A Study of Combinatorial Games

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0.1 Introduction

What is a game? The average person would probably say that it is a competition between groups, like Monopoly, or Chess. Mathematicians might create a somewhat sounder definition: A model of a competitive situation that identifies interested parties and stipulates rules governing all aspects of the competition, used in game theory to determine the optimal course of action for an interested party [1]. This fuller definition is needed in particular when we start discussing combinatorial games, where the key questions we ask about such games are, what are the optimal strategies, and what do these strategies guarantee?

This paper is broken down into four chapters. Chapter 1 is on positional games. A positional game is where two players alternate turns to claim different points on some finite board. The goal of each of the players is to occupy some winning set of points or to stop the opposite player from occupying a winning set of points. This chapter is a good introduction to combinatorial games because it gives us a good base as to how to begin analyzing games and optimal strategies. For example, Tic-Tac-Toe under the normal rules optimally will end in a draw. Suppose you alter those rules ever so slightly and say that a win for player two is if they stop player one from obtaining a winning line (This is called the Maker-Breaker version of a positional game). In this case player one has a winning strategy.

Chapter 2 deals with an interesting tool called the quasiprobabilistic method that can be helpful when trying to analyze a specific type of positional game. This technique is only useful for the Maker-Breaker version of a positional game. Notice in this set of rules there is no draw available, either player 1 occupies a winning set (in which case they win) or they don't (player 2 wins). The idea behind this method is to look at a random play of the game and who is expected to win. This first part gives no optimal strategy for either player however it does give a good guess as to who is more likely to have a winning strategy. Thus the second part of the method looks to see if one can somewhat derandomize these games to show an optimal strategy for player one or two, in which one can guarantee win.

The third chapter offers a study of a specific combinatorial game called the Voronoi Game. This game was created in response to an economic question. The question was the competitive facility location question. In other words, how can I best place my pieces to claim as much territory as possible? Thus the Voronoi game was created to help answer this question. The set-up of this game is there are two players, some (normally infinite) board, and n sites for each player. A point on the board belongs to the player who has the closest site to that point. Thus placing their sites alternatively the question is who can obtain more territory? This section looks at two cases. The first is a square arena for a board. The second looks at the perimeter of a circle for the board. In the first case it is unknown who has the better strategy, but a result is shown that gives an answer if the rules are slightly altered [3]. In the second case it is shown that player two has a winning strategy.

The final chapter analyzes a game called Revolutionaries and Spies. The idea behind this game is that there are two parties where the object of player one is to create a meeting of revolutionaries at some point away from a spy, and the spies are attempting to stop any such meeting. The question is given a certain number of revolutionaries what is the minimum number of spies on the board such that the spies can always protect against such a meeting. The analysis of this game proves to be very hard when played on the infinite integer lattice board in two dimensions and where the meeting size is 2. The main problem analyzed in this section is that given 8 revolutionaries, how many spies does one need so that a winning strategy exists for the spies to stop a meeting of 2 revolutionaries without a spy. This will give us a lower bound for any large number of revolutionaries.

Chapter 1

Positional Games

We begin with a discussion on positional games. We first examine the game of Tic-Tac-Toe as an example. The traditional game, as everybody knows, is played on a 2-dimensional on a 3x3 board. However we can extend this game to any dimension and any size n and say that a winner is anyone who obtains a winning set. We call this the n^d-game. A winning set in this case is a "multidimensional" line with n points. In other words a sequence of n points such that at each coordinate the number of the coordinate is either increasing to n, decreasing to 1, or staying constant.

Def.
A **graph** (V, F) is a set of points or vertices, V, and a collection or family, F, of sets such that if a player occupies one of these sets they have won the game. These sets are called **winning sets**.

Def.
A **positional game** is a game played on a graph (V, F) where on each turn a player chooses a vertex to call his own and either i.) both players are trying to obtain a winning set in F (Symmetric game). ii.) player one is trying to obtain a winning set from F and the other is trying to stop him. (Maker-Breaker game)

In positional games, it often helps to alter the rules slightly to give us more of an understanding of the game. Thus we create the Maker and Breaker version of a positional game. We can see that in the traditional positional game of tic-tac-toe, each player is trying to occupy a winning position before the other. Now instead of players trying to occupy a winning set, we look at this game as Maker trying to occupy a winning set, with Breaker trying to stop them. Clearly, there is no way to draw in this game. Either Maker occupies a winning set after all pieces are taken or he doesn´t in which case Breaker wins.

Def.
Player 1 has a **weak-win** if he can always win as Maker in the Maker-Breaker version of a positional game regardless of Player 2´s actions. If no win exists then Player 2 (Breaker) is said to have a **strong draw**.

You can see that in the ordinary tic-tac-toe game (3^2-game) there is no win in the 3^2 game. However, this game has a weak-win. If you look at the Maker-Breaker version a winning strategy is available for player one. This is shown in Figure 1. The moves for each player are denoted by the subscripts. After Player one's center move, Player 2 has 2 options (either a corner or an outer middle point. In either case Player 1 can force the remaining moves of Player 2 resulting in a win for Maker in each case.

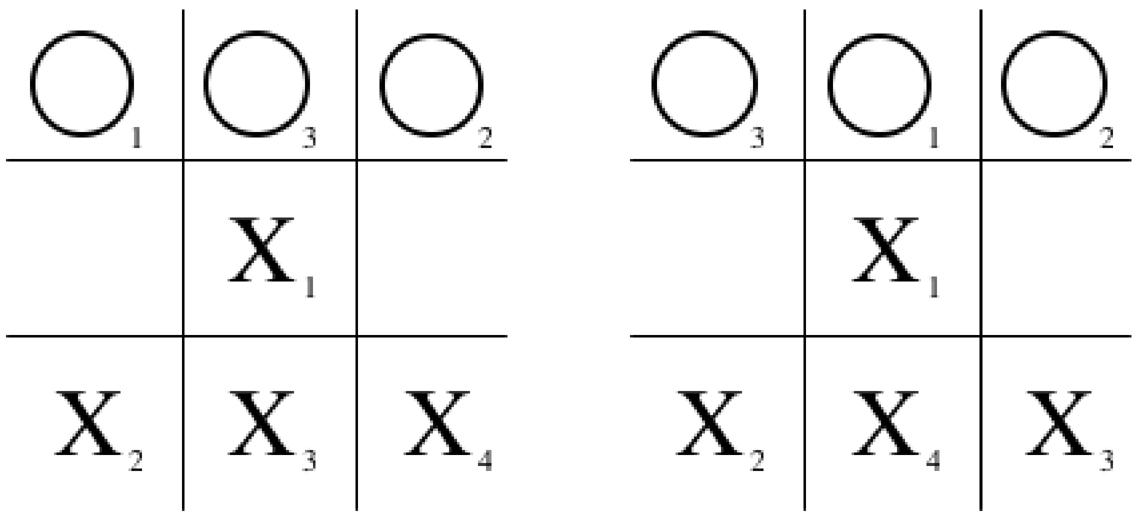

Figure 1.1: 3^2 Maker-Breaker game showing that Maker can always win

In Maker-Breaker version of the game, if Breaker can always force a win, then we call this a strong-draw in relation to the n^d-game. An example of this is the 5^2 version. You can enforce the pairing strategy listed below to stop 1st player from ever occupying a winning set:

