Revolutionaries and Spies

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Abstract

Let $G = (V, E)$ be a graph and let r, s, k be natural numbers. "Revolutionaries and Spies", $\mathcal{G}(G, r, s, k)$, is the following two-player game. The sets of positions for player 1 and player 2 are V^r and

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V^s respectively. Each coordinate in $p \in V^r$ gives the location of a "revolutionary" in G controlled by player 1. Similarly player 2 controls s "spies". We say $u, u' \in V(G)^n$ are adjacent, $u \sim u'$, if for all $1 \leq i \leq n$, $u_i = u'_i$ or $\{u_i, u'_i\} \in E(G)$. In round 0 player 1 picks $p_0 \in V^r$ and then player 2 picks $q_0 \in V^s$. In each round $i \geq 1$, player 1 moves to $p_i \sim p_{i-1}$ and then player 2 moves to $q_i \sim q_{i-1}$. Player 1 wins the game if he can place k revolutionaries on a vertex v in such a way that player 2 cannot place a spy on v in his following move. Player 2 wins the game if he can prevent this outcome.

Let $s(G, r, k)$ be the minimum value s such that player 2 can win $\mathcal{G}(G, r, s, k)$. We show that $\liminf_{r\to\infty} s(\mathbb{Z}^2, r, 2)/r \geq 3/4$. Here moves in $\mathbb{Z}^2$ are "king" moves as in chess.

1 Introduction

Let $G = (V, E)$ be a graph, possibly infinite, and let r, s, k be natural numbers. *"Revolutionaries and Spies"*, $\mathcal{G}(G, r, s, k)$, is the following two-player game, invented by Beck [?]. In round 0, player 1 places r markers called *revolutionaries* on the vertices of G. Then player 2 places s markers called *spies* on the vertices. There is no restriction on the number of spies and revolutionaries that may be placed on a vertex. For $i \geq 1$, round i begins with player 1 moving each revolutionary either to a vertex adjacent to its current vertex or by leaving it at its current vertex. Round i ends with player 2 moving his spies in the same fashion. Player 1 has a *meeting* of size k at a vertex v if k or more revolutionaries are present at that vertex. A set of vertices is *guarded* if a spy is present at some vertex in the set. Player 1 wins $\mathcal{G}(G, r, s, k)$ if he has a strategy to achieve an unguarded meeting of size k by the end of some round i. Otherwise player 2 has a strategy to prevent this for all time and we say player 2 wins $\mathcal{G}(G, r, s, k)$.

There is a similarity between this game and the game cops and robbers [**?**, **?**]. This is a pursuit game played by a cop and a robber on a graph G: the cop chooses a vertex, then the robber chooses a vertex, and players move alternately starting with the cop. A move consists of either staying at ones present vertex or else moving to an adjacent vertex; each move is seen by both players. The cop wins if he manages to occupy the same vertex as the robber, and the robber wins if he avoids this forever. The graphs on which the cop has a winning strategy have been characterized [**?**, **?**].

Let $k(G, r, s)$ be the maximum value of k such that player 1 wins $\mathcal{G}(G, r, s, k)$. We define $\mathcal{G}(G, r, s)$ to be $\mathcal{G}(G, r, s, k_0)$ where $k_0 = k(G, r, s)$. An *optimum* strategy for player 1 in $\mathcal{G}(G, r, s)$ is one eventually achieving an unguarded meeting of size k_0. Similarly an optimum strategy for player 2 is one preventing a meeting of size $k_0 + 1$. We sometimes describe these just as player 1's (player 2's) *strategies* in $\mathcal{G}(r, s)$. Let $s(G, r, k)$ be the minimum value s for which it is possible for player 2 to win $\mathcal{G}(G, r, s, k)$.

We record the following trivial observation.

Lemma 1.1 *If G has at least $s+1$ vertices, $k(G, r, s) \geq \left\lfloor \frac{r}{s+1} \right\rfloor$. Otherwise, $k(G, r, s) = 0$.*

Proof: In the first case, player 1 can win at the end of round 0 by placing at least $\lfloor \frac{r}{s+1} \rfloor$ revolutionaries on each of $s+1$ vertices. In the second case, player 2 can win by maintaining a spy at each vertex.

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This trivial lower bound is attained on the following classes of graphs.

Theorem 1.2 *If G is acyclic and has at least $s+1$ vertices then $k(G,r,s) = \lfloor \frac{r}{s+1} \rfloor$.*

This statement was originally proved by one of the authors [?]. Its proof will appear in another paper, currently in preparation, covering the revolutionaries and spies game on trees and unicyclic graphs [?].

If $v, w \in V(G)$ let $d_G(v,w)$ be the *distance* between v and w in G, i.e. the minimum length of an v-w path in G. If no such path exists we define $d_G(v,w) = +\infty$. Note $d_G(v,v) = 0$.

Let G and H be graphs. The *strong product* of G and H, denoted $G \boxtimes H$, is the graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$. Vertices (g,h) and (g',h') are adjacent in $G \boxtimes H$ if and only if $(g,h) \neq (g',h')$, $d_G(g,g') \leq 1$, and $d_H(h,h') \leq 1$ [?]. We denote by $\mathbb{Z}$ the graph $G = (V,E)$ with $V = \mathbb{Z}$ and $E = \{\{i, i+1\} : i \in \mathbb{Z}\}$. For $d \geq 1$, let $\mathbb{Z}^{\boxtimes d}$ be the d-fold strong product of $\mathbb{Z}$ with itself.

We study revolutionaries and spies on $\mathbb{Z}^{\boxtimes d}$, primarily for $d = 2$. Perhaps one of the most basic (yet non-trivial) quantities to study is the threshold $s(\mathbb{Z}^{\boxtimes d}, r, 2)$.

Theorem 1.3 *We have* $\liminf_{r\to\infty} s(\mathbb{Z}^{\boxtimes d}, r, 2)/r \geq \frac{3}{4}$.

