COOPERATIVE COLORINGS AND INDEPENDENT SYSTEMS OF REPRESENTATIVES

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Abstract. Two closely related notions (in fact, translatable into one another), are those of cooperative colorings and of independent systems of representatives (ISRs). A *cooperative coloring* of a family of (not necessarily distinct) graphs $G_1, G_2, \dots, G_k$ on the same vertex set V is a choice of independent sets A_i in G_i ($1 \le i \le k$) such that $\bigcup_{i=1}^k A_i = V$. An *ISR* for a partition of the vertex set of a graph G into sets $V_1, \dots, V_n$ is a choice of a vertex $v_i \in V_i$ for each i such that $\{v_1, \dots, v_n\}$ is independent in G. We study conditions on the maximal degrees of graphs guaranteeing the existence of ISRs and of cooperative colorings. Our main results are: (1) If for some $d \ge 3$, the graph G has maximal degree d and contains no $K_{d,d}$, and the partition (V_i) has $|V_i| \ge 2d-1$ for all i, then there exists an ISR, (2) If G is a disjoint union of cycles, and its vertex set is partitioned into triples, then an ISR is guaranteed if at most two cycles have length $1(\bmod 3)$, but may fail to exist otherwise, even if all cycles have the same length $\ell \equiv 1(\bmod 3)$, and (3) Three cycles on the same vertex set have a cooperative coloring. The proofs use topological tools. We also offer some remarks on the question: do every $d+2$ graphs of maximal degree d on the same vertex set have a cooperative coloring?

1. ISRs, cooperative colorings, and their interrelationship

The notion of ISR (independent system of representatives, sometimes also called "independent transversals") is a generalization of that of SDR (system of distinct representatives). As in the setting of SDRs, there are given sets $V_i,\ 1 \le i \le n$, and we wish to find distinct representatives $v_i \in V_i$. In the ISR setting another structure is added on $V = \bigcup_{i=1}^n V_i$ - in general, a simplicial complex, namely a closed down hypergraph, and it is required that $\{v_1, \dots, v_n\}$ should belong to the complex. In the case dealt with in this paper, the complex is the set $\mathcal{I}(G)$ of independent sets in a graph G.

Definition 1.1. A system $(G,\ (V_i)_{i=1}^n)$ consisting of a graph G and a partition of its vertex set $V(G)$ into independent sets $V_1, \dots, V_n$, is called an *ISR-system.* An independent set in G of the form $\{v_1, \dots, v_n\}$, where $v_i \in V_i$ for each i, is called an *ISR.*

Note that the more general case of overlapping and non-independent V_i's may be reduced to the situation considered in Definition 1.1 by the following trick. Form a graph G' by replacing every vertex v in G by a clique, consisting of copies v_i of v, one for each i for which $v \in V_i$. Connect u_i to v_j if and only if $i \ne j$ and u and v are connected in G. Finally, replace each V_i by $V_i' = \{v_i : v \in V_i\}$. Clearly, an ISR in the system $(G', (V_i'))$ corresponds to an ISR of $(G, (V_i))$.

The notion of cooperative coloring generalizes the usual graph coloring concept by considering not one but k graphs on the same vertex set V. We require that each of the k graphs contribute an independent set, which together cover V. In the special case where the k graphs are copies of the same graph G, this amounts to a k-coloring of G.

Definition 1.2. A *cooperative coloring* of a family of (not necessarily distinct) graphs $G_1, G_2, \dots, G_k$ on the same vertex set V is a choice of independent sets A_i in G_i ($1 \le i \le k$) such that $\bigcup_{i=1}^k A_i = V$.

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There is also a list version of cooperative coloring, where for each vertex $v \in V$ a list $L(v) \subseteq \{1, \ldots, k\}$ is specified, requiring that each v be covered by an independent set A_i in G_i such that $i \in L(v)$. This problem can be equivalently stated by restricting the vertex set of each G_i to $\{v : i \in L(v)\}$, leading to the following formulation.