12	1	8	1	11
6	2	2	9	10
3	7	*	9	3
6	7	4	4	10
11	5	8	5	12

Note: Whenever Maker plays on a specific position, Breaker can play on the other number (thus if Maker plays on 8 Breaker plays on the other copy of 8). If Maker plays on * or takes a number that Breaker already has, just choose another point at random. You can see that any winning set has a repeated number in it, which shows that Breaker can win.

I would like to now give a formal definition of what a strategy is.

Def.
A **strategy** for Player 1 is given any partial play $x_1, y_1, \ldots, x_n, y_n$ where each x_i is a move of Player 1, and y_i is a move of Player 2, $\exists$ an explicit legal move x_{i+1} for Player 1. A strategy for player 2 is defined similarly. A **winning (drawing) strategy** is a strategy such that regardless of the other player's moves the strategy will force a win (draw).

Thm.
In any finite positional game there exist 3 possibilities.
a.) There exists a winning strategy for player one.
b.) There exists a winning strategy for player two.
c.) There exists a drawing strategy for both players.

This makes logical sense in fact a proof through De Morgan´s Laws can be obtained.
Pf. (sketch)
$(\neg A)(\neg B) \Rightarrow C$

A theorem that immediately follows this one is the **Strategy Stealing Theorem.**

Thm.
In a finite symmetric positional game only option 'a' or option 'c' exists from the previous theorem.

Pf.
If player two had a winning strategy then taking the first move as arbitrary let player one employ player twos strategy. If player twos strategy requires moving in the position of a previously arbitrary move then just again choose an arbitrary move. The point is an arbitrary move does not hurt first player. This results in a win for Player 1, this is a contradiction. Thus at worst player 1 has a drawing strategy. $\square$

Def.
A k**-coloring** of a graph is where every vertex is assigned a color among k colors. A set is **Monochromatic** with respect to a coloring if every point in the set is colored with the same color.

Def.
A set is called k**-colorable** if there exists a coloring with these colors such that no monochro-

matic winning set exists.

Cor.
If no draw-end position is available then Player 1 has a winning strategy. In other words if it is impossible to 2 color the graph such that regardless of the two coloring a winning set will be monochromatic then Player one has a winning strategy.

The next theorem gives us a game that an explicit winning strategy for Maker.

Thm.
Given a finite graph (V, F) of chromatic number at least 3. Take a second copy of the graph (V', F'). Let $W = V \cup V'$ and $G = F \cup F'$. Then on the Maker-Breaker Game on (W, G), Maker has an explicit winning strategy.

Pf.
The idea is every time that Breaker makes a move Maker makes the opposite move in the other half of the graph. If this is not possible at any point choose a remaining point at random and this will not hurt Maker. Thus since the graph is at least three colorable then each half of the graph has a win in it. Thus if Breaker has a winning set on one (F or F') then Maker has a win on the other half set since F and F' are copies and Maker has the copy of Breaker's winning set in the opposite half. □

The most common way to guarantee a win or draw in a game is through a pairing strategy. In other words whenever a Player makes a move you have a counter move that somehow negates the other Players benefits of his previous move. We showed a couple pairing strategies above: the pairing strategy one can give to obtain a weak win in the Maker and Breaker version of 3^2 game (Tic-Tac-Toe). We also showed the pairing strategy for a strong draw in the 5^2 game.

Def.
A division of the vertices into pairs such that every winning set has at least one pair of vertices is called a **draw force pairing**.

If a positional game has a draw force pairing then a player can always block every winning set by choosing the other point in these pairs. The 5^2 game has a draw force pairing listed above.

Now we define a matching criterion theorem.

Thm.
Let F be a family of winning sets if F satisfies the following condition: $\forall G \subseteq F$, $2|G| \leq |\cup_{A \in G} A|$ then there is a draw force pairing.

Pf.
The König Hall Theorem applies here. Traditionally we think of a matching for König Hall, however in this case we look at 2 element representatives. Another way, a matching exists if we draw a bipartite graph between the set of vertices, V, and two copies of F. Thus we have 2 element representatives for each winning set. $\square$

Cor.
If F is a family of winning sets such that $|A| \geq n, \forall A \in F$. If V is the set of all points and for every $x \in V$, x is in at most $\frac{n}{2}$ sets then there exists a draw force pairing.

Pf.
For any subfamily $G \subseteq F$, we have $n|G| \leq \sum_{A \in G} |A| \leq |\cup_{A \in G} A| \frac{n}{2}$
Here we have given any $G \subseteq F$, the size of the families contained in G are at least $\sum_{A \in G} |A|$ since each $A \in G$ is of size at least n. Now looking at the points contained in these families in G the number of times that each one is counted is at most $\frac{n}{2}$. Thus we have $\sum_{A \in G} |A| \leq |\cup_{A \in G} A| \frac{n}{2}$. Multiplying both sides by $\frac{2}{n}$ we apply the previous theorem. $\square$

One can classify positional games into 5 classes.
1.) There does not exist a draw end position. (2^2 and the 3^3 of the n^d game)
2.) There exists a draw ending position but still Player one has a winning strategy (4^3 game)
3.) The symmetric game has a drawing strategy but there exists a weak-win (3^2 ordinary tic-tac-toe)
4.) There is a strong drawing strategy for Player two in the symmetric game (4^2) game
5.) There exists a draw force pairing for Player two (n^2 for all $n > 5$) [2]

Chapter 2

Quasiprobabilistic Method

After defining the Maker-Breaker version of a game and understanding its significance in a positional game. We introduce a surprising method that helps to solve the Maker-Breaker game for certain cases. The method is called the quasiprobabilistic method. The idea behind it is to look over all possible plays of the game at random. Thus if the board size is N there are $N!$ different plays of the game. Since each is equally likely the probability of an occurrence of a particular game is $1/N!$. If the large majority favors player one not receiving a winning set then it seems as though Breaker would have a winning strategy. Similarly, if the overwhelming majority of plays end with Maker receiving a winning set, Maker is likely to have a winning strategy. Unfortunately playing randomly does not give an overwhelming majority in the symmetric game. Thus, this method gives no help in the symmetric version of positional games.

