# A CHARACTERIZATION OF PARTIALLY ORDERED SETS WITH LINEAR DISCREPANCY EQUAL TO 2

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Abstract. The linear discrepancy of a poset $\mathbf{P}$ is the least $k$ such that there is a linear extension $L$ of $\mathbf{P}$ such that if $x$ and $y$ are incomparable in $\mathbf{P}$, then $|h_L(x) - h_L(y)| \leq k$, where $h_L(x)$ is the height of $x$ in $L$. Tanenbaum, Trenk, and Fishburn characterized the posets of linear discrepancy 1 as the semiorders of width 2 and posed the problem of characterizing the posets of linear discrepancy 2. We show that this problem is equivalent to finding the posets with linear discrepancy equal to 3 having the property that the deletion of any point results in a reduction in the linear discrepancy. Howard determined that there are infinitely many such posets of width 2. We complete the forbidden subposet characterization of posets with linear discrepancy equal to 2 by finding the minimal posets of width 3 with linear discrepancy equal to 3. We do so by showing that, with a small number of exceptions, they can all be derived from the list for width 2 by the removal of specific comparisons.

## 1. Introduction

In [5], Tanenbaum, Trenk, and Fishburn introduced the notion of the linear discrepancy of a poset as the minimum over all linear extensions of the poset of the maximum distance between a pair of incomparable points. This property, which can be viewed as a measure of a poset's deviation from being a linear order, is equal to the bandwidth of the poset's co-comparability graph [2]. They showed that the posets of linear discrepancy equal to 1 are precisely the semiorders of width 2. (The forbidden induced subposets are thus $\mathbf{3}+\mathbf{1}$, $\mathbf{2}+\mathbf{2}$, and $\mathbf{1}+\mathbf{1}+\mathbf{1}$.) They also posed a series of eight challenges and questions about linear discrepancy, the first of which was to characterize the posets with linear discrepancy equal to 2. In [1], Chae, Cheong, and Kim introduced the notion of $k$-discrepancy-irreducibility to describe posets having the property that the deletion of any point reduces the linear discrepancy, which provides a nice framework in which to consider the problem posed by Tanenbaum, Trenk, and Fishburn. Since posets with linear discrepancy equal to 2 have width at most 3, in order to characterize those with linear discrepancy equal to 2 it suffices by our Lemma 11 below to identify the 3-discrepancy-irreducible posets of width 2 and 3. In [3], Howard showed that the 3-irreducible posets of width 2 are $\mathbf{1}+\mathbf{5}$, $\mathbf{2}+\mathbf{3}$, and, contrary to expectations, an infinite family $\mathscr{I}_3^2$, each member of which has an even number of points. This paper

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completes the characterization of linear discrepancy 2 by finding the 3-irreducible posets of width 3. We show that with four exceptions, the 3-discrepancy-irreducible posets of width 3 can all be derived from Howard's list for width two by the removal of comparisons meeting specific criteria. This allows us to answer the question of Tanenbaum, Trenk, and Fishburn with our primary result characterizing posets of linear discrepancy 2 in the theorem below.

**Theorem 1.** *A poset has linear discrepancy equal to* 2 *if and only if it contains* $\mathbf{1}+\mathbf{3}$*,* $\mathbf{2}+\mathbf{2}$*, or* $\mathbf{1}+\mathbf{1}+\mathbf{1}$ *and it does not contain any of the following:*

(i) $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$*;*
(ii) *any poset obtained from* $\mathbf{1}+\mathbf{5}$ *or* $\mathbf{2}+\mathbf{3}$ *by the removal of a (possibly empty) subset of cover relations;*
(iii) $\mathbf{S}_3$*,* $\mathbf{Q}_1$*,* $\mathbf{Q}_1^d$*, or* $\mathbf{Q}_2$*; or*
(iv) *any member of the families* $\mathscr{I}_3^2$ *and* $\mathscr{I}_3^3$*.*

The four exceptional posets in item iii of Theorem 1 are shown in Figure 1. In Section 2.1, we provide a description of Howard's infinite family $\mathscr{I}_3^2$ from [3] and describe two canonical linear extensions that demonstrate that these posets have linear discrepancy 3. Our description is made more natural than Howard's by taking advantage of the relationship between members of the family on $2n$ points. The family of 3-discrepancy-irreducible posets of width 3, denoted $\mathscr{I}_3^3$, is developed in Sections 3 and 4.

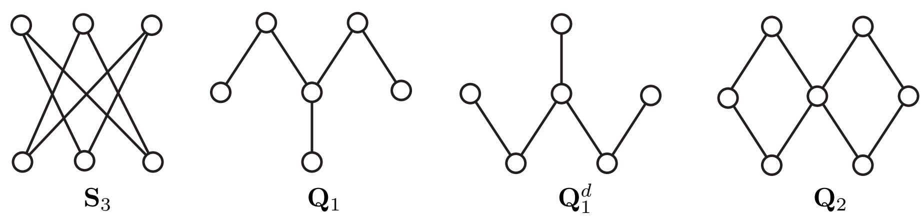


FIGURE 1. The exceptional 3-irreducible posets of width 3

The remainder of the paper is devoted to developing the components of the proof of Theorem 1. We use the canonical linear extensions for elements of $\mathscr{I}_3^2$ to characterize the comparisons that may be deleted from posets of $\mathscr{I}_3^2$ while maintaining 3-discrepancy-irreducibility. Theorem 5 asserts that other than the four exceptions mentioned above, the 3-discrepancy-irreducible posets of width 3 are all obtained in this manner. We show this via three lemmas, which describe when comparabilities can be inserted while retaining 3-discrepancy-irreducibility and then establish that if the comparabilities cannot be inserted, the poset is one of the four exceptional posets of Figure 1. We conclude the proof of Theorem 1 with a lemma that guarantees that any poset with linear discrepancy greater than 2 must contain a poset of linear discrepancy 3, and thus a 3-discrepancy-irreducible induced subposet. The paper concludes by discussing the viability of pursuing the analogous question for posets of higher linear discrepancy.

