A RAINBOW r-PARTITE VERSION OF THE ERDŐS–KO–RADO THEOREM

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Abstract. Let $[n]^r$ be the complete r-partite hypergraph with vertex classes of size n. It is an easy exercise to show that every set of more than $(k-1)n^{r-1}$ edges in $[n]^r$ contains a matching of size k. We conjecture the following rainbow version of this observation: If $F_1, F_2, \ldots, F_k \subseteq [n]^r$ are of size larger than $(k-1)n^{r-1}$ then there exists a rainbow matching, i.e. a choice of disjoint edges $f_i \in F_i$. We prove this conjecture for $r = 2$ and $r = 3$.

1. Motivation

1.1. An r-partite version of the Erdős–Ko–Rado theorem.

A *matching* is a collection of disjoint sets. As is customary, we write $[n]$ for the generic set of size n, $\{1, 2, \ldots, n\}$. The largest size of a matching in a hypergraph H is denoted by $\nu(H)$.

An r-uniform hypergraph H is called *r-partite* if $V(H)$ is partitioned into sets $V_1, \ldots, V_r$, called the *vertex classes* of H, and each edge meets every V_i in precisely one vertex. If all vertex classes are of the same size n, H is called *n-balanced*. The complete n-balanced r-partite hypergraph can clearly be identified with $[n]^{[r]}$, the set of all functions from $[r]$ to $[n]$, and in accord we denote it by $[n]^r$.

Observation 1.1. *If F is a set of edges in an n-balanced r-partite hypergraph and $|F| > (k-1)n^{r-1}$ then $\nu(F) \geq k$.*

Proof. The complete n-balanced r-partite hypergraph $[n]^r$ can be decomposed into n^{r-1} perfect matchings M_i, each of size n. Writing $F = \bigcup_{i \leq n^{r-1}} (F \cap M_i)$ shows that at least one of the matchings $F \cap M_i$ has size k or more. □

This observation can be viewed as an r-partite version of the celebrated Erdős–Ko–Rado problem, on the number of edges in the complete r-uniform hypergraph on n vertices needed to guarantee a matching of size k. The Erdős–Ko–Rado theorem settles this problem for $k = 2$.

The topic of this paper is a possible extension of Observation 1.1 to rainbow matchings.

Definition 1.2. Let $\mathcal{F} = (F_i \mid 1 \leq i \leq k)$ be a collection of hypergraphs. A choice of disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$.

Conjecture 1.3. *If $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ is a list of sets of edges in an n-balanced r-partite hypergraph and $|F_i| > (k-1)n^{r-1}$ for all $i \leq k$ then $\mathcal{F}$ has a rainbow matching.*

The case $k = 2$ is not hard, see [1]. The case $r = 2$ is also not hard, and though it will be subsumed by later results, we give here a short proof.

Theorem 1.4. *If $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ is a list of sets of edges in an n-balanced bipartite graph and $|F_i| > (k-1)n$ for all $i \leq k$ then $\mathcal{F}$ has a rainbow matching.*

Proof. Denote the vertex classes of the bipartite graph M and W. Since $\sum_{v \in M} \deg_{F_1}(v) = |F_1| > (k-1)n$, there exists a vertex $v_1 \in M$ such that $\deg_{F_1}(v_1) \geq k$. Let $F_2' = F_2 - v_1$ (namely the set of edges in F_2 not containing v_1). Since $\deg_{F_2}(v_1) \leq n$, we have $|F_2'| > (k-2)n$, and hence there exists a vertex $v_2 \neq v_1$ such that $\deg_{F_2}(v_2) \geq k-1$. Continuing this way we obtain a sequence $v_1, \ldots, v_k$ of distinct vertices in M, satisfying $\deg_{F_i}(v_i) > k - i$. Since $\deg_{F_k}(v_k) > 0$ there exists an edge $e_k \in F_k$ containing v_k. Since $\deg_{F_{k-1}}(v_{k-1}) > 1$

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there exists an edge $e_{k-1} \in F_{k-1}$ containing v_{k-1} and missing e_k. Since $\deg_{F_{k-2}}(v_{k-2}) > 2$ there exists an edge $e_{k-2} \in F_{k-2}$ containing v_{k-2} and missing e_k and e_{k-1}. Continuing this way, we construct a rainbow matching $e_1, \dots, e_k$ for $\mathcal{F}$. $\square$

Our main result is as follows.

Theorem 1.5. *Conjecture 1.3 is true for $r = 3$.*

2. Shifting

The proof in [4] uses an operation called "shifting", that has since become a main tool in the area. It is an operation on a hypergraph H, defined with respect to a specific linear ordering "$<$" on its vertices. For $x < y$ in $V(H)$ define $s_{xy}(e) = e \cup x \setminus \{y\}$ if $x \notin e$ and $y \in e$, provided $e \cup x \setminus \{y\} \notin H$; otherwise let $s_{xy}(e) = e$. We also write $s_{xy}(H) = \{s_{xy}(e) \mid e \in H\}$. If $s_{xy}(H) = H$ for every pair $x < y$ then H is said to be *shifted*.

Given an r-partite hypergraph G and a linear order on each vertex class, an *r-partite shifting* is a shifting s_{xy} where x and y belong to the same vertex class. G is said to be *r-partitely shifted* if $s_{xy}(H) = H$ for all pairs $x < y$ that belong to the same vertex class.

Given a collection $\mathcal{H} = (H_i,\ i \in I)$ of hypergraphs, we write $s_{xy}(\mathcal{H})$ for $(s_{xy}(H_i),\ i \in I)$.

Remark 2.1. Define a partial order on pairs of vertices by $(v_i, v_j) \leq (v_k, v_\ell)$ if $i \leq k$ and $j \leq \ell$. Write $(v_i, v_j) < (v_k, v_\ell)$ if $(v_i, v_j) \leq (v_k, v_\ell)$ and $(v_i, v_j) \neq (v_k, v_\ell)$. A set F being shifted is equivalent to its being closed downward in this order, which in turn is equivalent to its complement being closed upward.

As observed in [3] (see also [2]) shifting does not increase the matching number of a hypergraph. This can be generalized to rainbow matchings.

Lemma 2.2. *Let $\mathcal{F} = (F_i \mid\ i \in I)$ be a collection of hypergraphs, sharing the same linearly ordered ground set V, and let $x < y$ be elements of V. If $s_{xy}(\mathcal{F})$ has a rainbow matching, then so does $\mathcal{F}$.*

Proof. Let $s_{xy}(e_i),\ i \in I$, be a rainbow matching for $s_{xy}(\mathcal{F})$. There is at most one i such that $x \in e_i$, say $e_i = a \cup \{x\}$ (where a is a set).