This method has two steps. The first is a probabilistic analysis of the randomized game. The second step is the derandomization of the game by potential techniques. Here we arrive at a new theorem.

<u>Thm.</u> (The Erdös-Selfridge Theorem)
This theorem states: If F is the family of winning sets and is n-uniform (every winning set has n elements) $\mid F \mid < 2^{n-1}$ then Breaker has a winning strategy in the Maker-Breaker game on F.

Before we prove this above though let us prove a helper theorem first. This is the first part of the quasiprobabilistic method. The proof of this theorem is going to look at the expected number of winning sets on the board and find that the value is less than 1. Which would lead us to believe that If F is n-uniform and $\mid F \mid < 2^{n-1}$ then the chromatic number of F is ≤ 2. This means that the graph is 2-colorable, there exists a play such that Maker loses (does not occupy a winning set).

Pf.

Let N be the number of spaces on the board. If the game is played randomly then the expected number of winning sets occupied by either player is $2* \mid F \mid *\frac{\binom{N-n}{[N/2]}}{\binom{N}{[N/2]}}$. Think of this as saying that for each player and each winning set the other player has N-n possible board positions that they can pick and they get $N/2$ choices. So the probability that you picked the winning set is $\frac{\binom{N-n}{[N/2]}}{\binom{N}{[N/2]}}$. Note: $[N/2]$ is the largest integer that is less than or equal to $N/2$. $2* \mid F \mid *\frac{\binom{N-n}{[N/2]}}{\binom{N}{[N/2]}} = 2* \mid F \mid *\frac{[N/2]}{N} * \frac{[N/2]-1}{N-1} * \frac{[N/2]-2}{N-2} * \cdots * \frac{[N/2]-n+1}{N-n+1} \leq 2 * |F| * 2^{-n} < 1$ Thus the expected number is less than one so there exists a win for Breaker. □

Note: Though this theorem is not needed for the proof of the Erdös-Selfridge Theorem. Going through this proof gives us an idea that perhaps we would want to try to prove a win for Breaker, through a sort of derandomization.

Pf.

We give an explicit winning strategy for Breaker. After Maker's initial move, Breaker will try to minimize the "danger" of Maker's subsequent moves. Breaker does this by looking at which points possess the most "danger" and choosing one for the next move. Let us define S (the set of survivor sets) to be the set of all winning sets not containing a point belonging to Breaker (this set gets smaller throughout play). The danger function at round i (where a round is a move for Breaker followed by a move for Maker) is $D_i = \sum_{s \in S} 2^{-u_s}$. In other words it is the sum over all winning sets where each set contributes 2^{-u_s} and u_s is the number of remaining unoccupied points in set s. If at the end of play $\sum D_{end} \geq 1$ the set of survivor sets is non-empty, thus Maker has completely occupied some winning set. Otherwise $D_{end} < 1$ so there are no survivor sets and Breaker has won. If Breaker can keep $D_i < 1 \ \forall i$, then Breaker would have a winning strategy. Well after Maker's initial move(x_1) we have $D_0 = \sum_{A: x_1 \in A \in F} 2^{-n+1} + \sum_{A: x_1 \notin A \in F} 2^{-n} \leq |F| * 2^{-n+1} < 1$. Imagine worst case where x_1 is in every set well even in this case the value of D_0 is still less than 1. Now look at D_i at any round. Well Breaker will choose the value that lowers this function the most, and Maker will attempt to maximize it to finish the round. Well notice that any particular point will double the value of any survivor set containing it for Maker and destroy all those sets for Breaker. This means that a best move for Breaker is the same as a best move for Maker since any point will lower or raise the danger function by the same amount. Well this in turn means that $D_i \leq D_{i+1}$. Thus the danger function whose initial starting value is less than one and Breaker has a winning strategy. □

Let's now shift to a different problem that involves a similar argument to this potential argument above. The following puzzle is called Solitaire Army. Many people have visited restaurants and played peg solitaire where pegs get hopped over and removed and the object is to have one peg left at the end. Well the difference in this puzzle is that instead of having to remove all but one peg, one must get a peg to a certain position on the board.

In the solitaire army puzzle the board is the integer lattice and there exists a horizontal line such that all the men (pegs) are below this line. Now the question is how many men are required to send man 1, 2, 3, 4 or 5 holes up into the half plane. It turns out that, for the first 4 cases the number of jumps needed is respectively 2, 4, 8, 20.

Figure 2.1: Set-up for a 4-hole jump. [2]

However, sending a man five holes takes a few more men then twenty.

Thm.
It is impossible to send a man five holes into the plane with a finite number of jumps.

Pf.
First assign values to each hole in the plane such that if H_1, H_2, and H_3 are 3 consecutive holes, and $v(H_1)$, $v(H_2)$, $v(H_3)$ are their values. Then $v(H_1)+v(H_2) \geq v(H_3)$. Thus when a man from H_1 jumps over a man at H_2 we can think of this as replacing those two men with a man with value $v(H_3)$ at hole H_3. This change can never be an increase in value of the position. What this set-up guarantees is that no play is possible from an initial position to a target position if the position has a higher value.

Thus let us define w to be a positive number such that $w + w^2 = 1$. If we assume that we can jump a man 5 spaces forward into the half plane. Write a 1 five spaces into the half plane, and extend it in the following way.

Now if you sum up the value of the top line of the plane we have $w^5+2w^6+\ldots = w^5+2*\frac{w^6}{1-w} = w^5 + 2 * \frac{w^6}{w^2} = w^5 + 2 * w^4 = w^3 + w^4 = w^2$. This means the value of the upper half plane is: $w^2 * (1 + w + w^2 + \ldots) = w^2 * \frac{1}{1-w} = 1$. This means that no finite number of points will send a man five holes up into the half-plane. □ [2]

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & & w & & & & & \\
 & & & & & w^2 & & & & & \\
 & & & & & w^3 & & & & & \\
 & & & & & w^4 & & & & & \\
\cdots & w^9 & w^8 & w^7 & w^6 & w^5 & w^6 & w^7 & w^8 & w^9 & \cdots \\
\cdots & w^{10} & w^9 & w^8 & w^7 & w^6 & w^7 & w^8 & w^9 & w^{10} & \cdots \\
\cdots & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^8 & w^9 & w^{10} & w^{11} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Figure 2.2: [2]

Chapter 3

Voronoi Game

The Voronoi game is an idea made up to study Competitive Facility location. The idea behind it is that there are 2 players and a playing area. The object of each player is to take as much area as possible. The way the play goes is that there are some number n pieces for each player and each player alternates placing points on the board. At the end of placing these points, the area owned by player 1, are those points that are located closer to a point owned by player 1 than player 2. The remainder of area is given to player 2 (points closer to player 2).