Definition 1.3. A family of (not necessarily distinct) graphs $G_1, G_2, \ldots, G_k$ with respective vertex sets $V(G_1), V(G_2), \ldots, V(G_k) \subseteq V$ is called a *CO-system*. The *multiplicity* of the system, denoted by m, is the minimum over all $v \in V$ of $|\{i : v \in V(G_i)\}|$. A *cooperative coloring* is a choice of independent sets A_i in G_i $(1 \leq i \leq k)$ such that $\bigcup_{i=1}^{k} A_i = V$.

The usual concept of list-coloring of a graph is obtained when $G_1, \ldots, G_k$ are all induced on their respective vertex sets by the same graph on V. Note that the multiplicity m corresponds to the minimal size of the lists. Observe also that Definition 1.2 is the special case $m = k$ of Definition 1.3. We shall refer to such CO-systems as *full*.

The problem of finding a cooperative coloring can be stated in ISR form. Given a CO-system $G_1, G_2, \ldots, G_k$ with vertex set V, let G be the disjoint union of the system: its vertex set consists of all ordered pairs (i, v) such that $v \in V(G_i)$, and for each i, vertices (i, u) and (i, v) are connected if and only if u and v are connected in G_i. Consider the partition (V_v) where each part is indexed by a vertex in V and consists of all its copies. Then, clearly, an ISR for the system $(G, (V_v))$ is equivalent to a cooperative coloring of $G_1, G_2, \ldots, G_k$. Observe that under this reduction, the multiplicity m becomes the minimal size of the V_v's. We write $\mathcal{J}(G_1, \ldots, G_k)$ for the ISR-system constructed in this way.

The translation between cooperative coloring problems and ISR problems goes also in the other direction. Let $(G, (V_i)_{i=1}^n)$ be an ISR-system. Form a full CO-system of graphs $CO(G, (V_i)_{i=1}^n)$ as follows. First, augment every V_j to V_j^* of size $k := \max\{|V_i| : 1 \leq i \leq n\}$, by adding $k - |V_j|$ vertices that are connected to all vertices of all other V_ℓ^*'s. Denote by $(G^*, (V_i^*)_{i=1}^n)$ the augmented system. Then let $G_1, \ldots, G_{k-1}$ be identical graphs, each consisting of disjoint cliques, a clique on each V_j^*. Let $G_k = G^*$. An independent set taken from G_i $(i < k)$ can contain at most one vertex from each V_j^*. Therefore a cooperative coloring must have at least one vertex from each V_j^* belonging to the independent set taken in G_k, and these vertices form an ISR in $(G^*, (V_i^*)_{i=1}^n)$ and hence in $(G, (V_i)_{i=1}^n)$. Conversely, an ISR in the latter produces a cooperative coloring of $G_1, \ldots, G_k$.

The notion of ISR has been studied by many authors, under various terminologies and with many applications. Cooperative colorings are introduced here for the first time, as far as we know.

2. Topological tools

One of the most effective tools for dealing with ISRs is that of topological connectivity. A simplicial complex $\mathcal{C}$ is called *k-connected* if for every $-1 \leq j \leq k$, every continuous function $f : S^j \to ||\mathcal{C}||$ can be extended to a continuous function $\tilde{f} : B^{j+1} \to ||\mathcal{C}||$ (here $||\mathcal{C}||$ is the underlying space of the geometric realization of $\mathcal{C}$). This means that there is no hole of dimension $k+1$ or less. We define $\eta(\mathcal{C})$ to be the largest k for which $\mathcal{C}$ is k-connected, plus 2 (the addition of 2 simplifies the statements of the theorems). The topological meaning of $\eta(\mathcal{C})$ is that it is the smallest dimension of a hole in $||\mathcal{C}||$, and ∞ if there is no such hole.