If there is no edge e_s containing y, then replacing e_i by $a \cup \{y\}$ as a representative of F_i, leaving all other e_s as they are, results in a rainbow matching for $\mathcal{F}$. If there is an edge e_s containing y, say $e_s = b \cup \{y\}$, then there exists an edge $b \cup \{x\} \in F_s$ (otherwise the edge e_s would have been shifted to $b \cup \{x\}$). Replacing then e_i by $a \cup \{y\}$ and e_s by $b \cup \{x\}$ results in a rainbow matching for $\mathcal{F}$. $\square$

3. A Hall-type size condition for rainbow matchings in bipartite graphs

In this section we prove a result on the existence of rainbow matchings for a collection of bipartite graphs, all sharing the same vertex set and bipartition, that will be later used for the proof of Theorem 1.5. This condition is not formulated in terms of the sizes of the individual graphs, but (somewhat reminiscent of the condition in Hall's theorem) in terms of the sizes of subsets of the collection of graphs.

Theorem 3.1. *Let $F_i,\ i \leq k$ be subsets of $E(K_{n,n})$. If*

$$\sum_{i \in I} |F_i| > n|I|(|I| - 1) \text{ for every } I \subseteq [k] \tag{1}$$

then the system $\mathcal{F} = (F_1, \dots, F_k)$ has a rainbow matching.

Sharpness of this bound is shown by the example of k sets F_i, each consisting of all edges incident with a set of $k - 1$ vertices in one side of the bipartite graph. The analogous result for $r = 1$ can be proved directly, or using Hall's theorem. For $r \geq 3$ the analogous result, suggested by the same example, is that if $\sum_{i \in I} |F_i| > n^2|I|(|I| - 1)$ for all I then the system $(F_1, \dots, F_n)$ has a rainbow matching. But this is false, as shown by the pair F_1, F_2 in which F_1 consists of a single edge and F_2 the set of all edges meeting this edge. Then $|F_2| = n^3 - (n-1)^2$, $|F_1| + |F_2| = 3n^2 - 3n$, which for $n > 3$ is larger than $2n^2$, and there is no rainbow matching. It is not clear what is the right condition for general r.

3.1. **An algorithm.** The proof of Theorem 3.1 is algorithmic. As before, we assume that each side of the bipartite graph is linearly ordered, say $M = (m_1 < m_2 < \ldots < m_n)$ and $W = (w_1 < w_2 < \ldots < w_n)$.

Definition 3.2. Two edges e, f are said to be *parallel* if the order between their M vertices is the same as the order between their W vertices. If in this case the vertices of e precede those of f, we write $e < f$. Non-parallel edges are said to be *crossing*.

By Lemma 2.2, we may assume that all F_i are bipartitely shifted with respect to the given orders.

Order the sets F_i by their sizes,

$$|F_1| \leq |F_2| \leq \ldots \leq |F_k| \tag{2}$$

We choose inductively edges $e_i \in F_i$. As e_1 we choose a longest edge $(m_{c(1)}, w_{d(1)})$ in F_1, where the length of an edge (m_p, w_q) in this case is $|q - p|$. By the shiftedness of F_1, either $c(1) = 1$ or $d(1) = 1$.

Suppose that $e_1 \in F_1,\ e_2 \in F_2, \ldots, e_{t-1} \in F_{t-1}$ have been chosen. Let $Z_t = \bigcup_{j<t} e_j$. Let a_t the first index such that $m_{a_t} \notin Z_t$, and b_t be the first index such that $w_{b_t} \notin Z_t$. Let $R_t = \{m_1, \ldots, m_{a_t-1}\} \cup \{w_1, \ldots, w_{b_t-1}\}$ (R_1 is the empty set).

Let $\tilde{F}_t = F_t[V \setminus Z_t]$ (the set of edges in F_t not meeting Z_t). Define the *length* of an edge $(m_p, w_q) \in F_t$ as $|(q - b_t) - (p - a_t)|$ (this is the same as the above definition of "length", once the consecutive used vertices are removed). Assuming that $\tilde{F}_t \neq \emptyset$, choose e_t to be a longest edge in $\tilde{F}_t$. Since F_t is shifted, e_t must contain either m_{a_t} or w_{b_t}.

The fact that $e_t \in \tilde{F}_t$ implies inductively that the edges $e_i,\ i \leq t$, form a matching. The proof will be complete if we show that $\tilde{F}_t \neq \emptyset$ for all $t \leq k$.

The following example illustrates the way the algorithm proceeds. In it the inequalities of (1) are violated, and indeed the algorithm fails, although in fact there is a rainbow matching.

Example 3.3. Let $q < n$. Let $F_1 = \{m_c w_d \mid c, d \leq q\}$, and let $F_2 = F_3 = \ldots = F_{q+1} = \{m_c w_d \mid c \leq q,\ d \leq n\} \cup \{m_c w_1 \mid c \leq n\}$.
Here $|F_i| = (q+1)n - q$ for all $1 < i \leq q+1$, and hence $\sum_{i \leq q+1} |F_i| = q^2 + q[(q+1)n - q] = q(q+1)n$, so in this case (1) is violated, with equality replacing strict inequality. Indeed, as we shall see, the algorithm fails. Yet, there exists a rainbow matching: F_1 is represented by $m_1 w_q$, F_2 is represented by $m_n w_1$, and F_i is represented by $m_{i-1} w_{n-i+2}$ for $i > 2$.

Here is how the algorithm goes (we are assuming below that $q \geq 3$):

$$\begin{gathered} R_1 = \emptyset,\ e_1 = m_q w_1,\ R_2 = \{w_1\},\ e_2 = m_1 w_n,\ R_3 = \{m_1, w_1\},\ e_3 = m_2 w_{n-1}, \ldots, \\ e_q = m_{q-1} w_{n-q+2},\ R_{q+1} = \{w_1\} \cup \{m_c \mid c \leq q\}. \end{gathered}$$

After the choice of e_q there is no possible choice for e_{q+1} and the algorithm halts. Note that in the first step it was also legitimate to choose $m_1 w_q$, which would lead to a rainbow matching.

Let us now return to the proof. Suppose, for contradiction, that $\tilde{F}_m = \emptyset$ for some $m \leq n$. We shall show that this entails a violation of (1), for $I = [m]$.