The question one asks is, what are optimal playing strategies for each player, and who is guaranteed more area? A natural playing board for which one would ask this question is a square. In fact the answer as to optimal strategies is not known. However a paper called the One-Round Voronoi game [3] has shown results regarding this playing board. Summarizing this paper, the idea of alternating turns is abandoned and instead the alternation of rules is that player one plays all their pieces followed by player two. This paper's results show that player two has a winning strategy given a sufficient size n that guarantees at least a fraction of $\frac{1}{2} + \alpha$ where α is some constant independent of n. Though I will not reproduce the proof of this result, I will give a brief sketch. For the proof the authors assume that the square's length is $\sqrt{n}$ and that n is sufficiently large.

<u>Thm.</u>
For large enough n playing on a square arena in the One Round Voronoi Game, Player 2 can receive $\frac{1}{2} + \alpha$ fraction of the total area.
<u>Pf sketch</u>
The idea of the proof is broken down into stages. The first two stages are to show that a random point's expected value given n points from player 1 already played is at least $\frac{1}{2} + \beta$ for some $\beta > 0$ and not depending on n. The first stage looks at the playing board of a torus (as opposed to a square). The only difference between these two boards is that on a square

a "closest point" will not look at points wrapping around the edges of the square (thus the points $(0,0)$ and $(0,\sqrt{n})$ are the same point). The second stage creates this same result with boundary edges included. The third stage shows that one can select δn points where $\delta > 0$ does not depend on n, such that the expected value of these points is $(\frac{1}{2}+\beta_1)\delta n$. Not surprisingly the constant again does not depend on n. The final stage unsurprisingly shows the main result, by using these δ random points and then applying an area stealing strategy for large remaining regions. [3]

The main purpose of this portion of the paper is to show that if we play this game on the perimeter of a circle (as opposed to a square area we are trying to get as much of the perimeter as possible) player 2 can take over half the perimeter. Unfortunately after showing this result, it was found to already have been proven in another published paper [4]. We label the first player as Black and the second player as White. Though White can attain more perimeter than Black I show that Black can make the amount that White can win by arbitrarily small. There is one exception if we let n be the number of move for each player, and $n = 1$. Then clearly they split the circle's perimeter regardless of their moves. So for $n \geq 2$ the idea is as follows.

Let us first show that white can take more than $\frac{n}{2}$ of the circle.

Thm.

In the Voronoi game played on the perimeter of a circle, White (player 2) has an optimal strategy that guarantees them $\frac{1}{2}+\epsilon$ of the perimeter where $\epsilon > 0$ is a constant chosen by Black(player 1).

Pf.

First Black makes a move. Let $D = d_1, d_2, \ldots, d_n$ be a set of points called **dividing points** that divide the circle into n equal parts where d_1 is Black's first move. Thus for $n = 8$ we have a picture like Figure 1.

Stage 1: White's beginning moves are to take dividing points until none remain unoccupied. Figure 2 represents one possibility:

Stage 2: At this point look at the largest distance between two adjacent black points. This maximum distance is at most 1 since all the dividing points have been taken. This corresponds to two blacks being on adjacent dividing points. At this point white begins to insert white points into any point within these intervals with black endpoints having distance 1 (This will split the interval to give $\frac{1}{2}$ to white and $\frac{1}{2}$ to black). Now white has enough pieces to do this because black has at most $\lceil \frac{n}{2} \rceil - k - 1$ intervals, where k is the number of points Black did not place on these dividing points. The number of white points used for taking dividing points is $\lfloor \frac{n}{2} \rfloor + k$ Thus comparing these two values together we notice that white has at least 1 point remaining after we have split all these intervals.

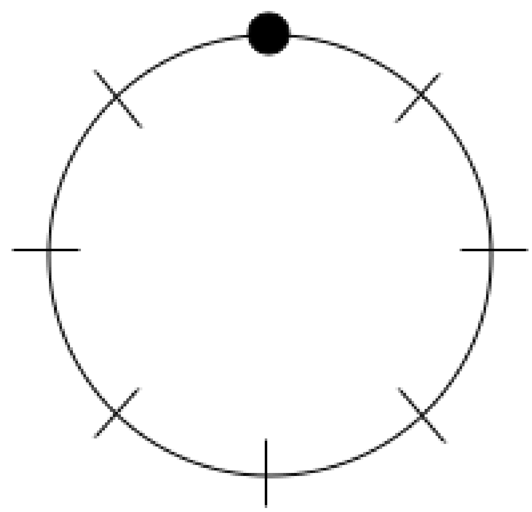

Figure 3.1: Black takes first spot and divide circle into equal intervals

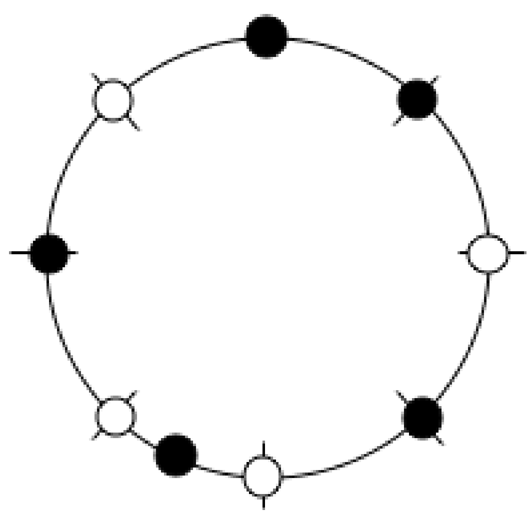

Figure 3.2: White takes a point on one of the dividing lines

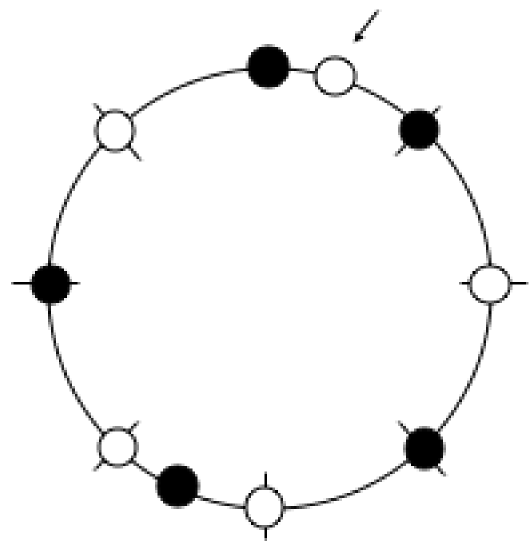

Figure 3.3: One such possibility of play where White breaks up Black's largest intervals

Stage 3: After all these large intervals are broken up, white will spend his remaining pieces (except the last one) in breaking up any intervals with two adjacent black pieces (this stage is skipped if white only has one piece remaining). White can always break up intervals because there is one more black piece to white piece on the board, which means there is always a black interval that can be broken with a white piece.