The best result obtainable using Lemma **??** is $\liminf_{r\to\infty} s(\mathbb{Z}^{\boxtimes d}, r, 2)/r \geq \frac{1}{2}$.

The organization of the paper is as follows. We present a number of basic definitions and results in Section 2 including a Hall-type condition on when it is possible to move from one position to another (Lemma **??**). We prove Theorem **??** in Section 3. In the final section we outline a few directions for further research in the concluding section.

2 Basic Results

Lemma 2.1 *Let G be a graph and let $r, r', s, s' \in \mathbb{N}$ with $r \leq r'$ and $s \leq s'$. Then*

1. $k(G, r, 0) = r$ *and* $k(G, r, r) = 0$.
2. $k(G, r', s) \geq k(G, r, s)$, *and*
3. $k(G, r, s') \leq k(G, r, s)$.

Proof: To prove $k(G, r, r) = 0$, player 2 can match each spy with a unique revolutionary. Each spy is then moved to its matched revolutionary's position at the end of each round. To prove statement 2, first note that player 2 has a strategy to prevent an unguarded meeting of size $k(r', s) + 1$ in $\mathcal{G}(r', s)$. Player 2 can then prevent a meeting of this size in $\mathcal{G}(r, s)$ by "pretending" that player 1 has an additional $r' - r$ revolutionaries left fixed at some arbitrary vertex. Player 2 then moves according to his strategy in $\mathcal{G}(r', s)$.

Similarly, to prove statement 3, we see that player 2 can keep $s' - s$ spies fixed at some arbitrary vertex and then play his optimum strategy in $\mathcal{G}(r, s)$ with his remaining s spies to prevent a meeting of size $k(G, r, s)+1$ in $\mathcal{G}(r, s')$.

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If $U, W \subseteq V(G)$, let $d_G(U, W) = \min_{u \in U, w \in W} d_G(u, w)$. If $W = \{v\}$ we write $d_G(S, v)$ for $d_G(U, W)$. Let $B_G(S, r) := \{v \in V(G) : d_G(S, v) \leq r\}$.

Lemma 2.2 *Let $r_1, \ldots, r_l, s_1 \ldots s_l \in \mathbb{N}$. Let $r = \sum_{i=1}^{\ell} r_i$ and $s = \sum_{i=1}^{\ell} s_i$. Then the following statements hold.*

1. *For all graphs G, $k(G, r, s) \leq \sum_{i=1}^{\ell} k(G, r_i, s_i)$.*

2. *Let $n \geq 0$ and suppose $G_1, \ldots, G_\ell$ are subgraphs of a graph G with $B_G(V(G_i), n)$ for $1 \leq i \leq \ell$ pairwise disjoint. Suppose that for all $1 \leq i \leq \ell$, player 1 has a strategy to win $\mathcal{G}(G, r_i, s_i, k_i)$ in at most n rounds with all revolutionaries starting and remaining in $V(G_i)$. Then $k(G, r, s + \ell - 1) \geq \min_{1 \leq i \leq \ell} k_i$.*

Proof: To prove statement 1, player 2 partitions player 1's revolutionaries into groups of sizes r_i for $1 \leq i \leq \ell$ and also partitions the spies into groups of size s_i. Thereafter he simultaneously uses his ith group of spies to prevent a meeting of size $k(G, r_i, s_i) + 1$ amongst the ith group of revolutionaries, for each $1 \leq i \leq \ell$. Clearly, player 1 cannot achieve a meeting of size $1 + \sum_{i=1}^{\ell} k(G_i, r_i, s_i)$.

To prove statement 2, we first note that the proof of statement 3 of Lemma **??** gives for each $1 \leq i \leq \ell$ a "unified" strategy for player 1 to achieve an unguarded meeting of size k_i by the end of round n, in $\mathcal{G}(G, r_i, t)$ for all $t \leq s_i$, using the *same* starting position in G_i. For each $t \leq s_i$ player 1 plays his strategy for $\mathcal{G}(G, r_i, s_i, k_i)$ all the while pretending that player 2 has an additional $s_i - t$ spies fixed at some arbitrary vertex . While player 2 moves his t spies, player 1 responds with moves from $\mathcal{G}(G, r_i, s_i, k_i)$. The initial position for player 1 is the same for all $t \leq s_i$.

Player 1 makes these uniform initial placements of r_i revolutionaries in each G_i. For some i, there must be $t \leq s_i$ spies placed in $B_G(V(G_i), n)$. Thus player 1 can achieve an unguarded meeting of size k_i in G_i in n rounds.

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The following statements are all easy corollaries of Lemma **??**, statement 1.

Corollary 2.3 *Let* $r, s \in \mathbb{N}$ *then*

1. $k(G, r, s) \leq k(G, r - a, s - a)$ *for all* $a \leq r, s$.

2. $k(G, r + r', s) \leq k(G, r, s) + r'$.

3. $k(G, ar, as) \leq ak(G, r, s)$ *for all* $a \geq 0$.

Lemma 2.4 *For all graphs* G *and for all* $R, r, s \in \mathbb{N}$ *we have* $k(G, R, s) \geq \left\lfloor \frac{R}{r} \right\rfloor k(G, r, s)$.

Proof: Let $R = R_1 + \cdots + R_r$ where $R_i \leq R_{i+1}$ for all $1 \leq i \leq r-1$ and $R_r \leq R_1 + 1$. Let player 1 partition his revolutionaries into r groups of sizes R_i and then place each member of the ith group on the position of revolutionary i in some optimum strategy for $\mathcal{G}(r,s)$. This strategy allows player 1 to achieve an unguarded meeting of size at least $R_1 k(G,r,s)$ in $\mathcal{G}(G,R,s)$. Note that $R_1 = \lfloor \frac{R}{r} \rfloor$.

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The next result follows directly from the definition of $k(G,r,s)$.

Lemma 2.5 *Suppose G is a graph whose components are $\{G_i : i \in I\}$. Then*

$$k(G,r,s) = \max_f \min_g \max_{i \in I} k(G_i, f(i), g(i))$$

where functions $f, g : I \to \mathbb{N}$ satisfy $r = \sum_i f(i)$, $s = \sum_{i \in I} g(i)$ respectively.