## 2. Survey of previous results

We begin with some terminology and notation. If $n$ is a positive integer, $\mathbf{n}$ will denote the linear order on $n$ elements. We will write $\mathbf{P} = (X, <_P)$ for a partially ordered set with ground set $X$ and order relation $<_P$. When the poset is clear, we will suppress the subscript. If $x, y \in X$ are comparable in $\mathbf{P}$, we will write $x \perp_P y$ or $x \perp y$. If they are incomparable, we will write $x \|_P y$ or $x \| y$. The set of all points incomparable to $x$ will be denoted $\mathrm{Inc}(x)$. If $x <_P y$ and there is no $z \in X$ such that $x <_P z <_P y$, we say that $x$ is *covered by* $y$ or $y$ *covers* $x$ and write $x \lessdot y$. If $\mathbf{P}$ and $\mathbf{Q}$ are posets with disjoint point sets, we will denote by $\mathbf{P} + \mathbf{Q}$ the disjoint union of $\mathbf{P}$ and $\mathbf{Q}$. A *linear extension* $L$ of a poset $\mathbf{P}$ is a linear order on $X$ such that if $x <_P y$, then $x <_L y$. We denote the down-set $\{y \in X \mid y < x\}$ by $D(x)$ and let $D[x] = D(x) \cup \{x\}$. The up-sets $U(X)$ and $U[x]$ are defined dually. We say $(x, y)$ is a *critical pair* in $\mathbf{P}$ if $x \|_P y$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$. For any unfamiliar terminology or notation, we refer the reader to [6].

**Definition 2.** The *linear discrepancy* of a poset $\mathbf{P}$ is

$$\mathrm{ld}(\mathbf{P}) := \min_{L \in \mathscr{E}(\mathbf{P})} \max_{x \|_P y} |h_L(x) - h_L(y)|,$$

where $\mathscr{E}(\mathbf{P})$ is the set of linear extensions of $\mathbf{P}$ and $h_L(x)$ is the height of $x$ in $L$.

From this definition it is clear that $\mathrm{ld}(\mathbf{P}) = 0$ if and only if $\mathbf{P}$ is a chain. Furthermore, the linear discrepancy of an $n$-element antichain is $n - 1$ and thus $\mathrm{ld}(\mathbf{P}) \geq \mathrm{width}(\mathbf{P}) - 1$. We show below in Lemma 11 that every poset with linear discrepancy greater than 2 must contain a poset of linear discrepancy 3. Thus, in order to characterize the posets of linear discrepancy equal to 2, it suffices to identify the minimal posets with linear discrepancy equal to 3. The most sensible definition of "minimal" in this context is that the removal of any point of the poset causes the linear discrepancy to drop, and so we have the following definition, introduced by Chae, Gheong, and Kim in [1].

**Definition 3.** We say that a poset $\mathbf{P}$ is *$k$-discrepancy-irreducible* (or simply *$k$-irreducible*) if $\mathrm{ld}(\mathbf{P}) = k$ and $\mathrm{ld}(\mathbf{P} - \{x\}) < k$ for all $x \in X$.

The shorter terminology $k$-irreducible is usually used in the context of dimension theory with a different meaning, but since there is little risk of confusion in this paper, we will use $k$-irreducible to mean $k$-discrepancy-irreducible unless there is a chance of confusion. Corollary 25 of [5] gives the forbidden subposet characterization of posets of linear discrepancy equal to 1, which Chae, Cheong, and Kim recast in terms of irreducibility in [1]. Therefore we have that the only 1-irreducible poset is $\mathbf{1}+\mathbf{1}$, and $\mathbf{2}+\mathbf{2}$, $\mathbf{1}+\mathbf{3}$, and $\mathbf{1}+\mathbf{1}+\mathbf{1}$ are the 2-irreducible posets. The natural hypothesis at this point seemed to be that there would be a finite list of 3-irreducible posets, and Rautenbach conjectured in [4] that the list of forbidden subposets for linear discrepancy 2 was in fact finite. However, in [3], Howard showed that this is not the case. He showed that not only are $\mathbf{1} + \mathbf{5}$ and $\mathbf{2} + \mathbf{3}$ 3-irreducible but also there is an infinite family of 3-irreducible posets having width 2, all of which have an even number of points greater than or equal to 8. We will denote this family by $\mathscr{I}_3^2$.

2.1. **The infinite family $\mathscr{I}_3^2$.** We will denote by $\mathbf{M}_{2n}$ ($n \geq 4$) a special member of $\mathscr{I}_3^2$ on $2n$ points and describe how the other members of the family on $2n$ points are obtained from $\mathbf{M}_{2n}$. Since $\mathrm{width}(\mathbf{M}_{2n}) = 2$, we consider it as being made of two chains, which we will call $L$ and $R$, with some comparisons added between the chains. The construction is dependent on the parity of $n$. For $n$ even, $L$ has $n$ points and $R$ has $n$ points, while for $n$ odd, $L$ has $n + 2$ points and $R$ has $n - 2$

points. Let the points of the chain $L$ be $l_1 \lessdot l_2 \lessdot \cdots$ and the points of the chain $R$ be $r_1 \lessdot r_2 \lessdot \cdots$. The covering relations we then add to construct $\mathbf{M}_{2n}$ are

$$l_3 \lessdot r_2 \lessdot l_5 \lessdot r_4 \lessdot \cdots \lessdot r_{n-4} \lessdot l_{n-1} \lessdot r_{n-2} \qquad \text{for } n \text{ even}$$

and

$$l_3 \lessdot r_2 \lessdot l_5 \lessdot r_4 \lessdot \cdots \lessdot l_{n-2} \lessdot r_{n-3} \lessdot l_n \qquad \text{for } n \text{ odd.}$$

The construction of $\mathbf{M}_{2n}$ is completed by adding all relations implied by transitivity after adding the covering relations above. For illustration, Figure 2 shows the posets $\mathbf{M}_8$, $\mathbf{M}_{10}$, and $\mathbf{M}_{12}$.

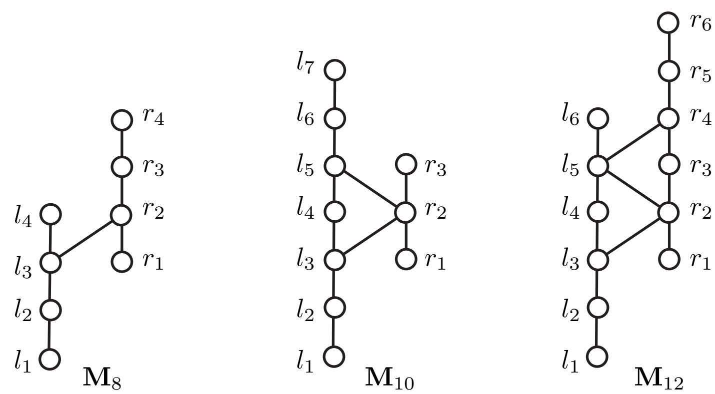


FIGURE 2. Three members of the infinite family $\mathscr{I}_3^2$

We obtain the remaining $2n$-element members of $\mathscr{I}_3^2$ from $\mathbf{M}_{2n}$ by removing any subset of the covering relations added above while retaining the comparabilities added due to transitivity. For example, Figure 3 shows the 3-irreducible poset of width 2 derived from $\mathbf{M}_8$ by removing the only possible covering relation and the 3-irreducible poset of width 2 derived from $\mathbf{M}_{10}$ by removing the covering relation $l_3 \lessdot r_2$.