Final Stage: Note that after black's initial move the dividing points D create n intervals and black cannot have a point interior to each interval since they at that point have only $n-1$ points remaining. Thus one interval must be left without any blacks in it. Now white does not have adjacent white pieces unless they are endpoints of an interval (they are both on dividing points). Also note at this point that there are no adjacent black intervals that have size ≥ 1. This means that there is at least one pair of adjacent dividing points that has no black point or white point between them, and that at least one endpoint must be white.

Now if both endpoints are white use the final move to split a black interval. Otherwise look at the longest interval that has two adjacent blacks. This must be strictly less than 1. Now we take our last point by placing it close enough to the black endpoint such that the distance between the adjacent whites is greater than this maximum adjacent black distance. Figure 4 shows the 2 possible endings.

If we look at all the intervals going around the circle we notice that for every interval that has a black and a white endpoint, the amount of the interval is shared equally between black and white. As for the adjacent white intervals they are all greater than the adjacent black intervals. The number of adjacent white intervals is equal to the number of adjacent black intervals and they are clearly both greater than 1. This means that white must hold more

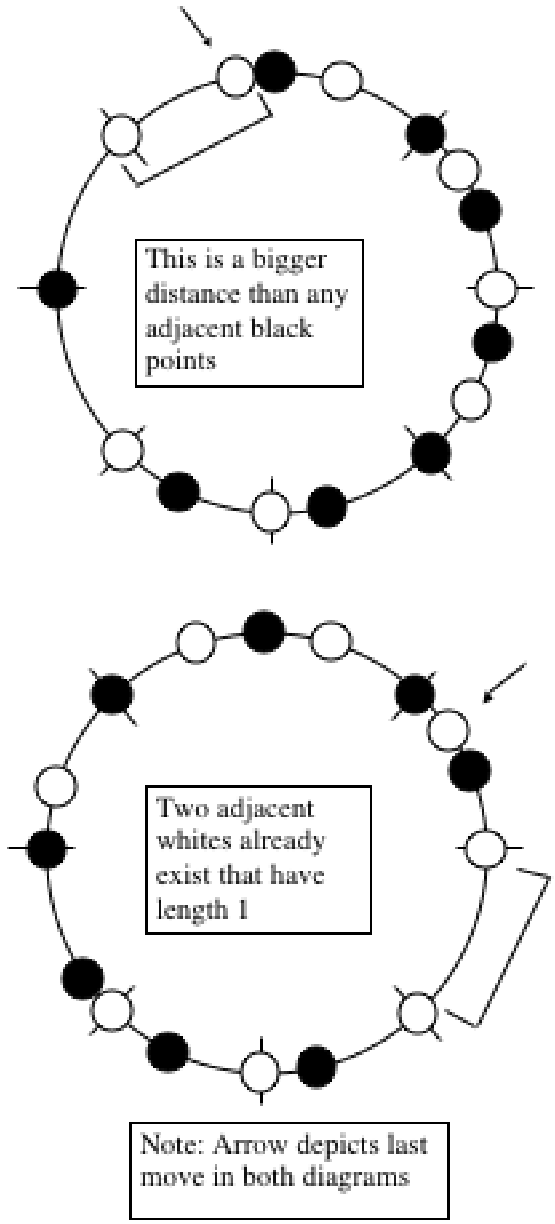

Figure 3.4: The top depicts the move if there are no adjacent whites. The bottom depicts the move otherwise.

than $\frac{n}{2}$ of the perimeter.

The proof to show that black can limit the amount that white wins by is much simpler.

<u>Thm.</u>
In the Voronoi game played on the perimeter of a circle, Black (player 1) has an optimal strategy that guarantees them $\frac{1}{2} - \epsilon$ of the perimeter where $\epsilon > 0$ is a constant chosen by Black.

<u>Pf.</u>
Let $\epsilon > 0$ be given. After black's initial move, split the board (just as white did) into n dividing points. Moving around the board clockwise place a black point that is between $(1 - \frac{2\epsilon}{n}, 1]$ of the previous black point. This makes the distance between any two black points greater than $(1 - \frac{2\epsilon}{n})$. This means that white regardless of placement either splits these intervals, or if more than one white is in an interval then black takes a whole interval somewhere else. This means that the amount that black claims regardless of white's placements is greater than $(1 - \frac{2\epsilon}{n})\frac{n}{2} = \frac{n}{2} - \epsilon$.

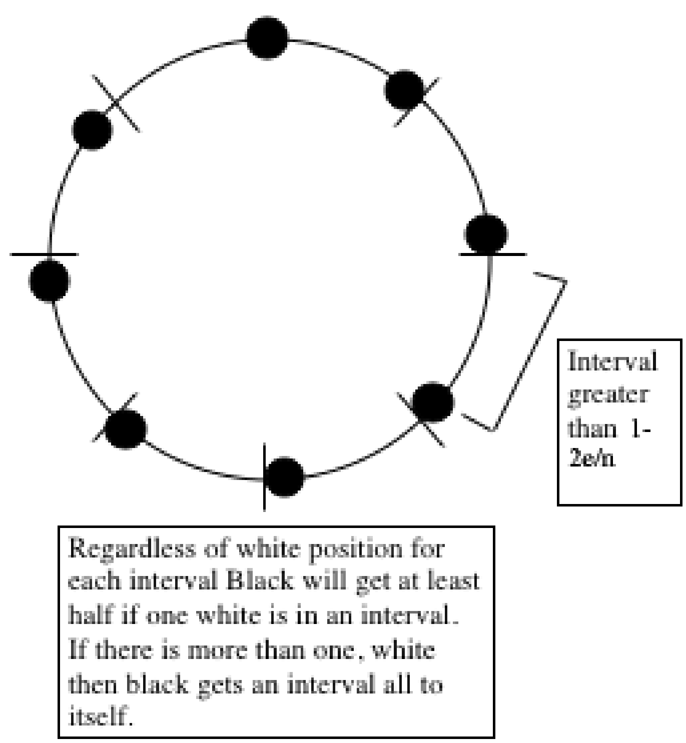

Figure 3.5: Black's play that guarantees $\frac{n}{2} - \epsilon$ of the perimeter

This ends the proof and shows that white can always win this perimeter game, however black can always decide how small they want the win to be. □

Chapter 4

Revolutionaries and Spies

This portion of the paper is dedicated to the study of a combinatorial game called Revolutionaries and Spies. This game is a two player game in which one player plays as the Revolutionaries and the other (not surprisingly) are the Spies. The Revolutionaries object is to bring a group together without a spy being present, and the Spies are trying to stop it from happening. The question that arises from this game is how many spies are required to prevent this revolutionary meeting? Well first let's define the set-up and how the moves (rounds) are defined in this game.