We shall use η as a graph invariant, by considering $\eta(\mathcal{I}(G))$, where $\mathcal{I}(G)$ is the complex of independent sets in G. When G is disconnected (as a graph), the complex $\mathcal{I}(G)$ is the join of the complexes corresponding to the components. By a standard result (see [5, p. 1848]) this yields:

Lemma 2.1. *If G consists of connected components H_i, $i = 1, \ldots, n$, then $\eta(\mathcal{I}(G)) \geq \sum_{i=1}^{n} \eta(\mathcal{I}(H_i))$.*

Notation 2.2. Given an ISR-system $(G, (V_i)_{i=1}^n)$ and a subset I of $[n] = \{1, \ldots, n\}$, we write V_I for $\bigcup_{i \in I} V_i$. We denote by $\mathcal{I}(G) \restriction V_I$ the complex of independent sets in the graph induced by G on V_I.

Aharoni and Haxell's theorem relating ISRs to connectivity is:

Theorem 2.3 ([3, 13]). *If $\eta(\mathcal{I}(G) \restriction V_I) \geq |I|$ for every $I \subseteq [n]$ then there exists an ISR.*

In order to apply Theorem 2.3, one needs combinatorial graph invariants that are lower bounds for $\eta(\mathcal{I}(G))$. One of the most general lower bounds is due to Meshulam [14] (see also [1]), and is denoted $\Psi(G)$. Its definition is conveniently expressed in terms of a game between two players, CON and NON, on the graph G. CON wants to show high connectivity, NON wants to thwart his attempt. At each step, CON chooses an edge e from the graph remaining at this stage, where in the first step the graph is G. NON can then either

(1) delete e from the graph (we call such a step a "deletion"),
  or
(2) remove the two endpoints of e, together with all neighbors of these vertices and the edges incident to them, from the graph (we call such a step an "explosion").

The result of the game (payoff to CON) is defined as follows: if at some point there remains an isolated vertex, the result is ∞. Otherwise, at some point all vertices have disappeared, in which case the result of the game is the number of explosion steps. We define $\Psi(G)$ as the value of the game, i.e., the result obtained by optimal play on the graph G.

Theorem 2.4 ([14, 1]). *$\eta(\mathcal{I}(G)) \geq \Psi(G)$.*

Meshulam calculated $\eta(\mathcal{I}(G))$ for paths P_n and cycles C_n on n vertices:

Theorem 2.5 ([14]). *$\eta(\mathcal{I}(P_n)) = \lceil \frac{n}{3} \rceil$ if $n \equiv 0$ or $2(\mathrm{mod}3)$ and $\eta(\mathcal{I}(P_n)) = \infty$ if $n \equiv 1(\mathrm{mod}3)$.*

Theorem 2.6 ([14]). *$\eta(\mathcal{I}(C_n)) = [\frac{n}{3}]$. (Here $[\alpha]$ is the rounding of α to the nearest integer.)*

3. ISRs in the absence of $K_{d,d}$

Our main concern in this paper is conditions for the existence of ISRs and cooperative colorings, that are formulated in terms of the maximal degree $\Delta(G)$. A basic result due to Haxell is:

Theorem 3.1 ([9, 10]). *Let $\Delta(G) = d$. If $|V_i| \geq 2d$ for all i, then there exists an ISR.*

Theorem 3.1 is sharp. Two types of examples were given, one in [11, 18] and the other in [16], of ISR-systems with V_i's of size $2d-1$, in which there is no ISR. In both types the graph in question consists of $2d-1$ disjoint copies of $K_{d,d}$. Our first result is that this is not a coincidence, but a must:

Theorem 3.2. *Let $\Delta(G) = d > 2$. If $|V_i| \geq 2d-1$ for all i, then there is an ISR unless G contains at least $2d-1$ connected components isomorphic to $K_{d,d}$.*

In order to prove Theorem 3.2, we establish the following lower bound on the connectivity of the independence complex of a graph:

Theorem 3.3. *Let $d > 2$. If G has maximal degree at most d and contains no $K_{d,d}$, then*

$$\eta(\mathcal{I}(G)) \geq \Psi(G) \geq \frac{|V(G)|}{2d-1}.$$

Before proving Theorem 3.3, let us show how it implies Theorem 3.2. Indeed, for every $I \subseteq [n]$, let k_I be the number of copies of $K_{d,d}$ contained in $G[V_I]$. Since $\eta(\mathcal{I}(K_{d,d})) = 1 = \frac{|V(K_{d,d})|-1}{2d-1}$, we obtain by Lemma 2.1 and Theorem 3.3

$$\eta(\mathcal{I}(G) \restriction V_I) \geq \frac{|V_I| - k_I}{2d-1} \geq |I| - \frac{k_I}{2d-1}.$$