Let $\mathcal{X} : V(G) \to \mathbb{R}$. We define $\mathcal{X}(S) := \sum_{v \in S} \mathcal{X}(v)$ for subsets $S \subseteq V(G)$ and $\mathcal{X}(H) := \mathcal{X}(V(H))$ for subgraphs $H \subseteq G$. The *weight* of $\mathcal{X}$ is $\mathcal{X}(G)$. We say $\mathcal{X}$ is finite if $\mathcal{X}$ has finite weight. If $\mathcal{X} : V(G) \to \mathbb{N}$ we call $\mathcal{X}$ a *position*. Let $\mathcal{P}(G,m)$ denote the set of all the positions of weight m on G. The set of all possible placements of r revolutionaries in $\mathcal{G}(G,r,s)$ is in one-to-one correspondence with the functions $\mathcal{R}$ in $\mathcal{P}(G,r)$. Namely, for each vertex v, let $\mathcal{R}(v)$ be the number of revolutionaries present at v. Similarly, we let $\mathcal{P}(G,s)$ represent the possible placements of the spies in $\mathcal{G}(G,r,s)$. If $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(G,m)$ and $\mathcal{X}'$ is one move from $\mathcal{X}$ then we denote this by $\mathcal{X}' \sim \mathcal{X}$.

Hall's theorem gives a characterization of when two positions are one move apart. Given $X \subseteq V(G)$ we define $X^{(1)} := \{x' \in V(G) : \exists x \in X \; d_G(x', x) \leq 1\}$.

Theorem 2.6 *Let $G = (V, E)$ be a graph and let $m \geq 0$. Let $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(G, m)$. Then the following are equivalent.*

1. $\mathcal{X} \sim \mathcal{X}'$
2. $\forall X \subseteq V(G) \; \mathcal{X}(X) \leq \mathcal{X}'(X^{(1)})$
3. $\forall X \subseteq V(G) \; \mathcal{X}'(X) \leq \mathcal{X}(X^{(1)})$

Proof: Note that it suffices to show statements 1 and 2 are equivalent. If we move from position $\mathcal{X}'$ to position $\mathcal{X}$, then $\mathcal{X}(X)$, the number of revolutionaries in X at the end of the move, must be less than or equal to $\mathcal{X}'(X^{(1)})$, the number of revolutionaries that may reach X in one move. It remains to show that statement 2 implies statement 1.

Let $U = \{(1, x_1), \ldots, (r, x_r)\}$ be a listing of the revolutionaries in $\mathcal{X}$ in some fixed order, i.e. for all $i \in [r]$ revolutionary i is at vertex x_i. Similarly let $W = \{(1, x'_1), \ldots, (r, x'_r)\}$ be some listing of the revolutionaries in $\mathcal{X}'$. Let $B(\mathcal{X}, \mathcal{X}') = (U \cup W, E)$ be the bipartite graph with bipartition (U, W) such that $\{(i, x_i), (j, x'_j)\} \in E$ if and only if $d_G(x_i, x'_j) \leq 1$. Clearly, $\mathcal{X} \sim \mathcal{X}'$ if and only if B has a perfect matching, i.e. each revolutionary in $\mathcal{X}$ has a unique target revolutionary in $\mathcal{X}'$ to which it can move. It is well known that this perfect matching exists if and only if Hall's condition [**?**] holds: $|N(S)| \geq |S|$

for all $S \subseteq U$, where $N(S) = \{w \in W : \exists s \in S \ \{w, s\} \in E(B)\}$. We show that statement 2 implies Hall's condition.

Let $S = \{(i_1, x_{i_1}), \ldots, (i_t, x_{i_t})\}$ be an arbitrary subset of U. Let $X = \{x_{i_1}, \ldots, x_{i_t}\}$. Note that we may have $|X| < t$ as there may be repetitions amongst the x_{i_j}. We have $\mathcal{X}(X) = \sum_{x \in X} |\{(i, x_i) \in U : x_i = x\}| \geq \sum_{x \in X} |\{(i, x_i) \in S : x_i = x\} = |S|$. These expressions for $\mathcal{X}(X)$ and $|S|$ are derived by partitioning the revolutionaries counted according to the vertex on which they lie. Similarly we get

$$N(S) = \bigcup_{x' \in X^{(1)}} \{(j, x'_j) \in W : x'_j = x'\}.$$

Using this expression we get $\mathcal{X}'(X^{(1)}) = \sum_{x' \in X^{(1)}} |\{(j, x'_j) \in W : x'_j = x'\}| = |N(S)|$. Thus $\mathcal{X}(X) \leq \mathcal{X}'(X^{(1)})$ implies $|S| \leq |N(S)|$ as desired.

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Given $\mathcal{X} \in \mathcal{P}(G, m)$ let $\operatorname{supp}(\mathcal{X}) := \{v \in V(G) : \mathcal{X}(v) > 0\}$. The proof of Lemma **??** shows that $\mathcal{X} \sim \mathcal{X}'$ if and only if $\mathcal{X}(X) \leq \mathcal{X}'(X^{(1)})$ for all $X \subseteq \operatorname{supp}(\mathcal{X})$.

If G is a graph the nth *power* of G is the graph G^n with $V(G^n) = V(G)$ and $E(G^n) := \{\{v, w\} : 0 < d_G(v, w) \leq n\}$ [?]. Let $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(G, m)$. Clearly $\mathcal{X}' \sim \mathcal{X}$ in G^n if and only if there is a sequence of positions $\mathcal{X}_i \in \mathcal{P}(G, m)$ for $0 \leq i \leq n$ with $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{X}_n = \mathcal{X}'$ such that $\mathcal{X}_i \sim \mathcal{X}_{i-1}$ in G for all $1 \leq i \leq n$.