This game can be played in any dimension however this paper deals with the 2-dimensional case, and the board is the integer lattice. Let r be the number of revolutionaries, and s be the number of spies. The first round of play begins by the revolutionaries placing their pieces on the board followed by the s spies being placed wherever on the board. Now note that there is no restriction as to where pieces can be played (thus multiple pieces can be placed on the same point). This is the end of round one. Every subsequent round works in the following way, the revolutionaries may move any number of their pieces, and their movement is just like a king's move in chess. Thus each piece at the end of the revolutionaries turn will be in one of 9 places based on the previous position (Each coordinate of the point can increase, decrease or stay the same). After this the spies move in the same way. Again the goal of the revolutionaries is to bring together some k revolutionaries to the same point without a spy being at the same coordinate given any finite number of rounds to do so. Thus the question is what is the minimum number s given k and r?

<u>Def.</u>
Let $spies(r, k)$ denote the minimum number of spies needed to prevent a meeting of size k given r revolutionaries in a finite number of rounds.

In dimension 1 it is easily showed that on the integer line for any goal k the number of spies needed to cover r revolutionaries is $\lfloor \frac{r}{k} \rfloor$ [5]. Here the idea is to equally split the group such that each spy is on every kth revolutionary moving left to right (thus each spy is on the kth, 2kth...) revolutionary, and the Spies can hold this invariant. In two dimensions this number does not work unfortunately. This paper explores that for $r = 8$ revolutionaries on a 2-dimensional board and a goal of $k = 2$ the minimum number s that is needed is 6. In fact for numbers 3 through 8 there needs to be at least $r - 2$ spies to stop the revolutionaries.

Thm.
$spies(r, 2) \leq r - 2$
Pf.
Note if it is true for $r = 3$ then we can place a spy on all but 3 revolutionaries and reduce it to that case. Thus I show $r = 3$.
Given any position of the revolutionaries, place the spy in the "middle" of all of them. In other words look at the x-coordinate of all three and choose the middle value, and do the same for the y-coordinate. You'll notice that no matter how the revolutionaries move you can always keep this invariant since the middle only alters by at most 1 for each coordinate. Finally, note that if the revolutionaries were ever to meet the middle would be located where the meeting place was. □

This shows that $r - 2$ is good enough for 3. However interestingly one can quickly see that if the spy were to ever deviate from this strategy, in the case of 3 revolutionaries, two revolutionaries could meet. There are other somewhat short proofs for the other numbers up to 7. When it comes to 8 though a short proof seems difficult to find. Without further ado I prove this result.

Thm.
$spies(8, 2) = 6$ Thus $spies(r, 2) \geq 6\lfloor \frac{r}{8} \rfloor$ for $r \equiv 0$ or 1 mod 8. Also $spies(r, 2) \geq 6\lfloor \frac{r}{8} \rfloor + t - 2$ for $r \equiv t$ mod 8, $t \neq 0, 1$ and $t < 8$.

Pf. Enough to show that $spies(8, 2) > 5$
Let us first suppose that 5 spies are enough to prevent any 2 of 8 revolutionaries grouping together without a spy present. The proof that follows will be a breakdown into different cases eventually showing that none are possible. Let us start by showing an initial position of the revolutionaries.

In figure 1 the X's represent revolutionaries, and to ease explanation let us suppose that the center dot is in position (0,0). The Box B_1 represents the rectangle $[-2, 2] \times [-2, 2]$. The inner box B_2 is the box $[-1, 1] \times [-1, 1]$.

Case 1:
Suppose ≤ 1 spies are outside B_1. Then either all spies must be strictly left of the line $x = 3$, or strictly right of the line $x = -3$. Let's suppose the former case. This means that the spy at $(3, 3)$ and $(3, -3)$ can meet at $(6, 0)$ in three moves. No spy can reach this because at

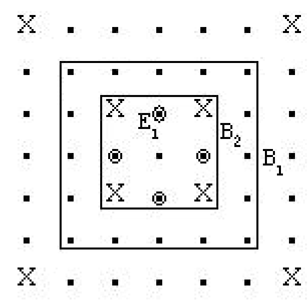

Figure 4.1: The starting position

most it has x-coordinate 2 which means in 3 moves it could only reach as far as $x = 5$. The latter case is symmetric.
Now ≥ 2 spies outside B_1.

Case 2:
Suppose ≤ 1 spies are within B_2. Since at most 3 spies reside within B_1 and outside B_2 (by case 1) these spies can reach at most 3 of the points $\{(0,1),(1,0),(-1,0),(0,-1)\}$ (circles with dots in the diagram) within one move. Without loss of generality let E_1 (point $(0,1)$) be a spot which they can't cover on the initial move. The top 2 revolutionaries in B_2 at the points $\{(-1,1),(1,1)\}$ will move to E_1 on their first move. The bottom 2 revolutionaries at the points $\{(-1,-1),(1,-1)\}$ will move to the center point $(0,0)$. Thus E_1 and the center point $(0,0)$ must both be covered by a spy. This is impossible since at most one spy is in B_2 and only that spy can reach those points which means one of these points would mean a win for the revolutionaries.
Now ≥ 2 spies in B_2.

Case 3:
Suppose there are 3 spies are strictly outside B_1. First note that one spy must be placed on an outside revolutionary (i.e. on one of the points $\{(-3,-3),(-3,3),(3,-3),(3,3)\}$). The reason is because within 3 moves, a meeting of revolutionaries can occur at the points $\{(6,0),(-6,0),(0,6),(0,-6)\}$ and the only way a spy can cover 2 of these points is if he is located directly on one of the outside revolutionaries. I will assume the top right point has a spy $(3,3)$ without loss of generality. In fact for all remaining cases I will assume a spy located at this specific point because we have already proven there are ≤ 3 spies outside B_2.

Case $3a_1$:
Suppose the remaining 2 spies are not located in opposite corners of the box B_2. In this case at least one of the 4 internal walls must be left uncovered by an internal spy. An **internal wall** in this case are the points along the edge of the box B_2 (i.e. the east internal wall are the points $\{(1,-1),(1,0),(1,1)\}$). A spy is on 2 internal walls if and only if it is in a corner of B_2, so since at least one spy is not on a corner by assumption at most 3 internal walls can be occupied by a spy. Now notice that both spies cannot be on the same internal wall

inside B_2 else the 2 revolutionaries not on that internal wall can immediately run together and create a win (thus if 2 spies were on the north internal wall, the two revolutionaries on the south internal wall could meet at point $(0, -1)$ to create a win. So now let's suppose the east internal wall is uncovered as in Figure 2 (which is identical to the north internal wall being uncovered in the diagram just flip position about $y = x$ to obtain this).