Therefore, if G contains fewer than $2d-1$ copies of $K_{d,d}$, then $k_I < 2d-1$, and thus $\eta(\mathcal{I}(G) \restriction V_I) > |I| - 1$, and since η is integral this means that $\eta(\mathcal{I}(G) \restriction V_I) \geq |I|$. By Theorem 2.3 there exists an ISR.

Proof of Theorem 3.3. We may clearly assume that G is connected. We will prove the theorem by showing that in the game that defines $\Psi(G)$, CON can force NON to either create an isolated vertex or spend at least $\frac{|V(G)|}{2d-1}$ explosion steps to destroy all vertices.

In the first step $2d$ vertices may be removed in an explosion, but from that step on there is at each step a vertex of degree at most $d-1$, and as we may assume that it is not isolated, CON can choose an edge uv with $|N(u) \cup N(v)| \leq 2d-1$. This implies that $\Psi(G) \geq \frac{|V(G)|-1}{2d-1}$, and we may assume that equality holds (otherwise we are done). Therefore, the first step must be an explosion destroying $2d$ vertices, and any subsequent explosion step must remove $2d-1$ vertices. We may assume that G is d-regular and triangle-free, otherwise CON can choose his first edge so as to prevent NON from exploding $2d$ vertices in the first step.

Suppose first that $\Psi(G) = 1$. Then $|V(G)| = 2d$. Taking an arbitrary edge uv, we can partition $V(G)$ into $A = N(u)$ and $B = N(v)$, each of size d. By triangle-freeness these sets are independent, and d-regularity implies that G is the complete bipartite graph with parts A, B. This contradicts our no $K_{d,d}$ assumption.

Suppose next that $\Psi(G) > 1$. Consider the position after $\Psi(G) - 1$ explosion steps. The remaining graph has $2d-1$ vertices. If it has an edge whose explosion would remove fewer that $2d-1$ vertices, CON can choose it and force NON to delete it. By iteratively choosing and deleting such edges, we are left with a graph H in which every edge uv satisfies $|N_H(u) \cup N_H(v)| = 2d-1$. An argument similar to the one above shows that H is a complete bipartite graph, this time with one part, say A, of size d and the other, B, of size $d-1$.

Returning to G, each $x \in A$ has precisely one neighbor in $V(G) \setminus V(H)$. This neighbor cannot be the same for all $x \in A$, or else a $K_{d,d}$ is formed. Let xy be an edge with $x \in A$ and $y \in V(G) \setminus V(H)$. Let $z \in A$ be a non-neighbor of y. CON can choose xy as the first step edge in the game, and exploding it removes all but at most one neighbor of z. If z is isolated, the result is ∞. If it still has a neighbor w, then CON can choose zw at the second step. NON must explode it (to avoid isolating z), but this explosion removes at most $d+1$ vertices, which is less than $2d-1$ (as we assume $d > 2$). □

To conclude this section, we remark that ISRs are known to exist even with V_i's significantly smaller than $2d$, where $d = \Delta(G)$, under suitable restrictive conditions on the ISR-system. For example, if G is chordal then $|V_i| \geq d+1$ suffices ([2]). Another type of restriction involves the local degree of $(G, (V_i))$, defined as the maximum of $|N(v) \cap V_i|$ taken over all vertices v and parts V_i. Asymptotically as $d \to \infty$, if the local degree is $o(d)$ then $|V_i| \geq (1+o(1))d$ suffices ([12]). It is not known whether $|V_i| \geq d+2$ suffices for an ISR, if the local degree is 1.

4. Degree conditions for cooperative coloring

A basic fact on standard graph coloring is that a graph of maximal degree d is $(d+1)$-colorable. Rephrased in our terminology, this says that if $G_1, G_2, \ldots, G_{d+1}$ are identical graphs of maximal degree d, then they have a cooperative coloring. Our first observation is that this is no longer true for non-identical graphs.