Howard also showed that there are two canonical linear extensions of an element of $\mathscr{I}_3^2$ that witness linear discrepancy 3. For convenience, we will often refer to a linear extension of $\mathbf{P}$ as a *labelling* of $X$ ($|X| = n$) using the elements of $[n] := \{1, 2, \ldots, n\}$ such that the ordering created by the labelling is a linear extension of $\mathbf{P}$. Because of the symmetry present in the members of $\mathscr{I}_3^2$, the way these labellings are created is effectively the same, in one case being generated by starting at the "bottom" of the poset with label 1 and in the other starting from the "top" of the poset with label $2n$. To construct the first labelling, which we call $f\colon X \to [2n]$, let $f(l_1) = 1$, $f(l_2) = 2$, and $f(r_1) = 3$. We then proceed back to label points two at a time from alternate chains until one chain is exhausted, at which point we complete the labelling of the remaining chain, so $f(l_3) = 4$, $f(l_4) = 5$, $f(r_2) = 6$, $f(r_3) = 7$, and so on. The second labelling $g$ uses the same pattern, labelling the top two elements of the first chain (if $n$ is even, $R$ is the first chain, and if $n$ is odd, $L$ is the first chain) with $2n$ and $2n-1$, then labelling the top element of the second chain with $2n-2$, and then returning to the first chain to establish the pattern of labelling two consecutive elements from each chain alternately. Figure 4 shows the labellings as ordered pairs $(f(x), g(x))$ for the two posets from Figure 3.

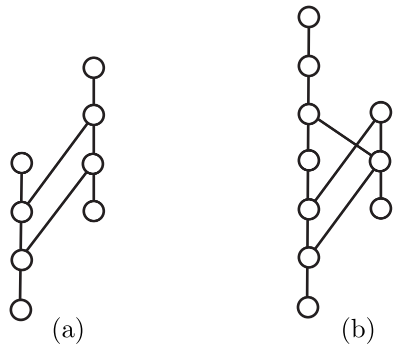


FIGURE 3. Members of the infinite family $\mathscr{I}_3^2$ derived from (a) $\mathbf{M}_8$ and (b) $\mathbf{M}_{10}$

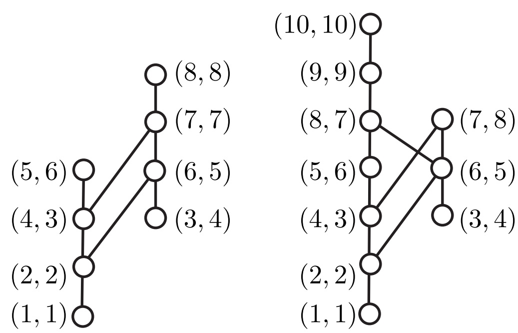


FIGURE 4. Examples of the two labellings of elements of $\mathscr{I}_3^2$

## 3. REMOVING COMPARABILITIES

The removal of comparabilities clearly cannot decrease the linear discrepancy of a poset. However, it is possible to remove comparabilities from a 3-irreducible poset and continue to have a 3-irreducible poset, as was seen in the previous section. In fact, there are more covering relations than just those inserted between the chains $L$ and $R$ to construct $\mathbf{M}_{2n}$ that can be removed while retaining 3-irreducibility. The cost we pay is an increase in width, but only to 3.

**Theorem 4.** *Let $f$ and $g$ be the two canonical labellings of $\mathbf{M}_{2n}$ that witness linear discrepancy equal to 3 as defined above. Let $C$ be the set of all covering relations $u \lessdot v$ in $\mathbf{M}_{2n}$ satisfying both $f(v) - f(u) \leq 2$ and $g(v) - g(u) \leq 2$. Then the poset $\mathbf{P}$ formed by removing the comparabilities of any subset $D$ of $C$ is 3-discrepancy-irreducible.*

*Proof.* Our proof is by induction on $|D|$. If $|D| = 0$, there is nothing to prove. Suppose that for some $k \geq 0$, if $|D| = k$ then deleting the comparabilities of $D$ creates a poset $\mathbf{P}$ that is 3-irreducible. Now consider $|D| = k + 1$. Fix $u, v$ such that $(u, v) \in D$. Let $D' = D - \{(u, v)\}$. By induction, deleting $D'$ gives a 3-irreducible poset $\mathbf{P}'$. We now consider the effect of removing the comparability $u \lessdot v$ from $\mathbf{P}'$, which gives the same poset $\mathbf{P}$ as removing all the comparabilities in $D$ from $\mathbf{M}_{2n}$. The labellings $f$ and $g$ witness $\mathrm{ld}(\mathbf{P}) = 3$, since the only pair of points that are incomparable in $\mathbf{P}$ that were comparable in $\mathbf{P}'$ is $\{u, v\}$, but our constraints on the labels of $u$ and $v$ in the two labellings ensures that this does not increase the linear discrepancy. To see that $\mathbf{P}$ is irreducible we consider the effect of deleting a point $w_0$. We begin by considering the poset $\mathbf{Q}$ formed by deleting $w_0$ from $\mathbf{M}_{2n}$. Since $\mathbf{M}_{2n}$ is an element of $\mathcal{I}_3^2$, it is 3-irreducible, and thus $\text{ld}(\mathbf{Q}) = 2$. We can construct a labelling that witnesses this from the labellings $f$ and $g$ defined above. There are precisely four points for which $f(w_0) = g(w_0)$. Two are located at the top of $\mathbf{M}_{2n}$ and two at the bottom. If $w_0$ is one of the two points at the top, use $f$ as the labelling, subtracting 1 above the deleted point, if there are any points above it. Similarly, if $w_0$ is one of the two points at the bottom, use $g$ as the labelling, subtracting 1 from all values higher than that of the deleted point. Since $f$ and $g$ each exhibit linear discrepancy 3 for precisely one pair of incomparable points and that pair is reduced to a difference of 2 under the modified labelling, this shows that $\mathbf{M}_{2n} - \{w_0\}$ has linear discrepancy equal to 2. Now suppose that $f(w_0) \neq g(w_0)$. Then there is a point $w_1$ such that (as sets) $\{w_0, w_1\} = \{l_k, r_{k-2}\}$ for some $3 \leq k \leq n$. Intuitively, $w_0$ occurs in a position in $\mathbf{M}_{2n}$ where it has the point $w_1$ opposite it in the other chain. We define a new labelling $f'$ as given below.