For each $i < m$ let $c(i),\ d(i)$ be such that $e_i = (m_{c(i)}, w_{d(i)})$. As already noted, by shiftedness either $c(i) = a_i$ or $d(i) = b_i$. We direct e_i, calling one of its endpoints "tail" and the other "head", as follows. If $c(i) = a_i$ we call m_{a_i} the *tail* of e_i, and $w_{d(i)}$ its *head*. Otherwise, we call $w_{d(i)}$ the tail, and $m_{c(i)}$ the head. We write $\text{tail}(e_i)$ for the tail, and $\text{head}(e_i)$ for the head. We clearly have:

Observation 3.4. *If $i < j$ then* $\text{tail}(e_i) \in R_j$. □

3.2. **Short edges.** We call the edges e_i contained in R_m *short* and an edge not contained in R_m *long*. Let $e_{i_j},\ j < p$, be the short edges, where $i_1 < i_2 < \ldots < i_{p-1}$ (so, there are $p - 1$ short edges). Define $i_0 = 0$ and $i_p = m$. To understand the significance of short edges, note that if there are no short edges then

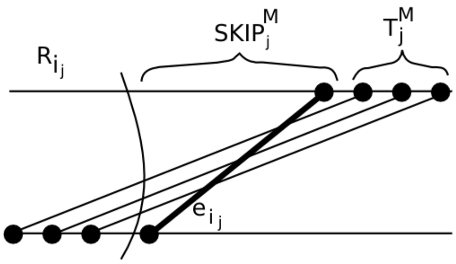

FIGURE 1. $SKIP_j^M$, T_j^M and e_{i_j}

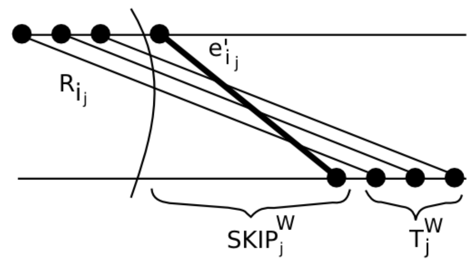

FIGURE 2. $SKIP_j^W$ and T_j^W. Here e'_{i_j} is the longest edge in $\tilde{F}_{i_j}$ starting at a_i.

$|R_m| = m-1$. Since $\tilde{F}_m = \emptyset$, the set R_m is a cover for F_m, and hence $|F_m| \leq (m-1)n$. By (2) this implies that $\sum_{i \leq m} |F_i| \leq m(m-1)n$, contradicting the assumption of the theorem.

Example 3.5. In Example 3.3 there is only one short edge, e_1.

For $j < p$ let ℓ_j^W be the length of the longest edge in $\tilde{F}_{i_j}$ containing $m_{a_{i_j}}$ and let ℓ_j^M be the length of the longest edge in $\tilde{F}_{i_j}$ containing $w_{b_{i_j}}$. Let $SKIP_j^M = \{m_{a_{i_j}}, m_{a_{i_j}+1}, \ldots, m_{a_{i_j}+\ell_j^M}\}$ and $SKIP_j^W = \{w_{b_{i_j}}, w_{b_{i_j}+1}, \ldots, w_{b_{i_j}+\ell_j^W}\}$.

We denote by T_j^M (resp. T_j^W) the longest contiguous stretch of vertices in $Z_{i_j} \cap M$ (resp. $Z_{i_j} \cap W$) starting right after $SKIP_j^M$ (resp. $SKIP_j^W$), and let $t_j^M = |T_j^M|$, $t_j^W = |T_j^W|$. See Figures 1 and 2.

4. BOUNDING $\sum |F_i|$ FROM ABOVE

4.1. **A toy case - one short edge.** Our aim is now to delve into calculations showing that under the negation assumption $\sum_{i \leq m} |F_i| < nm(m-1)$. To demonstrate the type of arguments involved in the general proof, let us consider separately the case in which there is only one short edge, say e_i. It may be worth following the arguments in Example 3.3, in which as mentioned above there is only one short edge.

Recall that either $c(i) = a_i$ or $d(i) = b_i$, and without loss of generality assume the latter, implying that $d(i) = \min\{j \mid w_j \notin R_i\}$.

Write ℓ for ℓ_1^M, namely the length of e_i (to understand the subscript 1 in ℓ_1^M remember that $i_1 = i$). The edge e_i skips ℓ vertices in R_m, each being matched by some edge e_j, $i < j < m$, and hence $\ell \leq m - i$.

Clearly, $|R_m| = m$, and since R_m is a cover for F_m it follows that $|F_m| \leq mn$. But in this calculation each of the ℓ edges $(m_{c(i)}, w_j)$ for $j = b_i, b_i+1, \ldots, b_i+\ell-1$, being contained in R_m, is counted twice, from the direction of $m_{c(i)}$ and from the direction of w_j. Thus we know the following:

$$|F_m| \leq mn - \ell$$

Since no edge e_q, $q < i$, satisfies $e_q < e_i$, we have $|R_i| = i-1$, and the number of edges in F_i incident with R_i is thus at most $(i-1)n$, and by the definition of ℓ we have $|F_i| \leq (i-1)n + \ell^2$. Hence

$$\sum_{q \leq m} |F_q| \leq i|F_i| + (m-i)|F_m| \leq i((i-1)n + \ell^2) + (m-i)(mn - \ell).$$

Hence

$$m(m-1)n - \sum_{q \leq m} |F_q| \geq m(m-1)n - [i((i-1)n + \ell^2) + (m-i)(mn-\ell)] = (i-1)(m-i)n + (m-i)\ell - i\ell^2$$
$$= [(i-1)(m-i)n - (i-1)\ell^2] + [(m-i)\ell - \ell^2]$$

Since $\ell \leq m-i$ and $\ell \leq n$ both bracketed terms are non-negative, so $m(m-1)n - \sum_{q \leq m} |F_q| \geq 0$, reaching the desired contradiction.

4.2. **Using the short edges as landmarks and a first point of reference.** Let us now turn to the proof of the general case. For $1 \leq j \leq p-1$ write $s_j = i_j - i_{j-1}$ and let $S_j = \{i_{j-1}+1, i_{j-1}+2, \ldots, i_j\}$, so that $|S_j| = s_j$.

By (2) $|F_k| \leq |F_{i_j}|$ for every $k \in S_j$, and hence

$$\sum_{k \leq m} |F_k| \leq \sum_{j \leq p} s_j |F_{i_j}| \tag{3}$$

The vertices in R_{i_j} are of degree at most n, and hence the number of edges in F_{i_j} incident with R_{i_j} is at most $n|R_{i_j}|$. We use $n|R_{i_j}|$ as a baseline estimate on $|F_{i_j}|$. In this estimate we are ignoring the edges of F_{i_j} not incident with R_{i_j}, and also the double counting of edges.