Theorem 4.1. *For every $d \geq 2$, there exists a full CO-system $G_1, G_2, \ldots, G_{d+1}$ with $\Delta(G_i) = d$ for $i = 1, \ldots, d+1$, that does not have a cooperative coloring.*

Proof. First, we construct an ISR-system $(G, (V_i)_{i=1}^n)$ with $\Delta(G) = d$, $|V_i| = d+1$ for all i, having no ISR (as mentioned above, this can be done even with $|V_i| = 2d-1$, but for completeness we give the easy construction that we need here). Let $n = 4$, and $V_i = U_i \cup \{v_i\}$ where $|U_i| = d$ and $v_i \notin U_i$, $i = 1, \ldots, 4$. Let G have three connected components: a $K_{d,d}$ with sides U_1, U_2, a $K_{d,d}$ with sides U_3, U_4, and a $K_{2,2}$ with sides $\{v_1, v_2\}, \{v_3, v_4\}$. Clearly, this system has no ISR. From this system, we pass to a CO-system $CO(G, (V_i)_{i=1}^4)$ which, as shown in Section 1, has no cooperative coloring. Each of the first d graphs in the CO-system is a disjoint union of K_{d+1}'s, and the last graph is G, so they all have maximal degree d, as required. □

If $d+1$ graphs of maximal degree d do not suffice for a cooperative coloring, how many of them are needed? More generally, for CO-systems that may not be full, how high does the multiplicity need to be (in terms of the maximal degree d) in order to guarantee the existence of a cooperative coloring? We get an upper bound by applying the reduction of CO-systems to ISR-systems described in Section 1, and invoking Theorems 3.1 and 3.2.

Corollary 4.2. *Any CO-system of multiplicity $2d$ or more, in which the graphs have maximal degree d, has a cooperative coloring. Moreover, for $d > 2$ multiplicity $2d-1$ suffices, unless the graphs in the system have between them at least $2d-1$ copies of $K_{d,d}$.*

The two parts of the corollary bear a similarity to the known facts about standard (or list) graph coloring: there, $d+1$ colors (or lists of size $d+1$) are needed to color a graph of maximal degree d, and Brooks' theorem asserts that one color (or list element) may be saved in the absence of K_{d+1}, for $d > 2$. But, contrary to this analogy to graph coloring, and unlike the situation for Theorems 3.1 and 3.2 from which it was derived, Corollary 4.2 is not sharp for general d. The reason is that the reduction of CO-systems to ISR-systems described in Section 1 yields ISR-systems with local degree 1. An asymptotic result of [12] mentioned in Section 3 implies that for such systems V_i's of size $(1+o(1))d$ suffice for an ISR. In particular, multiplicity $(1+o(1))d$ in a CO-system suffices for a cooperative coloring.

However, it is not known whether multiplicity $d+O(1)$ suffices for a cooperative coloring. In view of Theorem 4.1, $d+1$ does not suffice, but the question for $d+2$ is open. In particular, do every $d+2$ graphs of maximal degree d on the same vertex set have a cooperative coloring? The following theorem shows that no counterexample to the latter can exist, in which d of the graphs are identical (as was the case in the construction of the counterexample for $d+1$ graphs above).

Theorem 4.3. *$d+2$ graphs of maximal degree d on the same vertex set have a cooperative coloring, provided that d of the graphs are identical.*

The proof of Theorem 4.3 requires the following lemma, in which we speak of a *partial* ISR for a system $(G,(V_i)_{i=1}^n)$: this is a choice of independent representatives from some of the V_i's, and is expressed as a function h with domain $dom(h) \subseteq [n]$.