$$f'(x) = \begin{cases} f(x) & x = l_i, i < k, \text{ or } x = r_j, j < k-2; \\ g(x) - 1 & x = l_i, i > k \text{ or } x = r_j, j > k-2; \\ \min(f(x), g(x)) & x = w_1. \end{cases}$$

Since $f$ only exhibits linear discrepancy 3 at the top of the poset, where we do not use it in $f'$, and $g$ only exhibits linear discrepancy 3 at the bottom of the poset, where we do not use it in $f'$, we have constructed a labelling that demonstrates that $\text{ld}(\mathbf{M}_{2n} - \{w_0\}) \leq 2$. By induction, this labelling demonstrates that $\text{ld}(\mathbf{P}' - \{w_0\}) \leq 2$. If $u, v \neq w_0$, then removing the covering relation $u \lessdot v$ to form $\mathbf{P}$ is clearly allowed. On the other hand, if $w_0 \in \{u, v\}$, we can still remove the relation $u \lessdot v$, as $f'(w_1)$ agrees with both the labelling used from below and the labelling used from above, adjusting the assignment by subtracting 1 on one side to adjust for the deletion of $w_0$. Therefore $\mathbf{P}$ is 3-irreducible as desired. □

Theorem 4 demonstrates that there are an infinite number of 3-irreducible posets of width 3, and we will denote this entire class as $\mathscr{I}_3^3$. The next section will demonstrate that all elements of $\mathscr{I}_3^3$ arise via the approach of Theorem 4.

It is also worthwhile to note that the covering relations described in Theorem 4 are the *only* covering relations that can be deleted while maintaining 3-irreducibility. It appears that it may be possible to delete a covering relation $u \lessdot v$ for which one of $f(v) - f(u)$ and $g(v) - g(u)$ is equal to 3. However, this is not the case, as the resulting poset is not irreducible, since it contains an induced copy of a smaller 3-irreducible poset.

## 4. Three-irreducible posets of width 3

We are now prepared to provide the complete catalog of 3-discrepancy-irreducible posets of width 3. There are four such posets on five points. They are $\mathbf{1} + \mathbf{1} + \mathbf{3}$, $\mathbf{1} + \mathbf{2} + \mathbf{2}$, and the pair of dual posets formed from $\mathbf{2} + \mathbf{3}$ by removing the top comparability and the bottom comparability from the 3-element chain. They are shown, along with the 5-point, 3-irreducible poset of width 2 in Figure 5, where each poset is enclosed in a box to indicate which components belong to each poset.

On six points, there are 15 disconnected, 3-irreducible posets of width 3. They are all of the form $\mathbf{1}+\mathbf{P}$ where $\mathbf{P}$ is a connected poset on five points with width($\mathbf{P}$) $= 2$ and $\mathbf{P}$ derived from $\mathbf{5}$ by deleting a nonempty subset of the cover relations of $\mathbf{5}$. It is easy to verify that these are all 3-irreducible. In Figure 1 (in Section 1), we give the Hasse diagrams of the remaining three six-point 3-irreducible posets of width 3 (the standard example $\mathbf{S_3}$, $\mathbf{Q}_1$, and $\mathbf{Q}_1^d$) along with the only seven-point 3-irreducible poset of width 3, which we call $\mathbf{Q}_2$. Again, it is easy to verify that these posets are all 3-irreducible.

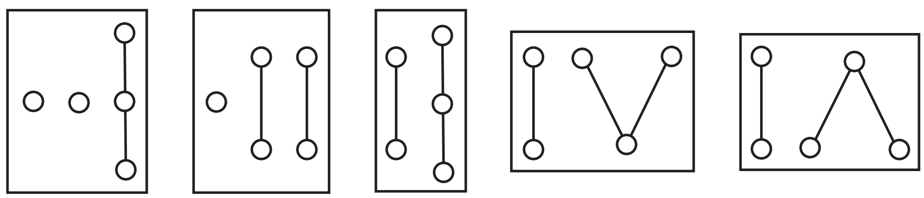

FIGURE 5. The 3-irreducible posets on five points

In the previous section, we saw that we could remove certain covering relations from $\mathbf{M}_{2n}$ to obtain more 3-irreducible posets. We also have the following theorem, which effectively says that the process is reversible and that all members of $\mathscr{I}_3^3$ arise this way.

**Theorem 5.** *For $n \geq 4$, every 3-discrepancy-irreducible poset $\mathbf{P}$ of width 3 with $2n$ points can be obtained from $\mathbf{M}_{2n}$ by the removal of comparabilities. Furthermore, there are no 3-discrepancy-irreducible posets on $2n+1$ points.*

We will prove Theorem 5 via the following lemmas. The general idea is to make pairs of points comparable, reversing the removal described in the previous section. If this cannot be done and the poset is of width 3, the poset in question is either one of those shown in Figure 1 or else is not irreducible.

**Remark 6.** Before beginning the proof, we observe that if $|\mathbf{P}| > 6$, $\mathrm{ld}(\mathbf{P}) = 3$, and there exists $x \in \mathbf{P}$ such that $|\operatorname{Inc}(x)| > 4$, then $\mathbf{P}$ is not irreducible, as it contains one of the disconnected 3-irreducible posets.

Our lemmas focus on 3-element antichains with particular properties, first showing what can be done if such antichains exist and then focusing on posets that do not include such antichains.

**Definition 7.** A 3-element antichain $A = \{x, y, z\}$ is called a 3-*critical antichain* if $|\operatorname{Inc}(x)| = 4$ and $(y, z)$ is a critical pair.

**Lemma 8.** *Let $\mathbf{P}$ be a 3-discrepancy-irreducible poset of width 3 on at least six points. If $\mathbf{P}$ contains a 3-critial antichain $A = \{x, y, z\}$, then the poset obtained by adding the cover relation $y < z$ to $\mathbf{P}$ is also 3-discrepancy-irreducible.*

*Proof.* By Remark 6, we may assume that $|\operatorname{Inc}(x)| = 4$. Let $\operatorname{Inc}(x) = \{y, z, v, w\}$. Since $A$ is an antichain, we know that $y \| z$. The proof proceeds based on the various relations that $v$ and $w$ can have to $y$ and $z$. Up to duality, there are five cases we must consider as shown in Figure 6. Case I is illustrated as having $v \| w$, but this is not used in our argument, so Case I also includes the case where $v \perp w$. We show that the first two configurations cannot exist in a 3-irreducible poset, the third either allows the addition of $y < z$ or contains a copy of the excluded configuration of Case II, and the final two readily allow for the insertion of the relation $y \lessdot z$.