If there are no short edges then $|R_m| = m-1$, and hence $|F_m| \leq n|R_m| = (m-1)n$. Since $|F_i| \leq |F_m|$ for all $i \leq m$, we have $\sum_{i \leq m} |F_i| \leq m(m-1)n$, a contradiction. We shall use this calculation as a first point of reference, and to get the real quantities we shall measure the deviations from the estimate $|F_i| = (m-1)n$.

The existence of short edges affects the estimate of $\sum_{j \leq p} s_j |F_{i_j}|$ in two ways - adding something to it, and deducting something. The first we call "loss", since it takes us further away from the desired contradiction, and the second is called "gain". We shall associate a gain G_j and a loss L_j with each short edge e_{i_j}, and we shall show that $G_j \geq L_j$ for every $j \leq p$. Note that our calculation is not uniquely determined, since adding the same number to G_j and to L_j does not change the total balance.

Clearly, $|R_{i_j}|$ is $i_j - 1$, plus the number of short edges contained in R_{i_j}. Compared with the estimate $|R_{i_j}| = m-1$ above, the estimate $|R_{i_j}| = i_j - 1$ gives a gain of $m - i_j$ on $|R_{i_j}|$, yielding a gain of $n(m-i_j)$ on the estimate $n|R_{i_j}|$ of $|F_{i_j}|$, which yields a total gain of

$$s_j(m-i_j)n$$

in (3). In order to obtain an estimate serving as a second point of reference, we assume that $e_{i_j} \subseteq R_{i_k}$ for all $k > j$. This entails a loss of $s_k n$ in (3) for each such k, so altogether there is a loss of

$$n(s_{j+1} + s_{j+2} + \ldots + s_p) = n(m - i_j).$$

So, the net gain with respect to the baseline estimate is so far $s_j(m-i_j)n - n(m-i_j) = (s_j - 1)(m-i_j)n$. Writing

$$G_j^{BASIC} = (s_j - 1)(m - i_j)n \tag{4}$$

we can use G_j^{BASIC} as a baseline gain.

4.3. **The loss on edges outside R_{i_j}.** In the above calculation there is an overoptimistic assumption: that all edges in F_{i_j} are incident with R_{i_j}. In fact this is false for all $j < p$. By shiftedness and the definition of ℓ_j^M, ℓ_j^W, T_j^M and T_j^W there can be at most $(\ell_j^M + t_j^M)(\ell_j^W + t_j^W)$ edges that are not incident with R_{i_j}.

Remembering that $|F_{i_j}|$ is multiplied by s_j in (3), this entails a possible loss of

$$L_j := s_j(\ell_j^M + t_j^M)(\ell_j^W + t_j^W) \tag{5}$$

This is the only loss we encounter, besides the loss incurred by short edges being contained in sets R_{i_j}, that has already been subsumed in G_j^{BASIC}.

4.4. **Two types of regains.** We shall use two types of regains, related to two ways in which $|F_{i_j}|$ was overestimated.

(1) Gains on procrastination. If $k < j$ we were assuming above that $e_{i_k} \subseteq R_{i_j}$. When this does not happen we say that j *procrastinates with respect to* k (meaning that R_{i_j} is late to capture the edge e_{i_k}), and then $|R_{i_j}|$ was overestimated by 1, giving rise to a gain of n in $|F_{i_j}|$, and to a gain of $s_j n$ in the total sum.
(2) Gains on double counting. In the basic estimate $n|R_{i_j}|$ of the number of edges incident with R_{i_j} there is an overestimate of 1 on each pair (u, v) of vertices in R_{i_j}, where $u \in M$ and $v \in W$. This entitles us to a gain of s_j in the total sum.

4.5. **A first gain on double counting, and a first offset with L_j.**

Without loss of generality we may (and will) assume that $\ell_j^M \geq \ell_j^W$, and that $\mathrm{tail}(e_{i_j}) \in W$. Then

$$L_j \leq s_j[\ell_j^M(\ell_j^W + t_j^M + t_j^W) + t_j^M t_j^W] \tag{6}$$

Here we turn to our first gain on double counting. Let $E_j = \{e_i \mid i < i_j\}$ be the partial rainbow matching chosen so far. Let $\bar{T}_j^M = E_j[T_j^W]$ (namely the set of vertices in M matched by E_j to T_j^W), and let $\bar{T}_j^W = E_j[T_j^M]$. The edges of $\bar{T}_j^M \times \bar{T}_j^W$ were counted twice in the estimate nR_{i_j} of the number of edges incident with R_{i_j}. This entitles us to a gain of $t_j^M t_j^W$ in the calculation of $|F_{i_j}|$, which results in a regain of $s_j t_j^M t_j^W$ in (3). Offsetting this with part of L_j as appearing in (6), and writing

$$\lambda_j := \ell_j^W + t_j^M + t_j^W, \tag{7}$$

this leaves us with a loss of at most

$$L_j^r := s_j \ell_j^M \lambda_j \tag{8}$$

The superscript r stands for "remaining". This loss should be offset by G_j^{BASIC} and by other gains.

Observation 4.1. $\lambda_j < n$.

Proof. This follows from the fact that $W \setminus R_{i_j}$ contains two disjoint sets: $SKIP_j$, which is of size ℓ_j^W, and T_j^W, which is of size t_j^W; and $W \cap R_{i_j}$ contains $\bar{T}_j^W$, which is of size t_j^M. Thus $\lambda_j = \ell_j^W + t_j^M + t_j^W \leq |W| = n$. We may assume strict inequality, since otherwise W is completely matched by E_j.

□

5. Gains associated with vertices in $SKIP_j^M$

5.1. **Six types of vertices in $SKIP_j^M$ and the regains associated with them.**

Notation 5.1. For $v \in R_m$ let $i(v)$ be the index i for which $v \in e_i$, and let $k(v)$ be the index $k < p$ such that $i(v) \in S_k$.

Notation 5.2. Let Σ_j be the set of short edges contained in R_{i_j}, and let $|\Sigma_j| = \sigma_j$. Also let $M_j = M \cap R_{i_j}$ and $\mu_j = |M_j|$.

Notation 5.3. Let $\omega = \omega(j) = \min(k : R_{i_k} \supseteq e_{i_j})$.

Lemma 5.4. *If $k < j$ and* $\mathrm{head}(e_{i_k}) \in SKIP_j^M$ *then* $\lambda_k < \mu_j + \ell_j^M$.