Lemma 2.7 *For all graphs G and all $n, r, s \in \mathbb{N}$, $k(G^n, r, s) \leq k(G, r, s)$.*

Proof: If player 2 has a strategy to prevent an unguarded meeting of size $k+1$ in G, then it can prevent such a meeting in G^n as follows. Initially, player 2 places the spies according to an optimum strategy in $\mathcal{G}(G, r, s)$. Suppose $\mathcal{R}$ and $\mathcal{S}$ are the positions of the revolutionaries and the spies, respectively, at the beginning of some round. Suppose player 1 moves from $\mathcal{R}$ to $\mathcal{R}'$ in G^n. Player 2 constructs a sequence of positions $\mathcal{R} = \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_n = \mathcal{R}'$ with $\mathcal{R}_{i+1} \sim \mathcal{R}_i$ in G for $0 \leq i < n$. Let $\mathcal{S}_0 = \mathcal{S}$ and for $1 \leq i \leq n$ let $\mathcal{S}_i$ be the position played by the spies in response to $\mathcal{R}_i$ according to the strategy in $\mathcal{G}(G, r, s)$. Clearly $\mathcal{S}' = \mathcal{S}_n$ is one move from $\mathcal{S}$ in G^n and continues to prevent an unguarded meeting of size $k+1$ by $\mathcal{R}'$.

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Lemma 2.8 *For graphs G, H, we have*

$$k(G \boxtimes H, r, s) \geq \max(k(G, r, s), k(H, r, s)).$$

Proof:

Fix $h_0 \in V(H)$. Given $\mathcal{X}_0 \in \mathcal{P}(G, m)$ we define $\mathcal{X}_1 = L(\mathcal{X}_0) \in \mathcal{P}(G \boxtimes H, m)$ as follows. For all $(g, h) \in V(G \boxtimes H)$, $\mathcal{X}_1((g, h)) = \mathcal{X}(g)1_{h=h_0}$ where $1_{h=h_0} = 1$ if $h = h_0$, 0 otherwise. Given $\mathcal{X}_1 \in \mathcal{P}(G \boxtimes H, m)$ we define $\mathcal{X}_0 = P(\mathcal{X}_1) \in \mathcal{P}(G, m)$ by $\mathcal{X}_0(g) = \mathcal{X}_1(\{g\} \times H)$.

Let $\mathcal{X}_0, \mathcal{X}_0' \in \mathcal{P}(G, m)$ and let $\mathcal{X}_1 = L(\mathcal{X}_0), \mathcal{X}_1' = L(\mathcal{X}_0')$. Since $G \times \{h_0\}$ is an isomorphic copy of G in $G \boxtimes H$, it is clear that $\mathcal{X}_0 \sim \mathcal{X}_0'$ implies $\mathcal{X}_1 \sim \mathcal{X}_1'$ in $G \boxtimes H$.

Let $\mathcal{X}_1, \mathcal{X}_1' \in \mathcal{P}(G \boxtimes H, m)$ and let $\mathcal{X}_0 = P(\mathcal{X}_1)$, $\mathcal{X}_0' = P(\mathcal{X}_0')$. Suppose $\mathcal{X}_1 \sim \mathcal{X}_1'$. Then for all $X \subseteq V(G)$ we have $\mathcal{X}_0(X) = \mathcal{X}_1(X \times H) \leq \mathcal{X}_1'((X \times H)^{(1)}) = \mathcal{X}_1'(X^{(1)} \times H) = \mathcal{X}_0'(X^{(1)})$ and so by Lemma **??**, $\mathcal{X}_0 \sim \mathcal{X}_0'$.

Let $k = k(G \boxtimes H, r, s)$. It suffices to prove $k(G, r, s) \leq k$.

Given a position $\mathcal{R}_1 \in \mathcal{P}(G \boxtimes H, r)$ for player 1, let $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{R}_1) \in \mathcal{P}(G \boxtimes H, s)$ be an optimum response for player 2 in $\mathcal{G}(G \boxtimes H, r, s)$. That is, by playing $\mathcal{S}_1$ player 2 will prevent a meeting of size $k + 1$. We claim that if $\mathcal{R}_0 \in \mathcal{P}(G, r)$ is a position for player 1 then it is always possible for player 2 to play $\mathcal{S}_0 = \mathcal{S}_0(\mathcal{R}_0) := P(\mathcal{S}_1(L(\mathcal{R}_0))) \in \mathcal{P}(G, s)$ and that this will prevent a meeting of size $k + 1$ in $\mathcal{G}(G, r, s)$. Let $\mathcal{R}_0, \mathcal{R}_0' \in \mathcal{P}(G, r)$ and let $\mathcal{S}_0 = \mathcal{S}_0(\mathcal{R}_0), \mathcal{S}_0' = \mathcal{S}_0(\mathcal{R}_0')$. It is clear from the definitions that $\mathcal{S}_0, \mathcal{S}_0' \in P(G, s)$. It is also clear from the statements proven in the previous paragraphs that if $\mathcal{R}_0 \sim \mathcal{R}_0'$, then $\mathcal{S}_0 \sim \mathcal{S}_0'$. Furthermore if $\mathcal{R}_0(g) > k$ then $L(\mathcal{R}_0)((g, h_0)) = \mathcal{R}_0(g) > k$ and $\mathcal{S}_1(L(\mathcal{R}_0))((g, h_0)) \geq 1$, as $\mathcal{S}_1$ is a response that prevents meetings of size $k + 1$. Thus $\mathcal{S}_0(\mathcal{R}_0) = P(\mathcal{S}_1(L(\mathcal{R}_0)))(g) \geq 1$.

$\blacksquare$

3 Proof of Theorem ??.

As a warmup, we prove

Theorem 3.1 *For all $r, s \in \mathbb{N}$, $k(\mathbb{Z}, r, s) = \left\lfloor \frac{r}{s+1} \right\rfloor$.*

We need a lemma first. Given $\mathcal{X} \in \mathcal{P}(\mathbb{Z}, m)$ let $f_i(\mathcal{X})$ be the *ith order statistic* of $\mathcal{X}$, i.e. $f_i(\mathcal{X}) = j$ if and only if $\mathcal{X}((-\infty, j]) \geq i$ and $\mathcal{X}((-\infty, j)) < i$.