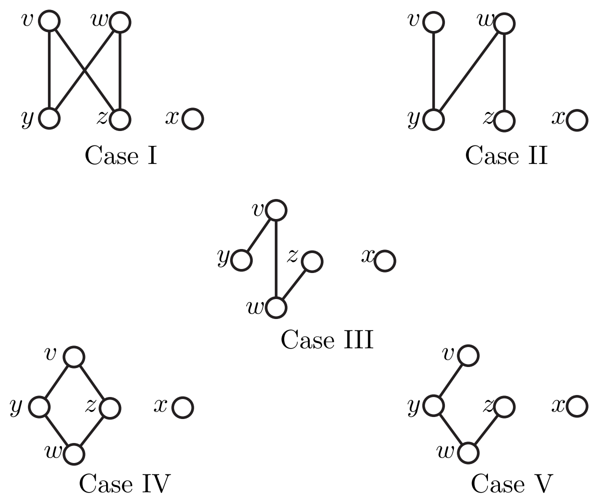


Figure 6. The five cases for Lemma 8

**Case I**. Suppose that $v$ and $w$ are both greater than $y$ and $z$. Then the 3-irreducibility of $\mathbf{P}$ implies that the deletion of a point results in a poset of linear discrepancy equal to 2. We have that $v$ and $w$ are over all elements of $D[A] - \{x\}$. This is because, for example, the deletion of $w$ forces $x$, $y$, and $z$ to be consecutive (in some order) to maintain linear discrepancy equal to 2, and $v$ must then be over all of them in any linear extension witnessing linear discrepancy 2. Thus $v$ cannot be incomparable with anything less than $x$, $y$, or $z$. The argument for $w$ is analogous. Furthermore, we have that $y$ and $z$ are under all elements of $U[\{v, w, x\}] - \{x\}$. This is because any $u \in U[\{v, w, x\}] - \{x\}$ must be greater than $x$ and therefore over $z$, since if it is incomparable with $z$ we have a 3-irreducible poset on 5 points induced by $\{u, x, v, w, z\}$. Since $(y, z)$ is a critical pair, we also get $u > y$. Now the 3-irreducibility of $\mathbf{P}$ implies that the posets $\mathbf{U}$ induced by $U[\{x, v, w\}]$ and $\mathbf{D}$ induced by $D[A]$ each have linear discrepancy 2. Without loss of generality, we may assume that the linear extension of $\mathbf{D}$ that witnesses linear discrepancy 2 has $x$ at the top followed immediately by $y$ and $z$ (in some order) and the one for $\mathbf{U}$ has $x$ at the bottom with $v$ and $w$ (in some order) immediately above. (This is because $A$ and $\{x, v, w\}$ are three element antichains, which must be kept consecutive in an extension witnessing linear discrepancy 2, and $x$ is comparable to all points other than $y, z, v, w$.) Since $v$ and $w$ are both over both $x$ and $y$, we can form a linear extension of $\mathbf{P}$ by starting with the one for $\mathbf{D}$ and then the one for $\mathbf{U}$ and it witnesses that $\mathrm{ld}(\mathbf{P}) \leq 2$, a contradiction.

**Case II**. Suppose that $y < v$, $y < w$, $z < w$, and $z \| v$. As in the previous case, we have that $v$ and $w$ are over all elements of $D[A] - \{x\}$ and that $y$ and $z$ are under all elements of $U[\{v, w, x\}] - \{x\}$. Again, the 3-irreducibility of $\mathbf{P}$ implies that the posets $\mathbf{U}$ induced by $U[\{x, v, w\}]$ and $\mathbf{D}$ induced by $D[A]$ each have linear discrepancy 2. Since $D(y) \subseteq D(z)$, we may assume without loss of generality that an optimal linear extension of $\mathbf{D}$ ends with $y < z < x$. If there is an optimal linear extension of $\mathbf{U}$ that starts $x < v < w$, then we can use the linear extension of $\mathbf{D}$ followed by the linear extension of $\mathbf{U}$ to find a linear extension of $\mathbf{P}$ that witnesses linear discrepancy 2, since the only pair of points we must check is $z$ and $v$, which are two apart. This contradiction implies that every linear extension of $\mathbf{U}$ that witnesses linear discrepancy 2 must place $w < v$. Now consider an optimal linear extension $L$ of $\mathbf{P} - \{y\}$, which has linear discrepancy 2. Since restricting $L$ to the points of $\mathbf{U}$ would give an optimal linear extension of $\mathbf{U}$, we must have $w <_L v$. Since $\{v, w, x\}$ is an antichain, we must have that they appear consecutively in $L$. But since $x \| z$, we cannot have $w <_L v <_L x$, otherwise we would have linear discrepancy 3. We also must have $z <_L x$ to keep $x$ with $v$ and $w$, but then $x$ and $w$ are between $z$ and $v$, again creating linear discrepancy 3, a contradiction to the 3-irreducibility of $\mathbf{P}$.

**Case III**. Suppose that $y < v$, $w < z$, and $w < v$ but $v \| z$ and $y \| w$. (Note here that $w < v$ is forced by the other four relationships, as if this is not true, we have an induced $\mathbf{1} + \mathbf{2} + \mathbf{2}$, contrary to 3-irreducibility.) Form $\mathbf{P}'$ by adding the cover relation $y \lessdot z$ to $\mathbf{P}$. If $\mathrm{ld}(\mathbf{P}') = 3$, we are done, as its irreducibility follows from that of $\mathbf{P}$. Thus, suppose that $\mathrm{ld}(\mathbf{P}') = 2$. If there is an optimal linear extension $L$ of $\mathbf{P}'$ with $w <_L y <_L x <_L z <_L v$, then we have a contradiction, as the same linear extension witnesses that $\mathrm{ld}(\mathbf{P}) = 2$. Thus, without loss of generality (by considering the dual poset if necessary), we may assume that any optimal linear extension $L$ of $\mathbf{P}'$ orders these five points consecutively as $w, y, x, v, z$ or $y, w, x, v, z$. There must be a reason that $z$ is forced to be last among these five, so there is another point $u$ that is incomparable to $z$, and thus $\mathrm{Inc}(z) = \{x, y, v, u\}$. We know that $u$ is over $x$, and the ordering of points in $L$ (witnessing linear discrepancy 2 for $\mathbf{P}'$) also implies that $u > y$. In fact, $L$ forces $y$ to be under everything in $U(A)$, and thus $U(x) \subseteq U(y)$. Furthermore, it is clear that $D(y) \subseteq D(x)$. Thus, $(y, x)$ is a critical pair in $\mathbf{P}$ and $|\mathrm{Inc}(z)| = 4$. Since we have $y < v$, $y < u$, and $x < u$, we are in the excluded configuration of Case II with $z$ playing the role of $x$, so we are done.

