $b \in U$ such that

$$\{b\} = V(0 : f) \in \big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^+ = X^+ \cap C_{0,\emptyset}(Y)^+,$$

and another constant function $g : \{b\} \underset{\emptyset}{\to} U$ and element $b' \in U$ such that $\{b'\} = V(0 : f)(0 : g) \in Y^+$. It follows that $g \circ f : V \underset{\emptyset}{\to} U$ is a constant function such that $V(0 : g \circ f) = \{b'\} \in Y^+$. Hence

$$V \in C_{0,\emptyset}(X)^+ \cap C_{0,\emptyset}(Y)^+ = \big(C_{0,\emptyset}(X) +_\emptyset C_{0,\emptyset}(Y)\big)^+.$$

Conversely, suppose $V \in \big(C_{0,\emptyset}(X) +_\emptyset C_{0,\emptyset}(Y)\big)^+ = C_{0,\emptyset}(X)^+ \cap C_{0,\emptyset}(Y)^+$. Then there exist constant functions $f : V \underset{\emptyset}{\to} U$ and $h : V \underset{\emptyset}{\to} U$ as well as elements $b, b' \in U$ such that $\{b\} = V(0 : f) \in X^+$ and $\{b'\} = V(0 : h) = Y^+$. Let $g : \{b\} \underset{\emptyset}{\to} U$ be the function that maps b to b'. Then $h = g \circ f$, $V(0 : f)(0 : g) = V(0 : h) \in Y^+$, and $V(0 : f) \in C_{0,\emptyset}(Y)^+$. Thus

$$V(0 : f) \in X^+ \cap C_{0,\emptyset}(Y)^+ = \big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^+,$$

so $V \in C_{0,\emptyset}\big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^+$.

Now suppose $W \in C_{0,\emptyset}\big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-$. Then

$$U = W(0 : U) \in \big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^- = X^- \cup C_{0,\emptyset}(Y)^-.$$

If $U(0 : U) = U \in X^-$, then $U \in C_{0,\emptyset}(X)^-$. Hence

$$U \in C_{0,\emptyset}(X)^- \cup C_{0,\emptyset}(Y)^- = \big(C_{0,\emptyset}(X) \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-.$$

If $U \in C_{0,\emptyset}(Y)^-$, then

$$U \in C_{0,\emptyset}(X)^- \cup C_{0,\emptyset}(Y)^- = \big(C_{0,\emptyset}(X) \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-.$$

In either case, $W \in \big(C_{0,\emptyset}(X) \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-$.

Conversely, suppose

$$W \in \big(C_{0,\emptyset}(X) \cdot_\emptyset C_{0,\emptyset}(Y)\big)^- = C_{0,\emptyset}(X)^- \cup C_{0,\emptyset}(Y)^-.$$

If $W \in C_{0,\emptyset}(X)^-$, then

$$W(0 : U) \in X^- \cup C_{0,\emptyset}(Y)^- = \big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-,$$

so $W \in C_{0,\emptyset}\big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-$. If $W \in C_{0,\emptyset}(Y)^-$, then

$$W(0 : U)(0 : U) = W(0 : U) \in Y^-.$$

Hence $W(0:U) \in C_{0,\emptyset}(Y)^-$. Thus

$$W(0:U) \in X^- \cup C_{0,\emptyset}(Y)^- = \big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-.$$

Therefore $W \in C_{0,\emptyset}\big(X \cdot_\emptyset C_{0,\emptyset}(Y)\big)^-$.

(Q5) Suppose $V \in C_{0,\emptyset}\big(C_{0,\emptyset}(X)^\cup\big)^+$. Then $V(0:f) \in C_{0,\emptyset}(X)^-$ for some $f : V \underset{\emptyset}{\to} U$. Hence

$$U(0:U) = V(0:f)(0:U) \in X^-.$$

Thus $U \in C_{0,\emptyset}(X)^- = \big(C_{0,\emptyset}(X)^\cup\big)^+$, which implies $V \in \big(C_{0,\emptyset}(X)^\cup\big)^+$.

Conversely, suppose $V \in \big(C_{0,\emptyset}(X)^\cup\big)^+ = C_{0,\emptyset}(X)^-$. Then

$$U(0:U)(0:U) = V(0:U) \in X^-,$$

so

$$U(0:U) \in C_{0,\emptyset}(X)^- = \big(C_{0,\emptyset}(X)^\cup\big)^+.$$

Thus $U \in C_{0,\emptyset}\big(C_{0,\emptyset}(X)^\cup\big)^+$, which implies $V \in C_{0,\emptyset}\big(C_{0,\emptyset}(X)^\cup\big)^+$.

Now suppose $W \in C_{0,\emptyset}\big(C_{0,\emptyset}(X)^\cup\big)^-$. Then

$$U = W(0:U) \in \big(C_{0,\emptyset}(X)^\cup\big)^-,$$

which implies $W \in \big(C_{0,\emptyset}(X)^\cup\big)^-$. Conversely, suppose $W \in \big(C_{0,\emptyset}(X)^\cup\big)^- = C_{0,\emptyset}(X)^+$. Then there is a constant function $f : W \underset{\emptyset}{\to} U$ and an element $b \in U$ such that $\{b\} = W(0:f) \in X^+$. Let $g : U \underset{\emptyset}{\to} U$ be the constant function that sends every element to b. Then $U(0:g) = \{b\} \in X^+$, so $U(0:U) = U \in C_{0,\emptyset}(X)^+ = \big(C_{0,\emptyset}(X)^\cup\big)^-$. Thus $U \in C_{0,\emptyset}\big(C_{0,\emptyset}(X)^\cup\big)^-$. Therefore $\mathfrak{D}$ is a monadic Kleene algebra.

Next we will prove that $\mathfrak{D}$ satisfies the inequality

$$\text{(Q6)} \qquad \nabla(x \wedge \sim x) \leq \sim \nabla(x \wedge \sim x),$$

which is equivalent to an equation. Observe that for any double suit X,

$$X \cdot_\emptyset X^\cup = \big\langle \{\emptyset\}, X^+ \cup X^- \big\rangle \leq \Omega,$$

so by Lemma 2.5.3 we have

$$C_{0,\emptyset}(X \cdot_\emptyset X^\cup) \leq \Omega \leq \big(C_{0,\emptyset}(X \cdot_\emptyset X^\cup)\big)^\cup.$$

Finally, we exhibit a monadic Kleene algebra that does not satisfy (Q6). Let $\langle \underline{\mathbf{K}}, \nabla \rangle$ be the three-element Kleene algebra $\{0, a, 1\}$ equipped with the quantifier ∇ defined by

$$\nabla x = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

It is easy to check that $\langle \underline{\mathbf{K}}, \nabla \rangle$ is a monadic Kleene algebra. However,

$$\nabla(a \wedge \sim a) = 1 \not\leq 0 = \sim \nabla(a \wedge \sim a). \qquad \blacksquare$$

Conjecture 2.5.5. The class of monadic De Morgan reducts of double-suited DF_1 algebras generates the variety of all monadic Kleene algebras that satisfy

$$\text{(Q6)} \qquad \nabla(x \wedge \sim x) \leq \sim \nabla(x \wedge \sim x).$$

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