Lemma 3.2 *If $\mathcal{X}, \mathcal{X}' \in \mathcal{P}(\mathbb{Z}, m)$ and $\mathcal{X}' \sim \mathcal{X}$ then $|f_i(\mathcal{X}') - f_i(\mathcal{X})| \leq 1$ for all $1 \leq i \leq m$.*

Proof: Suppose $f_i(\mathcal{X}) = j$. Since $\mathcal{X}((-\infty, j]) \geq i$, $i \leq \mathcal{X}((-\infty, j]) \leq \mathcal{X}'((-\infty, j+1])$ and $f_i(\mathcal{X}') \leq j+1$. Also since $\mathcal{X}((-\infty, j)) < i$, $\mathcal{X}'((-\infty, j-1)) \leq \mathcal{X}((-\infty, j)) < i$ and $f_i(\mathcal{X}') \geq j-1$.

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Proof of Theorem ??. Clearly we have $k(\mathbb{Z}, r, s) \geq k := \left\lfloor \frac{r}{s+1} \right\rfloor$, by Lemma **??**. Since $r < r_0 := k(s+1)+s$, $k(\mathbb{Z}, r, s) \leq k(\mathbb{Z}, r_0, s)$, by Lemma **??**. Thus it suffices to show $k(\mathbb{Z}, r_0, s) \leq k$. If player 1's position is $\mathcal{R} \in \mathcal{P}(\mathbb{Z}, r_0)$, player 2's strategy is to play spy i at position $f_{i(k+1)}(\mathcal{R})$ for all $1 \leq i \leq s$. Since $r_0 = s(k+1)+k$, any vertex at which there is a meeting of $k+1$ revolutionaries must contain a spy. Let $\mathcal{R}' \sim \mathcal{R}$ be player 1's new position. Clearly, by Lemma **??**, each spy's new position is one move from its old position, because $|f_{i(k+1)}(\mathcal{R}') - f_{i(k+1)}(\mathcal{R})| \leq 1$.

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Recall that $a = (a_1, \ldots, a_d), b = (b_1, \ldots b_d) \in \mathbb{Z}^{\boxtimes d}$ are adjacent if and only $a \neq b$ and $|a_i - b_i| \leq 1$ for all $1 \leq i \leq d$.

Lemma 3.3 *If $m > 2k$, $k(\mathbb{Z}^{\boxtimes d}, m, m-2k) \leq k$.*

Proof: Clearly $k(\mathbb{Z}^{\boxtimes d}, 2k+1, 1) \geq \left\lfloor \frac{2k+1}{2} \right\rfloor = k$ by Lemma **??**. We give a strategy for the spies showing that $k(\mathbb{Z}^{\boxtimes d}, 2k+1, 1) \leq k$. Suppose player 1 is in position $\mathcal{R}$. Fix $1 \leq i \leq d$. Let $\mathcal{R}_i(j) := \mathcal{R}(\{x \in \mathbb{Z}^d : x_i = j\}) \in \mathcal{P}(\mathbb{Z}, r)$ be the the projection of $\mathcal{R}$ onto the ith coordinate axis. Let $c_i = f_{k+1}(\mathcal{R}_i)$ (see Lemma **??**). Player 2 's response to $\mathcal{R}$ is to move his spy to the vertex $c = (c_i : 1 \leq i \leq d)$. By Lemma **??** this is a playable strategy for player 2. Furthermore it guards all meetings of size $k+1$ or more. Clearly, if at least $k+1$ revolutionaries are at a single vertex $c' \in \mathbb{Z}_d$ then $c_i = c'_i$ for all i and $c = c'$, thus the spy is there also. This suffices to prove the theorem as Lemma **??** implies $k(\mathbb{Z}^{\boxtimes d}, m, m-2k) \leq k(\mathbb{Z}^{\boxtimes d}, 2k+1, 1) + k(\mathbb{Z}^{\boxtimes d}, m-2k-1, m-2k-1) = k + 0 = k$.

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Theorem **??** will follow from this next theorem (see Corollary **??**).

Theorem 3.4 *We have $k(\mathbb{Z}^{\boxtimes 2}, 8, 5) = 2$.*

Proof: By Lemma **??** and Lemma **??**, $k(\mathbb{Z}^{\boxtimes 2}, 8, 5) \leq k(\mathbb{Z}^{\boxtimes 2}, 7, 5) + k(\mathbb{Z}^{\boxtimes 2}, 1, 0) \leq 1 + 1 = 2$. We give a strategy for player 1, showing $k(\mathbb{Z}^{\boxtimes 2}, 8, 5) \geq 2$. In the first round, player 1 will place his revolutionaries on the eight positions $(\pm 1, \pm 1)$ and $(\pm 3, \pm 3)$ (see Figure **??**). In all of our figures the center point is position $(0, 0)$ and X's represent revolutionaries while O's represent spies. If 2 revolutionaries may reach a vertex in n rounds, there must be a spy within distance n of v to guard that potential meeting. We often describe this by saying that a spy must guard a meeting at v in n rounds.

Claim 1 (Box Property): One or more spies must begin in each of the four boxes $[1,3]\times[1,3]$, $[1,3]\times[-3,-1]$,$[-3,-1]\times[-3,-1]$, $[-3,-1]\times[1,3]$; we call these boxes B_1, B_2, B_3, and B_4 respectively (see Figure **??**).

Proof: By symmetry, we consider B_1. This box must contain a spy to prevent a win at $(2,2)$ in one round by the revolutionaries at $(3,3)$ and $(1,1)$.

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Let W_1 be the "wedge" of points $W_1 = \{(x,y) : y \geq 1, y \geq |x|\}$. We also consider the wedges obtained from W_1 by reflections in the lines $y = x$ and $y = -x$; in clockwise order from W_1, we call these W_2, W_3, and W_4 (see Figure **??**).