**Case IV**. Suppose that $w < y < v$ and $w < z < v$. Then adding the relation $y \lessdot z$ to $\mathbf{P}$ clearly forms a poset $\mathbf{P}'$ of linear discrepancy 3. If this were not the case, in order to have linear discrepancy 2 any optimal linear extension $L$ of $\mathbf{P}'$ would have $w <_L y <_L x <_L z <_L v$, and thus the removal of the covering relation $y \lessdot z$ would not increase the linear discrepancy. The irreducibility of $\mathbf{P}'$ follows trivially from the irreducibility of $\mathbf{P}$.

**Case V**. Suppose that $w < y < v$ and $w < z$ but $v \| z$. Form $\mathbf{P}'$ by adding $y \lessdot z$ to $\mathbf{P}$. Suppose for a contradiction that $\mathrm{ld}(\mathbf{P}') = 2$. Then if there is an optimal linear extension $L$ of $\mathbf{P}'$ with $w \lessdot_L y \lessdot_L x \lessdot_L z \lessdot_L v$, we have a contradiction, as then $L$ demonstrates that $\mathrm{ld}(\mathbf{P}) = 2$. The only other possible ordering for these five points in an optimal linear extension of $\mathbf{P}'$ is $w \lessdot_L y \lessdot_L x \lessdot_L v \lessdot_L z$. Now there must be a point $u$ incomparable to $z$ in $\mathbf{P}'$ (and therefore in $\mathbf{P}$ as well) that has forced $z$ into this position in $L$. Furthermore, we note that $x < u$ and $y < u$. Now the 3-irreducibility of $\mathbf{P}$ implies that any optimal linear extension of $\mathbf{P} - \{w\}$ must have $y, x, z, v, u$ appear consecutively in that order. If there were a point $u'$ other than $v$ and $u$ in $U(A)$, we could delete $u'$ and again force $y, x, z, v, u$ into order consecutively. We could then combine the linear extensions arrived at by deleting $w$ and $u'$ and witness that $\mathbf{P}$ has linear discrepancy 2, a contradiction. But now that $U(A) = \{u, v\}$, we cannot have drawn $u$ up far enough to require $z$ to appear last in the linear extension $L$ previously discussed (i.e., we could reverse $v$ and $z$ in $L$ without increasing the linear discrepancy). This contradiction finally shows that $\mathrm{ld}(\mathbf{P}') = 3$, and its irreducibility follows trivially from that of $\mathbf{P}$. $\square$

Our next step is to resolve the situation where there are no 3-critical antichains because in any antichain $\{x, y, z\}$ with $|\operatorname{Inc}(x)| = 4$, the points $y$ and $z$ do not form a critical pair. In this situation, it turns out that either the poset is not 3-irreducible or else forms one of our exceptional posets. More precisely, we have the following lemma:

**Lemma 9.** *Let $\mathbf{P}$ be a 3-discrepancy-irreducible poset of width 3 on at least six points. Suppose that $\mathbf{P}$ does not contain a 3-critical antichain but does contain a 3-element antichain $A = \{x, y, z\}$ such that $|\operatorname{Inc}(x)| = 4$. Then $\mathbf{P}$ is $\mathbf{Q}_1$, $\mathbf{Q}_1^d$, or $\mathbf{Q}_2$.*

*Proof.* Considering duality, there are essentially two possibilities for how we can ensure that neither $(y, z)$ nor $(z, y)$ is a critical pair. The first is to have points $w$ and $u$ such that $w > y > u$ but $w \| z$ and $u \| z$, and the second is to have points $w$ and $u$ such that $w > y$, $u > z$, $w \| z$, and $u \| y$. Since we require at least one additional point in our poset, in order to ensure 3-irreducibility, we must have in the first case that $w > x$, and in the second case we must have $w > x$ and $u > x$.

**Case I**. Here we have two subscases, depending on if $x \| u$ or $x > u$. We first consider $x \| u$. Since $|\operatorname{Inc}(x)| = 4$, there is one more point $v$ incomparable to $x$. We must have $v \perp z$, since otherwise $z$ would have five points incomparable to it. If $v > z$, then we also have $v > y$, as otherwise $w, y, x$ and $v, z$ form a 3-irreducible poset on five points. Then these points form $\mathbf{Q}_1$, so there are no more points since $\mathbf{P}$ is 3-irreducible. (Considering the dual case where $w \| x$ and $x > u$ yields $\mathbf{Q}_1^d$.) If $v < z$, we must have $v < y$, otherwise $y, u, z, v$, and $x$ induce a $\mathbf{2} + \mathbf{2} + \mathbf{1}$. But then we note that in any optimal linear extension, we must have $v, u, x, z, y$ appear consecutively in this fixed ordering, and thus either there are no more points above $A$ or else no more points below $A$, as if there are additional points on both sides of $A$ we can combine the linear extensions to demonstrate that $\mathrm{ld}(\mathbf{P}) = 2$. If there are no more points below $A$, consider $\mathbf{P} - \{v\}$, which must have linear discrepancy 2, and then add $v$ back at the bottom and witness $\mathrm{ld}(\mathbf{P}) = 2$. Similarly, if there are no more points above $A$, consider the result of removing $w$.

On the other hand, suppose that $x > u$. Again, there is a point $v$ incomparable to $x$, which must (by duality) be over $y$ and $z$ in order to maintain 3-irreducibility. By assumption there is another point $t$ incomparable to $x$. If $t < z$, then $t < y$ as well, for otherwise $\{y, u, x, t, z\}$ induces a smaller 3-irreducible poset. In this case, we have formed the poset $\mathbf{Q}_2$, so we are done. Thus, suppose that $t > z$. Again, we must have $t > y$ due to irreducibility, but then $\{v, t, z, w, x\}$ induce (regardless of whether $t$ and $v$ are comparable) a 3-irreducible poset on 5 points, showing that we cannot have $t > z$ and concluding this case.