Lemma 4.4. *If H_1, H_2 are two graphs on the same vertex set V having both degrees at most d, and $(V_i)_{i=1}^n$ is a partition of V into sets of size at least d, then there exist a partial ISR h_1 for $(H_1,(V_i))$ and a partial ISR h_2 for $(H_2,(V_i))$, such that $dom(h_1) \cup dom(h_2) = [n]$.*

Proof. We form an ISR-system $(G^*,(V_i^*)_{i=1}^n)$ on a vertex set V^* consisting of two disjoint copies of V. The graph G^* is the disjoint union of H_1 and H_2, placed on the two respective copies of V. Each V_i^* is the union of the two respective copies of V_i. By Theorem 3.1, there is an ISR for $(G^*,(V_i^*)_{i=1}^n)$; clearly, such an ISR decomposes into partial ISRs for H_1 and H_2 as desired. □

We can now prove Theorem 4.3. By Corollary 4.2, we may assume that $d \geq 3$. Say our full CO-system consists of d copies of the same graph G, and two additional graphs G_{d+1}, G_{d+2}, all of maximal degree d. Let $V_1,\dots,V_n,V_{n+1},\dots,V_r$ be the partition of the vertex set V into the connected components of G, enumerated so that $G[V_i]$ is d-colorable if and only if $n+1 \leq i \leq r$. Clearly, $\bigcup_{i=n+1}^r V_i$ can be covered by independent sets from the d copies of G. If $n=0$ we are done; if $n>0$ it suffices to find a cooperative coloring of $V' := \bigcup_{i=1}^n V_i$. By Brooks' theorem, each $G[V_i]$ $(1 \leq i \leq n)$ is a K_{d+1}. Applying Lemma 4.4 to the graphs $G_{d+1}[V'], G_{d+2}[V']$, we find independent sets in these two graphs whose union contains a vertex from each V_i, $i=1,\dots,n$. The remaining d vertices in each of these V_i's can be covered by independent sets from the d copies of G, thus obtaining the desired cooperative coloring.

5. The case $d=2$

The ISR problem for a 2-regular graph G and V_i's of size 3 is of particular interest. When G is just one cycle, the existence of an ISR is precisely the conjecture of Du, Hsu, and Hwang [6]. That conjecture, in a stronger 3-colorability version proposed by Erdős [7], became well known and got the name "the cycle-plus-triangles problem". It was proved by Fleischner and Stiebitz [8] and Sachs [15]. Counterexamples have been found, showing that the result does not extend to all graphs G that consist of several disjoint cycles. But there has not been a good insight as to what features of the decomposition into cycles are needed for a positive answer. We show here, using topological connectivity and Theorems 2.5 and 2.6, that the mod3 length of the cycles is crucial.

Theorem 5.1. *Let $\Delta(G) = 2$, and assume that at most two connected components of G are cycles of length $1(\text{mod}3)$. If $|V_i| \geq 3$ for all i, then there exists an ISR.*

Proof. By Theorem 2.3, it suffices to show that for every $I \subseteq [n]$, $\eta(\mathcal{I}(G) \restriction V_I)$ is at least $|I|$. Let $H_1, \ldots, H_r$ be the connected components of $G[V_I]$, with $n_1, \ldots, n_r$ vertices respectively. Note that $\sum_{j=1}^{r} n_j = |V_I| \geq 3|I|$. By our assumption on G, all but at most two of the H_j's are paths of any length or cycles of length 0 or 2 modulo 3; the exceptional cases are cycles of length $1(\text{mod}3)$. Applying Theorems 2.5 and 2.6, we have $\eta(\mathcal{I}(H_j)) \geq \frac{n_j}{3}$ for all j except possibly two j's for which $\eta(\mathcal{I}(H_j)) = \frac{n_j - 1}{3}$. Lemma 2.1 yields

$$\eta(\mathcal{I}(G) \restriction V_I) \geq \lceil \frac{\sum_{j=1}^{r} n_j - 2}{3} \rceil \geq \lceil \frac{3|I| - 2}{3} \rceil = |I|.$$

□

Theorem 5.1 is sharp: an example with three 4-cycles and V_i's of size 3 without an ISR is known. In order to show that it is not just 4-cycles, but cycles of length $1(\text{mod}3)$ in general, that hinder ISRs, we prove the following.