Claim 2 (Wedge Property): There must be at least two spies present in W_1; furthermore one of those spies must be distance 1 from $(0,2)$ and another distinct spy must be at distance 3 from $(0,6)$. We call this the wedge property for W_1. By symmetry (reflections through the lines $y = x$ and $y = -x$), analogous wedge properties hold for W_2, W_3, and W_4.

Proof: By symmetry we consider W_1. The revolutionaries at $(-1,1)$ and $(1,1)$ can form a meeting in one round at $(0,2)$ while, simultaneously, the revolutionaries at $(-3,3)$ and $(3,3)$ can form a meeting at $(0,6)$ in 3 rounds. Unless two spies are located as described, one of these meetings will be uncovered.

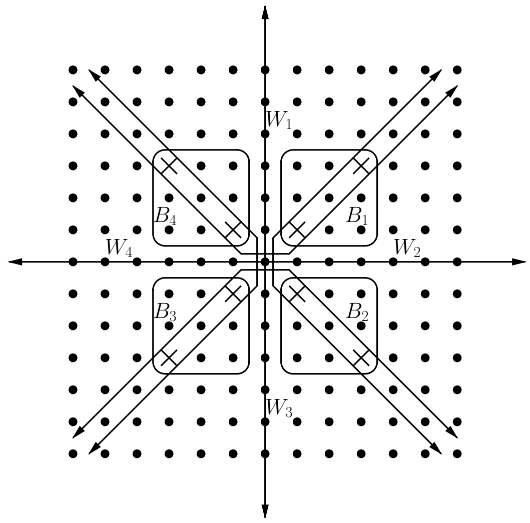

Figure 1: The initial position for player 1, the boxes B_i, and wedges W_i.

■

By symmetry and the pigeonhole principle we may assume that at least two spies, s_1' and s_1'' have positions (x, y) with $x \geq 0$ and $y \geq 0$. In fact we may assume that there are exactly two such spies, since each of the boxes B_2, B_3, and B_4 must contain a spy; we call these spies s_2, s_3, and s_4 respectively. Since the wedge property holds for W_3 and W_4, spy s_3 must lie in both wedges; furthermore its position must be either $(-3, -3)$ or $(-1, -1)$. This leads to the two cases below.

Case 1: s_3 **is at** $(-3, -3)$

The following discussion is illustrated in Figure **??**. Since s_3 is at $(-3, -3)$, the wedge property for W_4 implies that s_4 is in $[-3, -1] \times \{1\}$. Similarly s_2 is in $\{1\} \times [-3, -1]$. Furthermore, s_1' must be in $[0, 1] \times [0, 1]$ to guard against a meeting at $(0, 0)$ by the revolutionaries at $(-1, -1)$ and $(1, 1)$; neither s_2

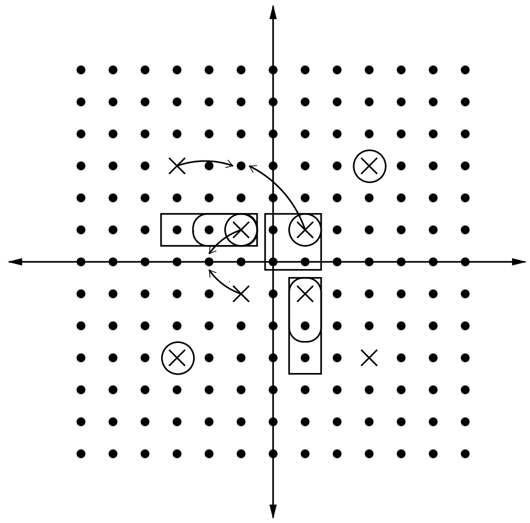

Figure 2: Case 1.

nor s_4 can guard $(0,0)$ since they must guard against the meetings at $(2,-2)$ and $(-2,2)$. Given these restrictions on s_1', s_2, and s_4 the wedge properties for W_1 and W_2 imply that s_1'' must be at $(3,3)$

Suppose the revolutionaries at $(-1,1)$ and $(-1,-1)$ form a meeting at $(-2,0)$ and those from $(-3,3)$ and $(1,1)$ form a meeting at $(-1,3)$. Spy s_4 must guard $(-2,0)$, so s_1' must guard $(-1,3)$; this means that s_1' has $y \geq 1$. By symmetry, s_1' has $x \geq 1$, so s_1' is at $(1,1)$. Now s_4 must be in $[-2,-1] \times \{1\}$ to guard against a meeting at $(-1,0)$ of revolutionaries from $(-1,1)$ and $(-1,-1)$. Similarly s_2 must be in $\{1\} \times [-2,-1]$.

We also must have $s_4 = (-1,1)$ or $s_2 = (1,-1)$. If not, the revolutionaries at $(-1,1)$ and $(-1,-1)$ can meet at $(0,0)$ while the revolutionaries at $(1,1)$ and $(1,-1)$ can meet at $(2,0)$. It will not be possible for the spies to guard both meetings. By symmetry we may assume $s_4 = (-1,1)$.

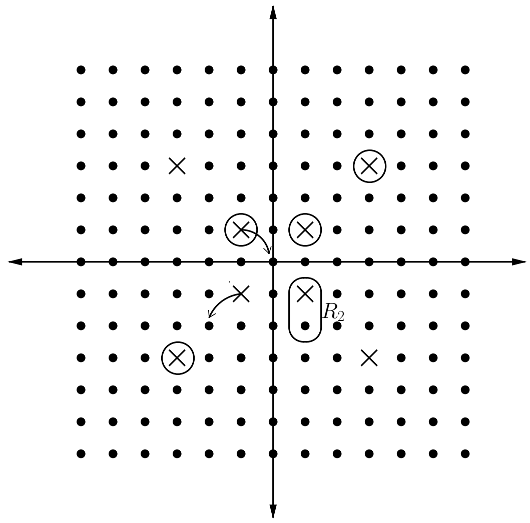

Figure 3: Case 1 after round 0.