Proof. This follows from the fact that

$$E_k[T_k^W] \cup SKIP_k^M \cup T_k^M \subsetneq (R_{i_j} \cap M) \cup SKIP_j^M$$

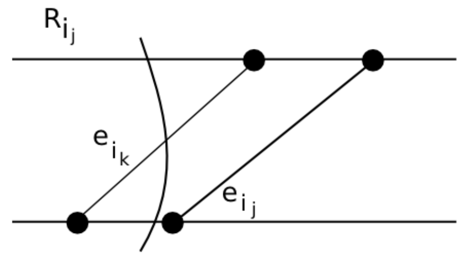

FIGURE 3. Case (1)

and on both sides the terms of the union are disjoint. The reason for the strict containment is that head(e_{i_j}) belongs to the right hand side and not to the left. In fact, the strict inequality in the lemma will not be used, it is only mentioned for clarification. □

Lemma 5.5. $\lambda_j \leq \mu_j + \ell_j^M + t_j^W$.

This follows from the fact that $\ell_j^W \leq \ell_j^M$ and $t_j^M \leq \mu_j$. □

Lemma 5.6. $R_{i_\omega} \supseteq T_j^M$

Proof. By the definition of T_j^M the vertex head(e_{i_j}) is adjacent to its first element, so the initial segment of head(e_{i_j}) in M, together with T_j^M, is an interval contained in Z_{i_j}. Applying the definition of R_{i_ω} yields the lemma. □

We write L_j^r as a sum:

$$L_j^r = L_j^a + L_j^b$$

where

$$L_j^a = \ell_j^M (s_j - 1)\lambda_j,\ L_j^b = \ell_j^M \lambda_j. \tag{9}$$

The expression (4) for G_j^{BASIC} explains why this splitting will be useful: G_j^{BASIC} will count towards offsetting L_j^a.

We shall have two "baskets" of gains for each j, which we shall call G_j^a (intended to compensate for L_j^a) and G_j^b (intended to compensate for L_j^b). To compensate for L_j^b, we need to assign to each of the ℓ_j^M vertices in $SKIP_j^M$ a gain of at least λ_j, which is given to G_j^b.

For the purpose of bookkeeping, we gather the vertices of $SKIP_j^M$ into six types, according to the conditions they satisfy. Vertices of types (2b) and (3) below will give rise to regains on double counting, while all other types will give rise to regains on procrastination. In all these cases a gain is given to G_h^b, where h is the smaller of j and k, namely if $j < k = k(v)$ at least λ_j is given to G_j^b, and if $k = k(v) < j$ at least λ_k is given to G_k^b.

In two of the cases, namely (2ai) and (1), the gain will be split between the two indices. The part given to the larger index will go to G^a of that index.

Here are the explicit classification and the rules by which gains are shared. The regain of λ_j for each vertex $v \in SKIP_j^M$ will be apparent in each of the cases, while the regains accumulating to G_j^a will be collected at the end.

(1) $k(v) < j$, implying that $v = \text{head}(e_{i_k})$ (see Figure 3). In this case j procrastinates with respect to k, entitling us to a gain of n on $|F_{i_j}|$, and $s_j n$ in total. This gain we split between G_j^b, G_k^a and G_k^b, as follows: G_k^a gets $(s_j - 1)(\mu_j + \ell_j^M)$, G_j^a gets $(s_j - 1)(n - \mu_j - \ell_j^M)$ and G_j^b gets n.

Denote by A_j the set of vertices of type (1), and let $\alpha_j = |A_j|$. The accumulating regain in G_j^a in this way is

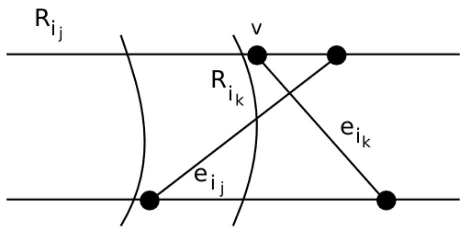

FIGURE 4. Case 2a(i), one type of crossing

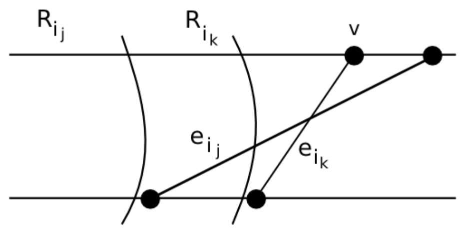

FIGURE 5. Case 2a(i), another type of crossing

$$(s_j - 1)(n - \mu_j - \ell_j^M)\alpha_j \tag{10}$$

G_j^b gets $n\alpha_j$, and since $\lambda_j < n$ this means that it gets more than λ_j for each vertex of this type, as promised.

(2) $j < k(v)$. This we divide into the following subcases:

(a) $e(v)$ is long and $k = k(v) < \omega$ and $e_{i_j} \not\subseteq R_{i_k}$, or $i(v) = i_k$ (see Notation 5.1 for the definition of $i(v)$). The latter means that $e(v) = e_{i_k}$.
In this case k procrastinates with respect to j, which entitles us to a regain of $s_k n$. Note that there are at most $s_k - 1$ vertices $v \in SKIP_j^M$ that are tails of long edges, and satisfy $k(v) = k$. So, distributing this gain among the vertices $v \in SKIP_j^M$ that are tails of long edges, and satisfy $k(v) = k$, each gets at least a gain of n. Remembering that $\lambda_j < n$, we are fulfilling the requirement of "λ_j gain in G_j^b for every vertex in $SKIP_j^M$".
The splitting of the gain between G_j and G_k is done in this case according to a still finer classification into subcases:

(i) e_{i_j} and e_{i_k} cross (see Definition 3.2. We do not discern in this case between the cases tail$(e_{i_k}) \in M$ and tail$(e_{i_k}) \in W$ - see Figures 4 and 5 for the two possibilities. We give a gain of $n-1$ to G_j^b, saving 1 for a fine point below (see remark after case (3)). By Observation 4.1 we are giving G_j^b at least λ_j, as required.

(ii) e_{i_j} and e_{i_k} are parallel. Here k is procrastinating with respect to j, and thus we are entitled to a gain of $s_k n$. This is the same as Case (1), with the roles of j and k reversed. This regain (that is shared between stages j and k) was considered in (1) for stage i_k, and hence we do not distribute regains for this case. But recall that G_j^b gets in stage i_k its share of $(s_k - 1)(\mu_k + \lambda_k)$ (keep in mind that the roles of the indices j and k are reversed). By Lemma 5.4 this quantity is at least λ_j for each such vertex.