Theorem 5.2. *For every $\ell \equiv 1(\text{mod}3)$, $\ell \geq 4$, there exists a graph, all of whose connected components are cycles of length ℓ, and a partition of its vertex set into sets of size 3, for which there is no ISR. Such an example exists with $\frac{\ell}{2} + 1$ cycles if ℓ is even, and with $\ell + 2$ cycles if ℓ is odd.*

The building block for the necessary construction is presented in the following lemma.

Lemma 5.3. *Let $\ell = 3r + 1$, $r \geq 1$. The vertices of a cycle of length ℓ can be partitioned into $r - 1$ sets $V_1, \ldots, V_{r-1}$ of size 3 and 2 sets U_0, U_r of size 2 each, so that there is no ISR.*

Proof. Let the vertices of the cycle be enumerated in cyclical order as $v_1, v_2, \ldots, v_\ell$. Let

$$V_i = \{v_{3i-1}, v_{3i+1}, v_{3i+3}\},\ i = 1, \ldots, r-1, \quad U_0 = \{v_1, v_3\}, \quad U_r = \{v_{\ell-2}, v_\ell\}.$$

Suppose, for the sake of contradiction, that A is an independent set in the cycle containing an element from each of $U_0, V_1, \ldots, V_{r-1}, U_r$. If the U_0 element of A is v_3, then the V_1 element must be v_6, the V_2 element must be v_9, …, the V_{r-1} element must be $v_{\ell-1}$, leaving no choice for the U_r element. A similar argument going backwards shows that the U_r element of A cannot be $v_{\ell-2}$. Thus the U_0 element must be v_1 and the U_r element must be v_ℓ, but these two are adjacent on the cycle. □

We give now the construction for Theorem 5.2. For even ℓ, we take $\frac{\ell}{2}$ cycles of length ℓ, and partition the vertices of each of them as in the lemma. This gives us a total of $\frac{\ell}{2}(r-1)$ sets of size 3, and ℓ sets of size 2. We add a new vertex to each of these ℓ sets, increasing their size to 3, and place a new cycle on the new vertices. The way we place that cycle is arbitrary, except that for one of the pairs U_0, U_r, the two vertices added to them, denoted u_0, u_r, are at distance 2 on the new cycle. By the lemma, any ISR of this system would have to include, for each of the pairs U_0, U_r, one of their two new vertices. This requires an independent set of size $\frac{\ell}{2}$ from the new cycle, but by construction, no such set contains exactly one of u_0, u_r.

For odd ℓ, we carry out a similar construction with ℓ original cycles of length ℓ, giving us a total of $\ell(r-1)$ sets of size 3, and 2ℓ sets of size 2. We add a new vertex to each of these 2ℓ sets, and place two new disjoint cycles on them arbitrarily. The non-existence of an ISR follows from the lemma and the fact that independent sets from the two new (odd) cycles of length ℓ can only have total size $\ell - 1$.

We note that Vandenbussche and West [17] gave a construction similar to the above, but only for $\ell \equiv 1(\bmod 6)$ - our odd case. This led them to conjecture that if a 2-regular graph G on V has girth at least $\sqrt{|V|}$, and V is partitioned into sets of size 3, then there is an ISR (with the exception of a particular 12-vertex example). Our construction above for even ℓ has girth ℓ and $|V| = \frac{\ell^2+2\ell}{2}$, so it disproves their conjecture by a factor of $\sqrt{2}$. The conjecture may still be true up to a constant factor.

Theorem 5.2 also serves to show that the condition $d > 2$ in Theorem 3.2 is necessary.

Turning now to cooperative coloring in the case $d = 2$, we have the following corollary of Theorem 5.1.

Corollary 5.4. *A CO-system of multiplicity* 3*, in which all graphs have degrees at most* 2*, and the total number of components which are cycles of length* $1(\mathrm{mod}3)$ *is at most* 2*, has a cooperative coloring.*

Theorem 5.2 also has a cooperative coloring counterpart.