Thus at the beginning of round 0, we may assume the spies and revolutionaries are located as in Figure **??**. This figure indicates that s_2 is located somewhere in $R_2 = \{1\} \times [-2, -1]$. Player 1's strategy is to move the revolutionary at $(-1, -1)$ to $(-2, -2)$ and the one at $(-1, 1)$ to $(0, 0)$, keeping the other revolutionaries in place.

We now analyze the positions that the spies must take at the end of round 1, see Figure **??**. Player 2 must keep the spies at $(3, 3)$ and $(-3, 3)$ fixed to continue guarding meetings at $(\pm 6, 0)$ and $(0, \pm 6)$. Besides these two spies, only the spy in R_2 (and only if it were located at $(1, -2)$) could be moved to help guard $(0, -6)$, but that spy can not assist as it must also guard the meeting of the revolutionaries at $(-2, -2)$ and $(1, -1)$ at the point $(0, -3)$.

Let (a, b) be the position of the spy in R_2 in round 0 and (a', b'), its position in round 1. We have $a' = 1$, since this spy must guard $(2, -2)$ and

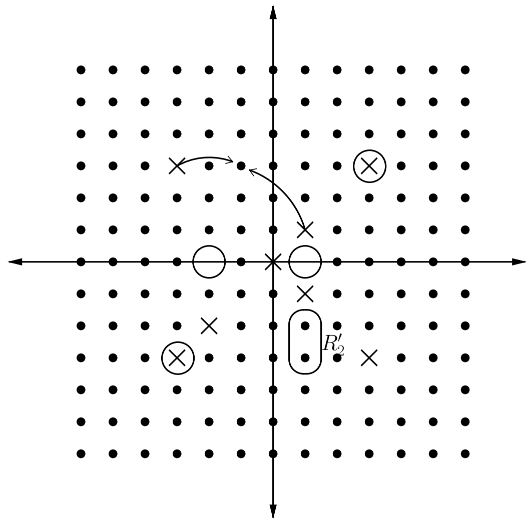

Figure 4: Case 1 after round 1.

also the meeting of $(-2,-2)$ and $(1,-1)$ at $(-1,-3)$. We must have $b' \leq -2$ as this spy must guard the meeting of the revolutionaries at $(-2,-2)$ and $(3,-3)$ at $(1,-5)$. Thus, player 2 must have a spy in $R_2' = \{1\} \times [-3,-2]$ at the end of round 1. (See Figure **??**.)

The spy located originally at $(1,1)$ cannot decrease its x-coordinate because it must guard $(2,0)$. This forces the spy originally located at $(-1,1)$ to decrease its y-coordinate to guard $(-1,-1)$. This same spy must also decrease its x-coordinate to guard against the meeting of the revolutionaries from $(-3,3)$ and $(-2,-2)$ at $(-5,1)$. This forces the spy at $(1,1)$ to move to $(1,0)$ to guard meetings at $(2,0)$, $(0,1)$, and $(0,-1)$. Note that the spy in R_2' cannot help guard these since it independently must guard the meeting at $(1,-5)$ by revolutionaries from $(-2,-2)$ and $(3,-3)$.

Now player 1 can win at $(-1,3)$ in 2 moves.

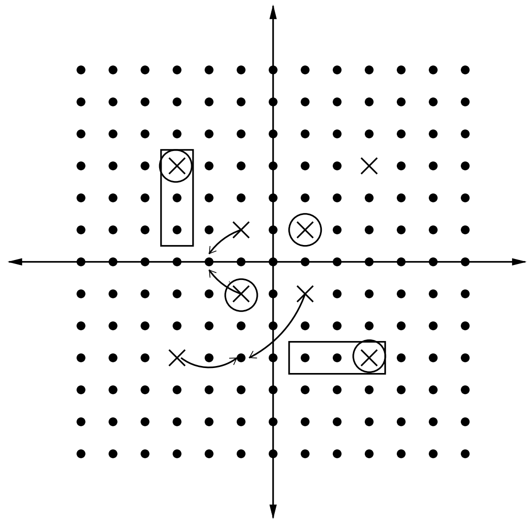

Figure 5: Case 2a.

Case 2: s_3 is at $(-1,-1)$

Since s_3 is at $(-1,-1)$ the wedge properties implies that s_2 is in $[1,3] \times \{-3\}$ and s_4 is in $\{-3\} \times [1,3]$. If s_2 is at $(3,-3)$ and s_4 is at $(-3,3)$, then the revolutionaries from $(-1,1)$ and $(-1,-1)$ can form a meeting at $(-2,0)$ which must be guarded by s_3. Simultaneously, the revolutionaries at $(1,-1)$ and $(-3,-3)$ can form a meeting at $(-1,-3)$ which will be unguarded (see Figure **??**).

Without loss of generality, suppose instead that s_2 is not at $(3,-3)$. By the wedge property for W_2, we must have s_1' is in $[1,3] \times [0,1]$ and s_1'' is in $[3,9] \times [0,3]$. (See Figure **??**). In fact s_1' must be at $(1,1)$ in order to guard the meeting at $(0,2)$. Now the revolutionaries at $(-1,-1)$ and $(1,-1)$ can form a meeting at $(0,-1)$. This must be guarded by s_3. Simultaneously the revolutionaries at $(-1,1)$ and $(-3,-3)$ can form a meeting at $(-3,-1)$ in

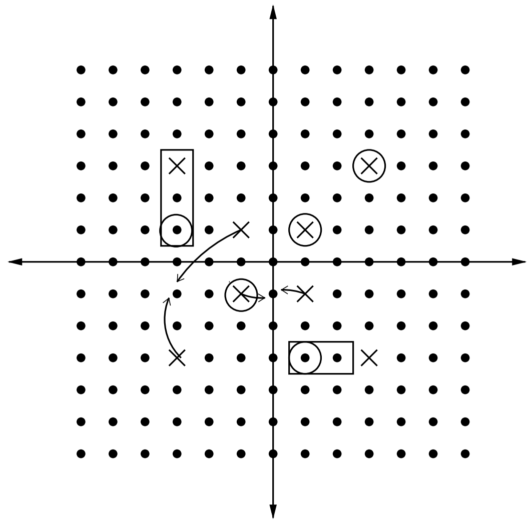

Figure 6: Case 2b.

two rounds; it can only be guarded if s_4 began at $(-3,1)$. By symmetry s_2 must begin at $(1,-3)$. Since now s_1'' must guard the meeting at $(0,6)$, it must be located at $(3,3)$.