(b) $e_{i_j} \subseteq R_{i_k}$. Then necessarily $k = \omega$ (see Figure 6).
With such vertices we associate a regain on double counting in the estimate $n|R_{i_k}|$ towards calculating $|F_{i_k}|$, of all edges in $(M_j \cup SKIP_j^M \cup T_j^M) \times \{\text{tail}(e_{i_j})\}$. The number of these edges is $\mu_j + \ell_j^M + t_j^M$. We give G_j^b the amount of $h(j)\lambda_j$, where $h(j)$ is the number of vertices in $SKIP_j^M$ having $k(v) = \omega(j)$.
Note that no regain of this type is counted more than once. To see this it is best to view the regain associated with each vertex v of this type as a regain on the calculation of $|F_{i(v)}|$ itself, rather than using the inequality $|F_{i(v)}| \leq |F_{i_k(v)}|$. Viewed this way, the sets of edges (which are actually stars) on which there is double counting in $|F_{i(v)}|$ are disjoint for different v's. Note also that by Lemma 5.5 for each vertex of the present type we are adding at least λ_j to G_j^b, as required.

(3) $k(v) = j$, meaning that $v = \text{head}(e_{i_j})$.

On this vertex we have the same regain as on vertices of type (2b), with a gain of $\mu_j + \ell_j^M + T_j^M$ given to G_j^b.

We have to be careful in this calculation, since in this case there is danger of considering the double counting of an edge twice. Here it may happen that for distinct j_1 and j_2 the vertices head$(e_{i_{j_1}})$ and head$(e_{i_{j_2}})$ both represent the same set of edges, namely $|F_{i_\omega}|$. Since one side in each edge considered is tail(e_{i_j}), this can happen only in one case: when $\omega(j_1) = \omega(j_2)$ for indices $j_1 \neq j_2$, and that tail$(e_{i_{j_1}})$

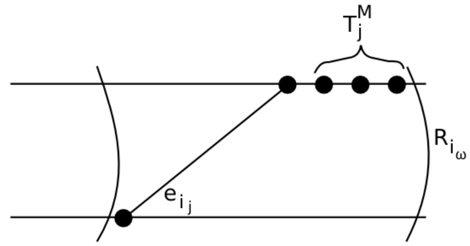

FIGURE 6. Case 2(b)

and tail($e_{i_{j_2}}$) are on different sides. In this case the double counting on the edge (tail($e_{i_{j_1}}$), tail($e_{i_{j_2}}$)) is taken into account twice, while it should have been taken only once. In this case we can compensate for this double-double counting in the following way. Without loss of generality assume that $j_1 < j_2$. Since $\omega(j_1) = \omega(j_2)$, the index j_2 procrastinates with respect to j_1, which means that we are entitled to a gain of n in the calculation of $|F_{i_{j_2}}|$, hence a gain of $s_{j_2}n$ in the total sum. We only used $s_{j_2}\lambda_{j_1}$, and since $s_{j_2} \geq 1$ and $\lambda_{j_1} < n$ (see Observation 4.1), we have the desired compensation.

Note that the regains given above to G_j^b cover all of L_j^b.

5.2. **Another regain on double counting.** We are entitled to another type of regain, on edges containing A_j vertices. In the calculation of $|F_{i_j}|$ all edges between $E_j[A_j]$ and $M \cap R_{i_j}$ are counted twice, so we are entitled to a regain of $\mu_j\alpha_j$ on $|F_{i_j}|$, and thus of $s_j\mu_j\alpha_j$ in total. In order to avoid considering this double counting more than once we do not take into account vertices contained in j-short edges - see Case (2b) above. Thus the regain is $s_j(\mu_j - \sigma_j)\alpha_j$, which for ease of later calculations we shall replace by the possibly smaller

$$(s_j - 1)(\mu_j - \sigma_j)\alpha_j \tag{11}$$

Example 5.7. To see why giving gains to the earlier indices is necessary, consider again Example 3.3. There $L_1 = q^2$, which is regained by a double count argument for i_2 (the second short edge). In the baseline argument $|F_{i_2}|$, and with it $|F_k|$, $k \in S_2$, $k > 2$ are estimated as $|R_{i_2}|n$. In this calculation all q edges $m_c w_1$ in R_{i_2} are double counted, so there is a gain of q in the calculation of $|F_{i_2}|$, resulting in a gain of $s_2 q = q^2$ in the baseline calculation - precisely L_1.

6. COLLECTING THE G_j^a GAINS

Lemma 6.1. $m - i_j \geq \ell_j^M - \alpha_j$.

Proof. This follows from the fact that every vertex in $SKIP_j^M \setminus A_j$ is matched by some edge e_i, $i \geq i_j$. □

By the lemma and the definition of G_j^{BASIC} (see (4)) we have:

$$G_j^{BASIC} \geq (s_j - 1)(\ell_j^M - \alpha_j)n \tag{12}$$

Lemma 6.2. $n \geq \ell_j^W + t_j^M + \sigma_j + t_j^W + \alpha_j$.

This follows from the fact that that $\alpha_j, t_j^M, t_j^W, \ell_j^W$ and σ_j are sizes of disjoint subsets of W, namely $A_j, SKIP_j^W, E_j[T_j^M], T_j^W$ and $\bigcup \Sigma_j \cap W$. □

The regain in (10), $(s_j - 1)(n - \mu_j - \ell_j^M)\alpha_j$, together with the regain of (11), $(s_j - 1)\alpha_j(\mu_j - \sigma_j)$, and G_j^{BASIC} sum up to

$$(s_j - 1)(\ell_j^M - \alpha_j)n + (s_j - 1)(n - \mu_j - \ell_j^M)\alpha_j + (s_j - 1)\alpha_j(\mu_j - \sigma_j)$$

and we need to show that this sum, that is a lower bound for G_j^a, is at least L_j^a. Namely, we have to show that

$$(s_j-1)(\ell_j^M-\alpha_j)n+(s_j-1)(n-\mu_j-\ell_j^M)\alpha_j+(s_j-1)\alpha_j(\mu_j-\sigma_j)\geq(s_j-1)\ell_j^M\lambda_j.$$

Canceling out the term s_j-1 and additive terms, we need to prove the following:

$$\ell_j^M(n-\alpha_j)-\sigma_j\alpha_j\geq\ell_j^M\lambda_j$$

By Lemma 6.2 $\lambda_j\leq n-\alpha_j-\sigma_j$. Thus it is enough to show that $\ell_j^M(n-\alpha_j)-\sigma_j\alpha_j\geq\ell_j^M(n-\alpha_j-\sigma_j)$, which follows from the fact that $\alpha_j\leq\ell_j^M$ (A_j being contained in $SKIP_j^M$).