**Case II**. We first consider where we can insert the two remaining points incomparable to $x$. Notice that if one of them is above $A$, then it must be over both $y$ and $z$ due to irreducibility, in which case we have formed $\mathbf{S}_3$. Since there is another point required to complete $\mathbf{P}$ ($|\operatorname{Inc}(x)| = 4$ by hypothesis), $\mathbf{P}$ is not 3-irreducible. Thus, the two other points $v$ and $t$ incomparable to $x$ are both under $A$, and in fact must both be under both $y$ and $z$ in order to avoid a $\mathbf{2} + \mathbf{3}$. Although we are not able to force a specific ordering on these seven points in an optimal linear extension, it is clear that $v$ and $t$ must appear in the first two positions (in some order), followed by $x$, followed by $y$ and $z$ (in some order), finally followed by $w$ and $u$ ($u$ first if $y$ comes before $z$ and $w$ first if $z$ comes before $w$). This again allows us to say that other than the points already under consideration, there are either no more points above $A$ or no more points below $A$. In the former case, delete $v$, find an optimal linear extension of linear discrepancy 2, and reinsert $v$ without increasing the linear discrepancy, a contradiction. In the latter instance, delete $w$, find an optimal linear extension with linear discrepancy 2, and then reinsert $w$ without increasing the linear discrepancy since it is clear that anything else below $A$ must be comparable to $x, y$, and $z$. □

Our final step is to show that with one additional exception, we are able to complete the process described in Lemma 8 to insert comparabilities, ultimately resulting in the reduction of the width of a 3-irreducible poset of width 3. We do this via the following lemma.

**Lemma 10.** *Let* $\mathbf{P}$ *be a poset of width* 3 *on at least* 6 *points. Assume that* $\mathrm{ld}(\mathbf{P}) = 3$*. If for all* 3*-element antichains* $A$ *of* $\mathbf{P}$*,* $|\operatorname{Inc}(x)| \leq 3$ *for all* $x \in A$*, then* $\mathbf{P}$ *is either* $\mathbf{S}_3$ *or is not irreducible with regard to linear discrepancy.*

*Proof.* As with the previous two lemmas, this proof proceeds by an analysis of cases. By way of contradiction, we assume that $\mathbf{P}$ is 3-irreducible but not isomorphic to $\mathbf{S}_3$. Here, however, we focus on the configuration of the minimal elements $M$ of $U(A)$ and their relationship to $A := \{x, y, z\}$. Since $\mathrm{width}(\mathbf{P}) = 3$, we know that $|M| \leq 3$. We may assume, by considering the dual if necessary, that $|M| > 0$ as well. Additionally, note that it is clear that if all elements of $M$ are comparable to all elements of $A$, then $\mathbf{P}$ is not irreducible, as there are no incomparbilities between points of $U[M]$ and points of $D[A]$, so one of these sets must induce a smaller 3-irreducible poset.

**Case I**. Suppose that $M = \{v\}$. Without loss in generality, $v > y$ and $v \| x$. The arguments for $v > z$ and $v \| z$ differ only slightly, so we will just give the argument for the former. Note that anything over $v$ must also be over $x$ since $|\operatorname{Inc}(x)| = 3$, and anything over $x$ is over $v$ by the fact that $M = \{v\}$. Thus we have that $v$ and $x$ must be maximal in $\mathbf{P}$, as otherwise we could obtain an optimal linear extension of $U(\{v, x\})$ and an optimal linear extension of $D[\{v, x\}]$ and combine them to obtain a linear extension of $\mathbf{P}$ with linear discrepancy 2. Having established that $v$ and $x$ are maximal in $\mathbf{P}$, consider the effect of deleting $v$. We may assume that the optimal linear extension $L$ (of linear discrepancy at most 2) does not place $x$ below both $y$ and $z$, since it is over all other elements of the poset, and thus we can place $v$ at the top of $L$ without increasing the linear discrepancy, contradicting that $\mathbf{P}$ is 3-irreducible.

**Case II**. Suppose that $M = \{v, w\}$. Here there are three configurations: $v$ and $w$ each over distinct two-element subsets of $A$, $v$ over all elements of $A$ and $w$ over two elements of $A$, and $v$ over all of $A$ and $w$ over one element of $A$. The arguments are all slight variations on the same theme, so we will provide the proof for the last scenario, supposing $w > x$. We first claim that in order to be 3-irreducible, we must have that either $U(\{v, w\})$ or $D(A)$ is empty. If not, deleting an element of $U(\{v, w\})$ or an element of $D(A)$ results in a linear extension $L$ witnessing linear discrepancy 2 and thus must have consecutively $x$, followed by $y$ and $z$ (in some order), followed by $w$ and then $v$. Because $y$ and $z$ are each already incomparable to three other points, they are comparable to all other points of the poset, and thus their ordering in $L$ can be freely interchanged. Therefore, we may use the linear extension from deleting an element of $U(\{v,w\})$ to order $D(A)$ and the linear extension from deleting an element of $D(A)$ to order $U(\{v,w\})$ and combine them by placing $x < y < z < w < v$ in between, achieving linear discrepancy 2.

Having established that either $v$ and $w$ are maximal or $A = \min(\mathbf{P})$, we proceed to argue that this cannot be the case. Suppose that $v$ and $w$ are maximal. If we delete $v$, we note that $\{y,z,x\}$ and $\{y,z,w\}$ are 3-element antichains, so to witness linear discrepancy 2 we must have $w$ as the last element of an optimal linear extension, which allows for placing $v$ at the top without increasing the linear discrepancy, since $v$ is comparable to all points but $w$. On the other hand, suppose that $A = \min(\mathbf{P})$. Consider the result of deleting $x$. Then $\{y,z,w\}$ is still a 3-element antichain, which must be kept consecutive in order to have linear discrepancy 2. Without loss of generality, we may assume that $y$ and $z$ come before $w$ in an optimal linear extension, as they are comparable to all points of the poset with $x$ deleted except $w$. Since $x$ is under all points except $y$ and $z$, we thus may safely add $x$ at the bottom of our linear extension without increasing linear discrepancy, which is our final contradiction.