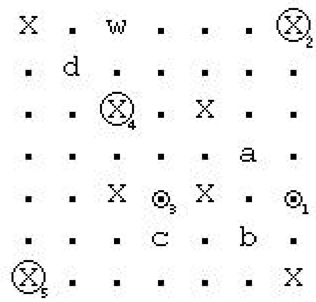

Figure 4.2: Given that there must be a spy on the north east outer corner, and supposing the east internal wall was left uncovered. Letters 'a' through 'd' represent immediate danger spots.

Given the east internal wall is open spy 1 must be placed on the east outer wall to be able to block wins 'a' (point $(2, 0)$) and 'b' (point $(2, -2)$). An **outer wall** corresponds to the points outside B_1 on one specific side (so the east outer wall would be the points $\{(3, -3), (3, -2), (3, -1), (3, 0), (3, 1), (3, 2), (3, 3)\}$. This forces spy 5 to be in the bottom left (point $(-3, -3)$) covering both west and south outside wins (wins at the points $\{(0, -6), (-6, 0)\}$ in 3 moves). Now spy 4 must be placed exactly in the top left inner corner of B_2 to prevent win 'd' (point $(-1, 1)$). Thus spy 3 must be placed at $(0, -1)$ since we already stated that it must be within the inner box B_2, it cannot be on the east internal wall or on the same internal wall as spy 4, and it must protect the win at 'c' $(0, -2)$. The winning strategy for the revolutionaries is moving the revolutionary at $(1, 1)$ and the revolutionary at $(-3, 3)$ towards position 'w' (position $(-1, 3)$), meanwhile running the two revolutionaries on the west internal wall together at point $(-2, 0)$. Since only spy 4 can have cover both positions $(-2, 0)$ and $(-1, 3)$ in time we have a win for the revolutionaries.

Case $3a_2$:
Now suppose the south internal wall was uncovered see Figure 3 which is identical to the west internal wall being uncovered (again just flip over $y = x$ to show this).

Now a spy has to be in L_1 to prevent the move 'd' (point $(-2, -2)$) and the winning move to the west (point $(-6, 0)$). Thus spy 1 is forced to its position on the bottom to protect wins 'b' and 'c' (points $(0, -2)$ and $(2, -2)$) and spy 4 must be at $(-1, 1)$ to protect win 'e'. Thus spy 3 must be exactly placed as such to stop win 'a' at point $(2, 0)$ and by our initial assumption it must be located on a different internal wall than spy 4 and it can not be placed in the corner by our current assumption that the south internal wall is uncovered.

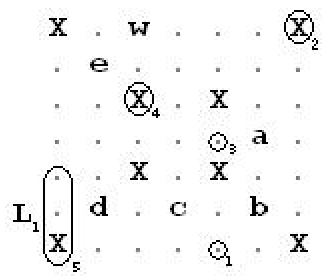

Figure 4.3: Just as Figure 2 except this time let the south internal wall be left unguarded.

Now the winning move for the revolutionaries is as before except that spy 4's revolutionary (point $(-1, 1)$) and the revolutionary below it at point $(-1, -1)$ move together at point (-1,0), again forcing 4 to cover both this point and 'w' (point $(-1, 3)$).

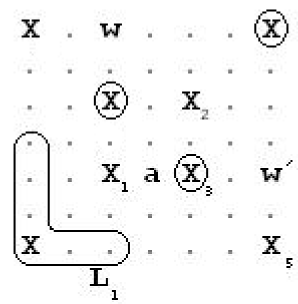

Figure 4.4: This is the case when the inner two spies are not aligned with the north east outer corner. The win is at 'w' or 'w´'

So there are 2 cases remaining here where there are 3 outside spies (spies located outside the box B_1). Either 1.) the spies inside B_2 are at positions $(-1, 1)$ and $(1, -1)$ or 2.) they are at positions $(1, 1)$ and $(-1, -1)$.

Case $3B_1$:
Well looking at the non-aligned case (Figure 4) we have 3 spies fixed at points $\{(-1, 1), (1, -1), (3, 3)\}$. One of the two remaining outside spies must be placed in L_1 to prevent the corner win at point $(-2, -2)$. The other can not cover 'w' and 'w´' at the same time in ≤ 2 moves. Without loss of generality let's suppose that the final outside spy does not cover 'w´'. Here the winning strategy is for revolutionaries X_1 and X_3 (points $(-1, -1)$ and $(1, -1)$) move to $(0, -1)$ (point 'a') forcing the spy at $(1, -1)$ to block that win, meanwhile points X_5 and X_2 race towards point 'w´' $(3, -1)$ which creates a win.

Case $3B_2$: (a little tricky)
In this case (Figure 5) both remaining spies must be in each of L_1 and L_2 to protect the corner wins at $(2, -2)$ and $(-2, 2)$. Now note if either one is not in circle 1 or 2 (points

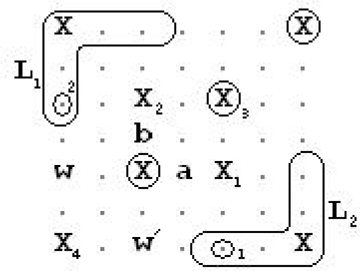

Figure 4.5: The aligned case where wins are at ‘w’ or ‘w′’, unless spies are at circles 1 and 2. In this case win is shown in Figure 6.

$(1, -3)$ and $(1, -3)$) respectively a win is immediate at either ‘w’ or ‘w′’ (point $(-3, -1)$ or $(-1, -3)$) in a similar fashion as in case $3B_1$. Thus suppose we have this initial condition with the remaining two spies outside B_1 at $(1, -3)$ and $(-3, 1)$.

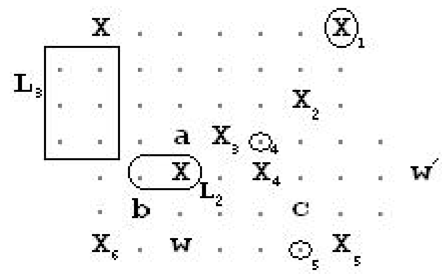

Figure 4.6: This is the forced board if you move the inner north two revolutionaries as such. The spies must move accordingly to prevent immediate wins. Still, a win is made available at position ‘w’.