Thus at the end of round 0, the spies and revolutionaries are positioned as in Figure **??**.

Player 1's strategy is to move the revolutionary at $(-1,1)$ to $(0,0)$ and the one at $(1,1)$ to $(2,1)$ while leaving all other revolutionaries unchanged. Figure **??** illustrates how the spies must be located at the end of round 1 in order to compensate. Player 2 must leave the spy at $(3,3)$ in place to guard $(6,0)$ and $(0,6)$. At the end of its move, the spy at $(-3,1)$ must be somewhere in $L_4 = [-4,-3] \times [0,2]$ as it must guard $(-6,0)$. Similarly the spy at $(1,-3)$ must remain in $L_2 = [0,2] \times [-4,-3]$.

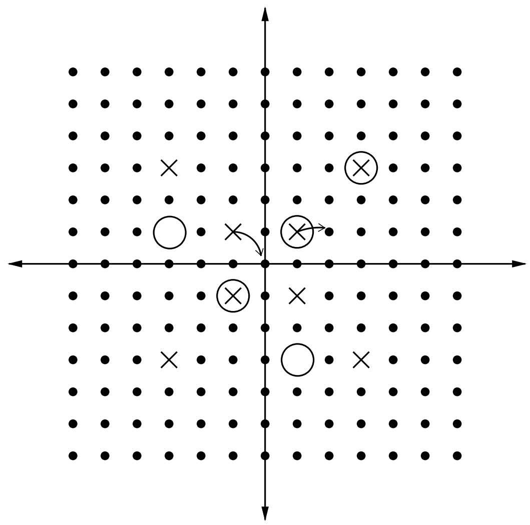

Figure 7: Case 2 after round 0.

The spy at $(-1,-1)$ must stay in place to guard meetings at $(-2,-2)$ and $(0,0)$ in one move. (Note: the spy at $(-1,-1)$ must guard $(0,0)$ as the spy at $(1,1)$ must guard against the potential meeting of the revolutionaries from $(2,1)$ and $(1,-1)$ at the point $(2,0)$.) Spy s_1' must move to $(1,0)$ to protect against meetings $(0,-1),(1,1),(2,1)$ in the next round (Spy s_3 cannot help as it must protect $(-2,-2)$).

Player 1's strategy is to simultaneously move revolutionaries $(0,0)$ and $(-1,-1)$ to $(-1,0)$, revolutionaries $(1,-1)$ and $(-3,-3)$ to $(-1,-3)$, and revolutionaries $(2,1)$ and $(3,-3)$ to $(4,-1)$. Only the spy at $(-1,-1)$ can guard the first meeting and consequently only the spy in L_2 can guard either of the other two meetings. Thus player 1 wins.

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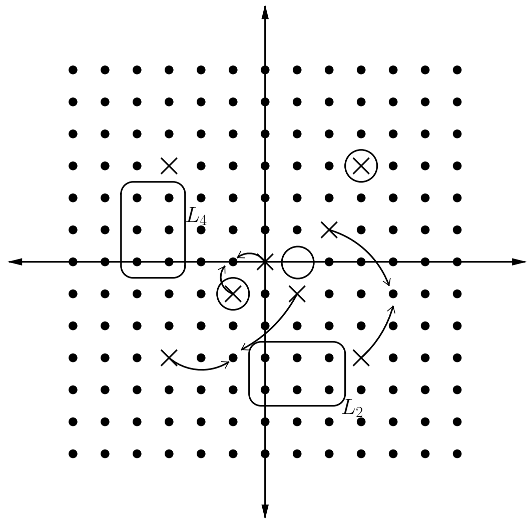

Figure 8: Case 2 after round 1.

Corollary 3.5 *Theorem* **??** *implies Theorem* **??**.

Proof: The winning strategy for player 1 in Theorem **??** does not involve movement of the revolutionaries outside of the box $[-6, 6]^2$ and takes no more than 5 moves to achieve a meeting of size 2. Letting G_i, $1 \le i \le \ell$ be sufficiently separated copies of $[-6, 6]^2$ in $\mathbb{Z}^{\boxtimes 2}$ and applying Lemma **??** gives $k(\mathbb{Z}^{\boxtimes 2}, 8\ell, 6\ell - 1) \ge k(\mathbb{Z}^{\boxtimes 2}, 8, 5) = 2$. Thus for all $\ell \ge 1$ and all $0 \le i < 8$ we have $s_2(8\ell + i) \ge 6\ell$. Hence $\liminf_{n\to\infty} s(\mathbb{Z}^{\boxtimes 2}, n, 2)/n \ge \frac{3}{4}$. By Lemma **??**, $k(\mathbb{Z}^{\boxtimes d}, r, s) \ge k(\mathbb{Z}^{\boxtimes 2}, r, s)$ for $d \ge 2$ and so $\liminf_{n\to\infty} s(\mathbb{Z}^{\boxtimes d}, n, 2)/n \ge \frac{3}{4}$ for $d \ge 2$.

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4 Conclusion

It would be of interest to get tight bounds on $k(\mathbb{Z}^{\boxtimes d}, r, s)$.

Continuous versions of this problem can also be considered. For example one could play the game in the plane, where each agent has the power to move to points within a Euclidean distance of 1 from their current position. This particular variant was suggested by Beck [**?**].

Theorem **??** has been generalized to the case of unicyclic graphs [**?**]. It would be interesting to characterize those graphs G for which $k(G, r, s) = \lfloor \frac{r}{s+1} \rfloor$.

The authors would like to thank the referees for their many helpful comments. We are grateful for the specific suggestions of one referee that greatly simplified our initial proof of Theorem **??**.

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