This shows that $G_j^a\geq L_j^a$, thereby completing the proof of Theorem 3.1.

7. Proof of Theorem 1.5

Let $\mathcal{F}$ be a collection of hypergraphs satisfying the condition of the theorem. Order the vertices of the first vertex class V_1 as $v_1,\dots,v_n$. By Lemma 2.2 we may assume that all F_i are shifted with respect to this order. Let i_1 be such that F_{i_1} has maximal degree at v_1 among all F_i's. Then we choose $i_2\neq i_1$ for which F_{i_2} has maximal degree at v_2 among all F_i, $i\neq i_1$, and so forth. To save indices, reorder the F_i's so that $i_j=j$ for all j. Let H_j be the set of 2-edges incident with v_j in F_j. It clearly suffices to show that the collection $\mathcal{H}=(H_j:j\leq k)$ of subgraphs of $K_{n,n}$ has a rainbow matching, so it suffices to show that $\mathcal{H}$ satisfies the conditions of Theorem 3.1. Assuming it does not, since the sizes $|H_j|$ are descending, $\sum_{k-t<j\leq k}|H_j|=\sum_{k-t<j\leq k}\deg_{F_j}(v_j)\leq t(t-1)n$ for some $t<k$. We shall reach a contradiction to the assumption that $|F_k|>(k-1)n^2$.

Write m for $|H_k|$. Clearly

$$\sum_{j\leq k-t}\deg_{F_k}(v_j)\leq(k-t)n^2$$

and by the order by which F_j were chosen

$$\sum_{k-t<j\leq k}\deg_{F_k}(v_j)\leq\sum_{k-t<j\leq k}\deg_{F_j}(v_j)\leq t(t-1)n$$

Since $\sum_{k-t<j\leq k}\deg_{F_k}(v_j)\geq mt$, this implies that $m\leq n(t-1)$.

By the shifting property,

$$\sum_{k<j\leq n}\deg_{F_k}(v_j)\leq m(n-k)\leq n(t-1)(n-k)$$

And so:

$$\sum_{j>k-t}\deg_{F_k}(v_j)\leq t(t-1)n+(t-1)n(n-k)=n(t-1)(t+n-k)\leq(t-1)n^2$$

Hence

$$|F_k|=\sum_{j\leq k}\deg_{F_k}(v_j)\leq(k-t)n^2+(t-1)n^2=(k-1)n^2$$

Which is the desired contradiction.

8. A remark and further conjectures

Not surprisingly, Conjecture 1.3 is easy for large n.

Remark 8.1. For every r and k there exists $n_0=n_0(r,k)$ such that Conjecture 1.3 is true for all $n>n_0$.

Proof. By Lemma 2.2 we may assume that all F_i's are shifted. Let A_i consist of the first $k-1$ vertices in V_i $(i \le r)$, and let $A = \bigcup_{i \le r} A_i$. Since the number of edges meeting A in two points or more is $O(n^{r-2})$, for large enough n for each i there exist at least $k-1$ points x in A such that $e \cap A = \{x\}$ for some $e \in F_i$. Hence we can choose edges $e_i \in F_i$ and distinct points $x_i \in A$ $(i \le k-1)$ such that $e_i \cap A = \{x_i\}$. Since the number of edges going through $x_1, \ldots, x_{k-1}$ is no larger than $(k-1)n^{r-1}$, there exists an edge e_k in F_k missing $x_1, \ldots, x_{k-1}$. Using the shifting property, we can replace inductively each edge e_i $, i \le k-1$, by an edge $e'_i \in F_i$ contained in A, missing e_k and missing all e'_j, $j < i$. This yields a rainbow matching for $F_1, \ldots, F_k$. □

Theorem 1.4 may be true also under the more general condition of degrees bounded by n.

Conjecture 8.2. *Let $d > 1$, and let $F_1, \ldots, F_k$ be bipartite graphs on the same ground set, satisfying $\Delta(F_i) \le d$ and $|F_i| > (k-1)d$. Then the system $F_1, \ldots, F_k$ has a rainbow matching.*

For $d = 1$ this is false, since for every $k > 1$ there are matchings $F_1, \ldots, F_k$ of size k not having a rainbow matching.

Theorem 3.1 has a simpler counterpart, which we believe to be true:

Conjecture 8.3. *If F_i, $i \le k$ are subgraphs of $K_{n,n}$ satisfying $|F_i| \ge in$ for all $i \le k$, then they have a rainbow matching.*

Theorem 8.4. *Conjecture 8.3 is true for $n > \binom{k}{2}$.*

Proof. As before, we assume that F_i are all shifted. Number one side of $K_{n,n}$ as $w_1, \ldots, w_n$. Let $d_{i,j} = \deg_{F_i}(w_j)$. Let M be a $k \times k$ 0,1 matrix, defined by: $m_{i,j} = 1$ if $d_{i,j} > k - j$ and $m_{i,j} = 0$ otherwise. It is enough to find a permutation $\pi : [k] \to [k]$ such that $m_{\pi(j),j} = 1$ for all j, since then one can match the vertices w_j in $F_{\pi(j)}$ greedily, one by one, starting at w_k: at the j-th step, when $w_k, \ldots, w_{k-j+1}$ have already been matched, since $\deg_{F_{\pi(j-k)}}(w_{k-j}) \ge j$ there exists at least one edge in $F_{\pi(k-j)}$ incident with w_{k-j} that can be added to the rainbow matching.

Assuming that there is no such permutation π, by Hall's theorem there is a set J of p columns of M and a set I of $k-p+1$ rows, such that $m_{i,j} = 0$ for all $i \in I$, $j \in J$. Let q be the largest element of I. Then $q \ge k-p+1$. We shall show that $|F_q| < n(k-p+1)$, contradicting the assumption of the conjecture.