Theorem 5.5. *For every* $\ell \equiv 1(\bmod 3)$*,* $\ell \geq 4$*, there exists a CO-system of multiplicity* 3*, in which all graphs are cycles of length* ℓ*, that does not have a cooperative coloring.*

Proof. Note that we cannot just apply to the ISR-system constructed in Theorem 5.2 the transformation to a CO-system described in Section 1 (because then only one of the graphs in the CO-system would have cycles of length ℓ, the others would have 3-cliques). We do start with the ISR-system $(G, (V_i)_{i=1}^n)$ produced in Theorem 5.2, but proceed differently. We augment each V_i by adding $\ell - 2$ new vertices, obtaining a set of size $\ell + 1$ written as $U_i = \{v_i^1, v_i^2, v_i^3, \ldots, v_i^{\ell+1}\}$, where the first 3 elements are those of V_i. Thus the vertex set of the CO-system will be $U = \bigcup_{i=1}^n U_i$, of size $n(\ell + 1)$. The graphs in the system are the cycles of G and $3n$ new cycles: for each $i = 1, \ldots, n$ we introduce the 3 cycles

$$C_i^1 := v_i^1, v_i^2, v_i^4, \ldots, v_i^{\ell+1}, \quad C_i^2 := v_i^2, v_i^3, v_i^4, \ldots, v_i^{\ell+1}, \quad C_i^3 := v_i^3, v_i^1, v_i^4, \ldots, v_i^{\ell+1}.$$

Clearly, the resulting CO-system has multiplicity 3. Assume, for the sake of contradiction, that it has a cooperative coloring. Because the original G has no ISR, the independent sets coming from the original cycles leave some U_i untouched. This U_i needs to be covered by independent sets A_1 from C_i^1, A_2 from C_i^2, A_3 from C_i^3. There are only two ways in which v_i^1, v_i^2, v_i^3 can all be covered: either $v_i^1 \in A_1, v_i^2 \in A_2, v_i^3 \in A_3$, or $v_i^1 \in A_3, v_i^2 \in A_1, v_i^3 \in A_2$. In the former case, none of the independent sets can contain $v_i^{\ell+1}$; in the latter case, the contradiction is obtained for v_i^4. $\square$

We remark that a CO-system in which all graphs are 4-cycles may be interpreted as a CNF formula, and in fact the $\ell = 4$ case of Theorem 5.5 is equivalent to a construction of an unsatisfiable (3,B2)-SAT formula given in [4].

We do not know if Theorem 5.5 can be strengthened to assert the existence of a *full* CO-system with these properties, i.e., three graphs on the same vertex set, each of them a disjoint union of ℓ-cycles, having no cooperative coloring. This question remains open, even for $\ell = 4$.

There is, however, an important special case of full CO-systems, in which a cooperative coloring is guaranteed regardless of the mod3 length of the cycles. Namely, when each of the three graphs is a cycle on the entire vertex set.

Theorem 5.6. *Three cycles on the same vertex set possess a cooperative coloring.*

Proof. Let G_1, G_2, G_3 be a system of three cycles on the same vertex set V. Choose any vertex v, and take it into the independent set chosen from G_1. Completing this choice to a cooperative coloring means finding an ISR in the ISR-system $\mathcal{K}$ obtained from $\mathcal{J}(G_1, G_2, G_3)$ by removing the set V_v, and removing from the G_1-copy of V the two vertices (that we shall name below u and w) adjacent to v in G_1. We shall prove that $\mathcal{K}$ satisfies the conditions of Theorem 2.3.

Write $V'_y = V_y$ for $y \in V \setminus \{v, u, w\}$, and let V'_u, V'_w be obtained from V_u, V_w, by removing the G_1-copies of u, w, respectively. Let H be the underlying graph of $\mathcal{K}$. Let X be a subset of $V \setminus \{v\}$. Because V_v

was removed, the three graphs induced on the three copies of X in H have only paths as their connected components. By Theorem 2.5, η of each path is at least a third of its number of vertices. The total number of vertices in $\bigcup_{x\in X} V'_x$ is at least $3|X|-2$. Thus we get by Lemma 2.1

$$\eta(\mathcal{I}(H) \restriction \bigcup_{x\in X} V'_x) \geq \lceil \frac{3|X|-2}{3} \rceil = |X|,$$

as required in order to apply Theorem 2.3. □

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