**Case III**. Suppose that $M = \{u,v,w\}$. Then we note that, by hypothesis, each element of $M$ is incomparable to at most one element of $\mathbf{P}$. Since we have also assumed that $\mathbf{P}$ is not $\mathbf{S}_3$, we thus have that at least one element of $A$ is comparable to all elements of $M$ and vice versa. Suppose that these elements are $w$ and $z$. Without loss of generality, take $u \| x$, since we know we are missing at least one comparability. It may also be that $v > y$ or $v \| y$, so we will suppose that $v \| y$, as the following argument only becomes simpler if $v > y$. Since $u$ and $x$ are comparable to all other elements of $\mathbf{P}$, consider the result of deleting $x$. This poset has linear discrepancy 2, and we may assume that our optimal linear extension $L$ places $u \lessdot v \lessdot w$, since $u$ and $v$ are comparable to all other points. Similarly, deleting $u$ results in an optimal linear extension $L'$ with $z \lessdot y \lessdot x$. Now use $L'$ to order $D[A]$ and $L$ to order $U[M]$ and we clearly have a linear extension that witnesses $\mathrm{ld}(\mathbf{P}) = 2$, a contradiction. □

Having established these three lemmas, we now have Theorem 5 as a consequence. Lemma 8 demonstrates that we are, under most circumstances, able to insert particular comparabilities in 3-irreducible posets of width 3. Repeating this process, we are able to reduce the poset to a 3-irreducible poset of width 2 unless we are in the setting of Lemmas 9 or 10, but in both of those cases, we must have an irreducible poset of at most 7 points. Thus, we are able to reduce the width of any 3-irreducible poset of width 3 on at least 8 points to a 3-irreducible poset of width 2, implying that there cannot be a 3-irreducible poset of width 3 on an odd number of points greater than seven. □

## 5. Conclusions and future work

Corollary 25 of [5] shows that a poset of linear discrepancy greater than 1 must contain an induced $\mathbf{1}+\mathbf{3}$, $\mathbf{2}+\mathbf{2}$, or $\mathbf{1}+\mathbf{1}+\mathbf{1}$. Now with Howard's result for width 2 and our Theorem 5 in hand, we are able to answer the first challenge posed by Tanenbaum, Trenk, and Fishburn in [5]. In order to do so, we first show that any poset of linear discrepancy greater than 2 must contain a poset of linear discrepancy

3, and therefore we have that the 3-discrepancy-irreducible posets are precisely the forbidden subposets for linear discrepancy equal to 2.

**Lemma 11.** *Let* $\mathbf{P}$ *be a poset with* $\mathrm{ld}(\mathbf{P}) > 2$*. Then* $\mathbf{P}$ *contains an induced subposet* $\mathbf{Q}$ *with* $\mathrm{ld}(\mathbf{Q}) = 3$*.*

*Proof.* Let $\mathbf{P}_1$ be the trivial poset induced by a minimal element of $\mathbf{P}$. Inductively form $\mathbf{P}_k$ from $\mathbf{P}_{k-1}$ by adding a minimal element of $\mathbf{P} - \mathbf{P}_{k-1}$ to $\mathbf{P}_{k-1}$. (We give $\mathbf{P}_k$ the induced ordering from $\mathbf{P}$.) In our construction, we always favor minimal elements of $\mathbf{P} - \mathbf{P}_{k-1}$ that keep $\mathrm{ld}(\mathbf{P}_k) \leq 2$. At some point, there is no such point, so we add any minimal element $x_{k+1}$ of $\mathbf{P} - \mathbf{P}_{k-1}$ to $\mathbf{P}_k$ and call this poset $\mathbf{P}'$. We will proceed to show that an induced subposet $\mathbf{Q}$ of $\mathbf{P}'$ has $\mathrm{ld}(\mathbf{Q}) = 3$.

Let $L$ be an optimal linear extension of $\mathbf{P}_k$. Let $L$ have

$$x_1 \lessdot x_2 \lessdot x_3 \lessdot \cdots \lessdot x_k.$$

If $x_{k+1}$ is incomparable to five or more points of $\mathbf{P}'$, then we are done, as $x_{k+1}$ and $\mathrm{Inc}_{P'}(x_{k+1})$ induce a poset of linear discrepancy 3. Thus, $x_{k+1}$ must be greater than $x_j$ for $j \leq k - 6$, since $x_k, x_{k-1}, x_{k-2}$, and $x_{k-3}$ are all greater than such $x_j$ in $\mathbf{P}$. Therefore we have that

$$\mathrm{Inc}_{P'}(x_{k+1}) \subseteq \{x_{k-5}, x_{k-4}, \ldots, x_k\}.$$

Now form a linear extension $L'$ of $\mathbf{P}'$ by inserting $x_{k+1}$ into $L$. If $\{x_{k-2}, x_{k-1}, x_k\} \subseteq \mathrm{Inc}(x_{k+1})$, we insert $x_{k+1}$ immediately above $x_{k-3}$. Then $x_{k+1}$ has been placed in the middle of the six candidates for elements of $\mathrm{Inc}_{P'}(x_{k+1})$, and thus $\mathrm{ld}(\mathbf{P}') = 3$. If $x_{k+1}$ must be placed over an element of $\{x_{k-2}, x_{k-1}, x_k\}$, we form $L'$ by doing so, placing $x_{k+1}$ as far down in $L'$ as possible. But then since $x_{k-2} > x_{k-5}$, $x_{k-1} > x_{k-4}, x_{k-5}$, and $x_k > x_{k-3}, x_{k-2}, x_{k-3}$, we have that the furthest $x_{k+1}$ could be placed from any element to which it is incomparable is a distance of 3, and in fact there must be such a pair, otherwise we would have $\mathrm{ld}(\mathbf{P}') \leq 2$, a contradiction to its construction. $\square$

Since any poset of linear discrepancy greater than 2 must contain an induced subposet of linear discrepancy 3, it must contain a 3-discrepancy-irreducible subposet, and thus we have completed the proof of the characterization of posets of linear discrepancy 2 given in Theorem 1.

Although Theorem 1 is perhaps disappointing in that there are infinitely many forbidden subposets required to characterize posets with linear discrepancy equal to 2, the set of posets that generate this collection is nicely describable. Furthermore, the results of Section 4 rely on the width and linear discrepancy in a way that may generalize to posets of higher linear discrepancy. In particular, it would be interesting to find a more general version of Lemma 8, as the role of critical pairs in linear discrepancy is much larger than previously recognized. Unfortunately, our proofs of these results require fairly intricate case analysis that would quickly become overwhelming with increasing linear discrepancy. Furthermore, computer investigations indicate that the number of "exceptional cases" increases quickly as linear discrepancy increases, which would make the arguments even more complex. Also, it appears that even if similar results can be proved for posets of higher linear discrepancy, the infinite families involved will require more posets to generate them.

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