Let $J = \{j_1, j_2, \ldots, j_p\}$, arranged in ascending order. Since $q \in I$, we have $d_{q,j_s} \le k - j_s$ for all $s \le p$. Since the sequence $d_{q,j}$ is non increasing in j, we have:

$$(13)\quad |F_q| = \sum_{j \le n} d_{q,j} \le n(j_1-1)+(j_2-j_1)(k-j_1)+(j_3-j_2)(k-j_2)+\ldots+(j_{p-1}-j_p)(k-j_{p-1})+(n-j_p+1)(k-j_p)$$

Call the right hand side of (13) $c(J)$. Suppose that there exists $s < p$ such that $j_s + 1 < j_{s+1}$. Then, moving j_s to the right, namely replacing j_s in J by $j_s + 1$, decreases $c(J)$ by 1 (the decrease in the term corresponding to j_s) and increases by $j_{s+1} - j_s - 1$ (corresponding to the increase in the terms between $j_s + 1$ and j_{s+1}). This means that $c(J)$ has not decreased. Hence, writing j for j_p, we have:

$$(14)\qquad c(J) \le c(\{j-p+1, j-p+2, \ldots, j\})$$

Writing $\gamma(j)$ for the right hand side of (14), we have:

$$\gamma(j) = n(j-p) + (k-j+p-1) + (k-j+p+2) + \ldots + (k-j) + (n-j)(k-j) =$$
$$\binom{p}{2} + p(k-j) + n(j-p) + (n-j)(k-j)$$

This is a quadratic expression in j, which attains its maximum at one of the two extremes, $j = p$ or $j = k$. In fact, for both values of j it attains the same value, $\binom{p}{2} + n(k-p)$. We have shown that $|F_q| < \binom{p}{2} + n(k-p)$. By the assumption $n > \binom{k}{2}$ this implies that $|F_q| < n(k-p+1)$, which is the desired contradiction.

□

To formulate yet another conjecture we shall use the following notation:

Notation 8.5.

(1) For a sequence $a = (a_i,\ 1 \le i \le k)$ of real numbers we denote by $\overrightarrow{a}$ the sequence rearranged in non-decreasing order.
(2) Given two sequences a and b of the same length k, we write $a \le b$ (respectively $a < b$) if $\overrightarrow{a}_i \le \overrightarrow{b}_i$ (respectively $\overrightarrow{a}_i < \overrightarrow{b}_i$) for all $i \le k$.

Given subgraphs $F_i,\ i \le k$ of $K_{n,n}$, define a $k \times n$ matrix $A = (a_{ij})$ as follows. Order one side of the bipartite graph as $v_1, v_2, \dots, v_n$, and let $a_{ij} = \deg_{F_i}(v_j)$. The i-th row sum $r_i(A)$ of A is then $|F_i|$. Thus, Theorem 3.1 can be formulated as follows:

Theorem 8.6. *If $\sum_{i \le j} \overrightarrow{r}_i > j(j-1)n$ for every $j \le k$ then there exists a permutation $\pi : [k] \to [k]$ such that $a_{i\pi(i)} \ge (1, 2, \dots, k)$.*

We believe that the following stronger conjecture is true:

Conjecture 8.7. *If $\sum_{i \le j} \overrightarrow{r}_i > j(j-1)n$ for every $j \le k$ then there exists a permutation $\pi : [k] \to [k]$ such that $\sum_{i \le j} \overrightarrow{a}_{i\pi(i)} > j(j-1)$ for every j.*

9. The case of the complete r-uniform hypergraph

As remarked above, the topic of this paper belongs to the family of Erdős–Ko–Rado problems. It is natural to assume that an analogous conjecture to Conjecture 1.3 is true in the more involved case of complete r-uniform hypergraphs, which is the topic of the EKR theory. Denote by $\binom{[n]}{r}$ the set of subsets of size r of $[n]$. Let $f(n, r, k)$ be the minimal number such that every hypergraph larger than $f(n, r, k)$ contained in $\binom{[n]}{r}$ contains a matching of size k. The EKR theorem states that if $r \le \frac{n}{2}$ then $f(n, r, 2) = \binom{n-1}{r-1}$.

Conjecture 9.1. *If $F_1, F_2, \dots, F_k \subseteq \binom{[n]}{r}$ are of size larger than $f(n, r, k)$ then there exists a rainbow matching, i.e. a choice of disjoint edges $f_i \in F_i$.*

For $k = 2$ this conjecture follows from results in [9, 11], that in [9] were also extended to two hypergraphs of different uniformities.

The following was proved in [5]:

Theorem 9.2. *If $n \ge kr$ then $f(n, r, k) \le (k-1)\binom{n-1}{r-1}$.*

A rainbow version of this theorem was proved in [8]:

Theorem 9.3. [8] *If $F_1, \dots, F_k$ are hypergraphs, where F_i is r_i-uniform and $n \ge \sum_{i \le k} r_i$ and $|F_i| > (k-1)\binom{n-1}{r_i-1}$ then the family $(F_1, \dots, F_k)$ has a rainbow matching.*

In [7] the case $r = 3$ of Conjecture 9.1 is solved for $n \ge 4k - 1$.

In [3] the value of $f(n, 2, k)$ was determined for all k:

Theorem 9.4. $f(n, 2, k) = \max\{\binom{2k-1}{2}, (k-1)(n-1) - \binom{k-1}{2}\}$.

In [2] this result was given a short proof, using shifting. Meshulam [10] noted that this proof yields also Conjecture 9.1 for $r = 2$:

Theorem 9.5. *Let $\mathcal{F} = (F_i,\ 1 \le i \le k)$ be a collection of subsets of $E(K_n)$. If $|F_i| > \max(\binom{2k-1}{2}, (k-1)(n-1) - \binom{k-1}{2})$ for all $i \le k$ then $\mathcal{F}$ has a rainbow matching.*

Proof. Enumerate the vertices of K_n as $v_1, v_2, \dots, v_n$. By Lemma 2.2 we may assume that all F_i's are shifted with respect to this enumeration. For each $i \le k$ let $e_i = (v_i, v_{2k-i+1})$. We claim that the sequence e_i is a rainbow matching for $\mathcal{F}$. Assuming negation, there exists i such that $e_i \notin F_i$. Since F_i is shifted, every edge (v_p, v_q) in F_i, where $p < q$, satisfies

(P) $p < i$ or $q < 2k - i + 1$.

The number of pairs satisfying $p < i$ is $(i-1)(n-1) - \binom{i-1}{2}$. The number of pairs satisfying $p \geq i$ and $q < 2k - i + 1$ is $\binom{2k-2i+1}{2}$, so

$$|F_i| \leq (i-1)(n-1) - \binom{i-1}{2} + \binom{2k-2i+1}{2}$$

This is a convex quadratic expression in i, attaining its maximum either at $i = 1$ (in which case $|F_i| \leq \binom{2k-1}{2}$) or at $i = k$ (in which case $|F_i| \leq (k-1)(n-1) - \binom{k-1}{2}$). In both cases we get a contradiction to the assumption on $|F_i|$. $\square$

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