The following strategy for the revolutionaries creates a picture as in Figure 6. Let X_2 move to the center (0,0) and X_3 move right 1 to $(2, 1)$. For the spies first note that Spy 1 must stay put at $(3, 3)$ to protect both north and east outer wall wins at points $(6, 0)$ and $(0, 6)$. Spy 3 (located originally at $(-3, 1)$) must be in L_3 to prevent the west outer wall win at $(-6, 0)$. Spy 4 (originally at $(1, 1)$) cannot decrease its x-coordinate because it must be able to protect against the win at $(2, 0)$ with X_4 and X_2 on the next move. This means that the spy located at $(-1, -1)$ will be in L_2 (thus either stay put or move one to the left) to protect against wins ‘b’ $(-2, -2)$ and ‘a’ $(-1, 0)$. Spy 4 must be located exactly as placed $(0, 1)$ because he alone must be able to protect against winning points at $(0, -1), (2, 0), (1, 1)$. The spy in L_2 might have to simultaneously protect the win at ‘b’. Spy 5 (originally located at $(1, -3)$ must move one to the right because he must protect against wins at c $(2, -2)$, ‘w′’ $(5, -1)$ and $(0, -6)$. This is the only spot reachable from which it can block all three of those winning spots in the necessary number of moves.

Revolutionaries at $(0,0)$ and $(-1,-1)$ move to point 'a' $(-1,0)$ and meanwhile move points X_4 and X_6 move to point w $(-1,-3)$. The spy in L_2 must protect the win at 'a' which allows the win at w.

We have now taken care of the cases where there are 3 spies outside our B_1 box construction. Now let's look at the case where there are 3 spies within B_1. First note given that there are only 2 spies on the outside, they must be located on opposite corners so that they can protect against all 4 outer wall wins mentioned above $\{(6,0),(0,6),(-6,0),(0,-6)\}$. We will say without loss of generality that the two spies are located at positions $(3,3)$ and $(-3,-3)$.

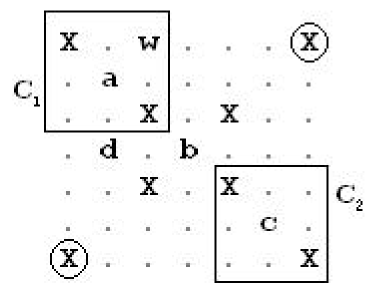

Figure 4.7: Here is the initial set-up now that we know that 2 must be located on the outside at the corners. One Spy must be in each or C_1 and C_2. Immediate danger spots are letters 'a' through 'd'. C_1 and C_2 must each contain a spy.

Case 4:
First let's note that one spy must be in each of C_1 and C_2 (boxes $[-3,-1]\times[1,3]$ and $[1,3]\times[-3,-1]$) to protect wins 'a' $(-2,2)$ and 'c' $(2,-2)$. This is shown in Figure 7. Without loss of generality we can assume the final spy satisfies $y \le x, y \ge -x$ (just rotate about the lines $y=x$ and $y=-x$ to obtain this).

Case 4a:
Let us suppose that the final spy is anywhere except $(1,1)$. The following provides a winning move for the revolutionaries. Move the two revolutionaries at points $(-1,1)$ and $(-1,-1)$ to point $(-2,0)$, this will force the spy in box C_1 to protect against this win. Meanwhile move revolutionaries at points $(-3,3)$ and $(1,1)$ to point $(-1,3)$ this will result in a win since no other spy can reach these in time.
We know have a spy at $(1,1)$.

Case 4b: (last case also a little tricky)
We are left with the position shown in Figure 8.

Now we have shown that at least 2 spies must be in B_2 and so we have pinned down 4 spies (Again we have symmetry argument and choose the top left to be fixed thus a spy will be in point $(-1,1)$). The last spy must be in R_2 (points $(1,-1)$ or $(1,-2)$) since it must be

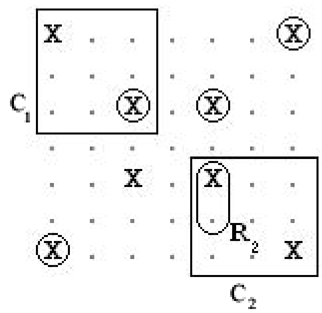

Figure 4.8: Last possible opening position for the spies.

in C_2 and it must stop the immediate win at point $(0, -1)$. Now we have a strategy for the revolutionaries. The first move (as shown in Figure 9) moves the revolutionary at point $(-1, -1)$ to $(-2, -2)$ and revolutionary at $(-1, 1)$ to the center $(0, 0)$.

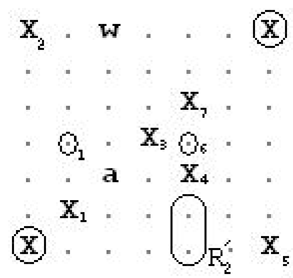

Figure 4.9: The moves by revolutionaries X_1 to point $(-2, -2)$ and X_3 to point $(0, 0)$ causes the spies to be in the pattern as shown.

First note that the two spies outside B_1 must remain in their initial positions to guard against wins at $(\pm 6, 0), (0, \pm 6)$. It is clear why $(3, 3)$ is fixed. Note the spy at position $(-3, -3)$ must still protect the win at $(0, -6)$ since the spy in R_2, that could have moved to the outer edge from $(2, -2)$, must protect against revolutionaries X_1 and X_4 meeting at $(0, -3)$ which means it cannot guard the other position. Now given that those are fixed the spy in R_2 may have been located at $(1, -1)$. If so it must decrease its y-coordinate by 1 otherwise X_1 and X_5 could meet in position $(1, -5)$ before the spy in R_2 or anyone else could get there. It must also stay within C_2 as before, and if it were to have moved right then you could let X_4 and X_1 meet at point $(-1, -3)$ meanwhile sending the west outer wall revolutionaries toward point $(-6, 0)$. Thus we have a spy in R_2'. The spy located originally at $(1, 1)$ cannot decrease it x-coordinate because it must prevent the win at point $(2, 0)$. This forces the spy originally located at $(-1, 1)$ to decrease its y-coordinate to protect win 'a' at $(-1, -1)$. This same spy must also decrease it x-coordinate to prevent revolutionaries X_1 and X_2 meeting at $(-5, 1)$. This finally forces the spy at $(1, 1)$ to move to point $(1, 0)$ because it must protect against the wins at $\{(2, 0), (0, 1), (0, -1)\}$. Note the spy in R_2 cannot help prevent these wins since independently it must protect against the meeting of X_1 and X_5 at point $(1, -5)$.

Now however we have opened up a win at point 'w' $(-1, 3)$, with revolutionaries X_2 and X_7.

This has exhausted all the possible plays of the spies given this initial opening set-up by the revolutionaries and in every case we have a win for the revolutionaries. By spacing out sections of size 8 a reasonable distance away from each other we can attain the desired bound for spies(n,2). □

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